Available online at http://scik.org
J. Math. Comput. Sci. 2 (2012), No. 2, 241-254
ISSN: 1927-5307

OSCILLATION OF IMPULSIVE NEUTRAL DIFFERENTIAL EQUATION WITH SEVERAL POSITIVE AND NEGATIVE COEFFICIENTS

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Abstract. This paper is concerned with the oscillation of solutions of impulsive neutral differential equation with several positive and negative coefficients of the form

$$[x(t) - R(t)x(t-\gamma)]' + \sum_{i=1}^{m} P_i(t)x(t-\tau_i) - \sum_{j=1}^{n} Q_j(t)x(t-\sigma_j) = 0, \quad t \geq t_0, t \neq t_k$$

$$x(t_k^+) = I_k(x(t_k)), \quad k = 1, 2, 3, \ldots.$$

Our results are generalization of some known results in literature.An example is also given to illustrate our results.

Keywords: Oscillation, neutral,impulsive,differential equation,coefficients.

2000 AMS Subject Classification:34A37,34C10

1. Introduction

In recent years, the theory of impulsive differential equations received much attention and a number of papers have been published in this field. This is due to wide possibilities for their applications in control theory, physics, biology,population dynamics, economics,

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Received January 19, 2012

etc. For further applications and questions concerning existence and uniqueness of solutions of impulsive differential equations one can refer [1, 2]. Oscillatory properties of linear impulsive differential equations with a single constant delay were studied by Gopalsamy and Zhang [3]. Later papers give more attention to oscillatory behaviour of linear or nonlinear impulsive differential equations include Bainov et al.[4] and Chen et al.[5]. In [6, 7], Luo et al. and Graef et al. investigated the oscillation of neutral impulsive differential equations with one or more delays. Recently, in [8, 9, 10],the authors studied the oscillations of solutions of first order impulsive differential equation with positive and negative coefficients. Motivated by the results of [9], in the present paper we obtain the oscillation of impulsive differential equation with several positive and negative coefficients. Our results are generalization of some known results in literature.

Consider following impulsive neutral differential equation with several positive and negative coefficients of the form

$$[x(t) - R(t)x(t-\gamma)]' + \sum_{i=1}^{m} P_i(t)x(t-\tau_i) - \sum_{j=1}^{n} Q_j(t)x(t-\sigma_j) = 0, \quad t \geq t_0, t \neq t_k \quad (1.1)$$

$$x(t_k^+) = I_k(x(t_k)), \quad k = 1, 2, 3, \ldots. \quad (1.2)$$

where

(A1) $\gamma > 0, \tau_i, \sigma_j \geq 0$;

(A2) $R \in PC([t_0,\infty),(0,\infty))$, $P_i, Q_j \in C([t_0,\infty),(0,\infty))$, $i = 1, 2 \ldots, m$ and $j = 1, 2 \ldots, n$;

(A3) $I_k(x)$ is continuous in $(-\infty,+\infty)$, and there exist positive numbers b_k^*, b_k such that $b_k^* \leq \frac{I_k(x)}{x} \leq b_k$ for $x \neq 0$ and $k = 1, 2. \ldots.$

2. Preliminaries

Throughout this paper, we always assume that (A1)-(A3) and

(A4) there exists a positive number $p \leq m$ and a partition of the set $\{1, 2, \ldots, n\}$ in to p disjoint subsets $J_1, J_2, \ldots, J_p$ such that $l \in J_i, \tau_i \geq \sigma_l$ with

$$H_i(t) = P_i(t) - \sum_{l \in J_i} Q_l(t - \tau_i + \sigma_l) \geq 0, \text{ for } i = 1, 2, \ldots p;$$

$$H_i(t) = P_i(t) \text{ for } i = p+1, \ldots m, H_i(t) \not\equiv 0 \ \text{ on } (t_{k-1}, t_k]\,(k \geq 1) \ \text{ hold.}$$

Let $\rho = max\{\gamma, \tau_i, \sigma_j\}$ and $\delta = min\{\gamma, \tau_i, \sigma_j\}$,$1 \leq i \leq m, 1 \leq j \leq n$. With equations (1.1)and (1.2), one associates an initial condition of the form

$$x(t_0 + s) = \phi(s), \ s \in [-\rho, 0], \tag{2.1}$$

where $\phi \in PC([-\rho, 0], R) = \{\phi : [-\rho, 0] \to R$ such that ϕ is continuous everywhere except at the finite number of points η and $\phi(\eta^+)$ and $\phi(\eta^-)$ exist with $\phi(\eta^+) = \phi(\eta^-)\}$.

A real valued function $x(t)$ is said to be a solution of the initial value problem (1.1),(1.2) and (2.1) if

(i) $x(t) = \phi(t - t_0)$ for $t_0 - \rho \leq t \leq t_0$, $x(t)$ is continuous for $t \geq t_0$ and $t \neq t_k$, $k = 1, 2, 3, \ldots$.

(ii) $[x(t) + R(t)x(t - \gamma)]$ is continuously differentiable for $t > t_0$, $t \neq t_k$, $t \neq t_k + \gamma$, $t \neq t_k + \tau_i$, $t \neq t_k + \sigma_j$, $k = 1, 2, 3, \ldots$ and satisfies (1.1).

(iii) for $t = t_k$, $x(t_k^+)$ and $x(t_k^-)$ exist with $x(t_k^-) = x(t_k)$ and satisfies (1.2).

A solution of (1.1)-(1.2) is said to be non oscillatory if the solution is eventually positive or eventually negative. Otherwise the solution is said to be oscillatory. Our results generalize the results of [8].

3. Main results

Lemma 3.1. *Assume that* $b_0 = 1$, $0 < b_k \leq 1$ *for* $k = 1, 2, 3, \ldots$ *and*

$$R(t_k^+) \geq R(t_k) \quad \text{for } k \in E_{1k} = \{k \geq 1, t_k - \gamma \neq t_{\overline{k}}, \overline{k} < k\} \tag{3.1}$$

$$\overline{b}_k R(t_k^+) \geq R(t_k) \quad \text{for } k \in E_{2k} = \{k \geq 1, t_k - \gamma = t_{\overline{k}}, \overline{k} < k\} \tag{3.2}$$

where $\overline{b}_k = b^*_{\overline{k}}$ *when* $t_k - \gamma = t_{\overline{k}}$ $(\overline{k} < k)$. *Let* $x(t)$ *be a solution of (1.1) and (1.2) such that* $x(t-\rho) > 0$ *for* $t \geq t_0$ *and let*

$$z(t) = x(t) - R(t)x(t-\gamma) - \sum_{i=1}^{p}\sum_{l\in J_i}\int_{t-\tau_i+\sigma_l}^{t} Q_l(s)x(s-\sigma_l)ds, \tag{3.3}$$

then $z(t)$ *is decreasing in* $[t_0,\infty)$ *and* $z(t_k^+) \leq b_k z(t_k)$ *for* $k = 1, 2, 3, \ldots$

Proof. From (1.1) and (3.3) we have

$$\begin{aligned} z'(t) &= (x(t) - R(t)x(t-\gamma))' - \sum_{i=1}^{p}\sum_{l\in J_i} Q_l(t)x(t-\sigma_l) + \sum_{i=1}^{p}\sum_{l\in J_i} Q_l(t-\tau_i+\sigma_l)x(t-\tau_l) \\ &= (x(t) - R(t)x(t-\gamma))' - \sum_{j=1}^{n} Q_j(t)x(t-\sigma_j) + \sum_{i=1}^{p}\sum_{l\in J_i} Q_l(t-\tau_i+\sigma_l)x(t-\tau_i). \end{aligned}$$

Hence,

$$z'(t) = -\sum_{i=1}^{p} P_i(t)x(t-\tau_i) - \sum_{i=p+1}^{m} P_i(t)x(t-\tau_i) + \sum_{i=1}^{p}\sum_{l\in J_i} Q_l(t-\tau_i+\sigma_l)x(t-\tau_i). \tag{3.4}$$

Using (A4) we get

$$z'(t) = -\sum_{i=1}^{m} H_i(t)x(t-\tau_i) \leq 0, t_k < t \leq t_{k+1}, k \geq 0. \tag{3.5}$$

From (3.3) it follows that

$$z(t_k^+) = x(t_k^+) - R(t_k^+)x(t_k-\gamma)^+ - \sum_{i=1}^{p}\sum_{l\in J_i}\int_{t_k-\tau_i+\sigma_l}^{t_k} Q_l(s)x(s-\sigma_l)ds. \tag{3.6}$$

If $k \in E_{1k}$, then

$$\begin{aligned} z(t_k^+) &= I_k(x(t_k)) - R(t_k^+)x(t_k-\gamma) - \sum_{i=1}^{p}\sum_{l\in J_i}\int_{t_k-\tau_i+\sigma_l}^{t_k} Q_l(s)x(s-\sigma_l)ds \\ &\leq b_k(x(t_k)) - R(t_k)x(t_k-\gamma) - \sum_{i=1}^{p}\sum_{l\in J_i}\int_{t_k-\tau_i+\sigma_l}^{t_k} Q_l(s)x(s-\sigma_l)ds \\ &\leq x(t_k) - R(t_k)x(t_k-\gamma) - \sum_{i=1}^{p}\sum_{l\in J_i}\int_{t_k-\tau_i+\sigma_l}^{t_k} Q_l(s)x(s-\sigma_l)ds \\ &= z(t_k) \end{aligned}$$

If $k \in E_{2k}$, then

$$\begin{aligned} z(t_k^+) &= I_k(x(t_k)) - R(t_k^+)x(t_k-\gamma)^+ - \sum_{i=1}^{p}\sum_{l\in J_i}\int_{t_k-\tau_i+\sigma_l}^{t_k} Q_l(s)x(s-\sigma_l)ds \\ &\le b_k(x(t_k)) - R(t_k^+)x(t_k^+) - \sum_{i=1}^{p}\sum_{l\in J_i}\int_{t_k-\tau_i+\sigma_l}^{t_k} Q_l(s)x(s-\sigma_l)ds \\ &\le x(t_k) - b_{\overline{k}}^{*}R(t_k^+)x(t_{\overline{k}}) - \sum_{i=1}^{p}\sum_{l\in J_i}\int_{t_k-\tau_i+\sigma_l}^{t_k} Q_l(s)x(s-\sigma_l)ds \\ &= x(t_k) - \overline{b}_k R(t_k^+)x(t_k-\gamma) - \sum_{i=1}^{p}\sum_{l\in J_i}\int_{t_k-\tau_i+\sigma_l}^{t_k} Q_l(s)x(s-\sigma_l)ds \\ &\le x(t_k) - R(t_k)x(t_k-\gamma) - \sum_{i=1}^{p}\sum_{l\in J_i}\int_{t_k-\tau_i+\sigma_l}^{t_k} Q_l(s)x(s-\sigma_l)ds \\ &= z(t_k) \end{aligned}$$

Since $E_{1k} \cup E_{2k} = \{1, 2, 3, \dots\}$ we get $z(t_k^+) \le z(t_k)$ $k = 1, 2, \dots$ This, together with (3.6) implies that $z(t)$ is decreasing in $[t_0, \infty)$.

Finally,since $b_k \le 1$, if $k \in E_{1k}$, then

$$R(t_k^+) \ge R(t_k) \ge b_k R(t_k) \tag{3.7}$$

If follows, from (3.5) and (3.6), that

$$\begin{aligned} z(t_k^+) &= I_k(x(t_k)) - R(t_k^+)x(t_k-\gamma)^+ - \sum_{i=1}^{p}\sum_{l\in J_i}\int_{t_k-\tau_i+\sigma_l}^{t_k} Q_l(s)x(s-\sigma_l)ds \\ &\le b_k(x(t_k)) - b_k R(t_k)x(t_k-\gamma) - b_k\sum_{i=1}^{p}\sum_{l\in J_i}\int_{t_k-\tau_i+\sigma_l}^{t_k} Q_l(s)x(s-\sigma_l)ds \\ &= b_k z(t_k) \end{aligned}$$

If $k \in E_{2k}$, then

$$\overline{b}_k R(t_k^+) \ge R(t_k) \ge b_k R(t_k). \tag{3.8}$$

Thus, we have from (3.5) and (3.7)

$$\begin{aligned}
z(t_k^+) &= I_k(x(t_k)) - R(t_k^+)x(t_k-\gamma)^+ - \sum_{i=1}^{p}\sum_{l\in J_i}\int_{t_k-\tau_i+\sigma_l}^{t_k} Q_l(s)x(s-\sigma_l)ds \\
&\le b_k x(t_k) - R(t_k^+)x(t_{\overline{k}}^+) - \sum_{i=1}^{p}\sum_{l\in J_i}\int_{t_k-\tau_i+\sigma_l}^{t_k} Q_l(s)x(s-\sigma_l)ds \\
&\le b_k x(t_k) - b_{\overline{k}}^* R(t_k^+)x(t_{\overline{k}}) - \sum_{i=1}^{p}\sum_{l\in J_i}\int_{t_k-\tau_i+\sigma_l}^{t_k} Q_l(s)x(s-\sigma_l)ds \\
&= b_k x(t_k) - \overline{b}_k R(t_k^+)x(t_k-\gamma) - \sum_{i=1}^{p}\sum_{l\in J_i}\int_{t_k-\tau_i+\sigma_l}^{t_k} Q_l(s)x(s-\sigma_l)ds \\
&\le b_k x(t_k) - b_k R(t_k)x(t_k-\gamma) - b_k\sum_{i=1}^{p}\sum_{l\in J_i}\int_{t_k-\tau_i+\sigma_l}^{t_k} Q_l(s)x(s-\sigma_l)ds \\
&= b_k z(t_k)
\end{aligned}$$

Therefore, $z(t_k^+) \le b_k z(t_k^+)$, $k = 1, 2, \dots$ and so the proof is complete.

Lemma 3.2. *Let the hypothesis of Lemma 3.1 hold and $z(t)$ is defined by (3.3). Furthermore, suppose that*

$$R(t) + \sum_{i=1}^{p}\sum_{l\in J_i}\int_{t-\tau_i+\sigma_l}^{t} Q_l(s)ds \le 1, t \ge t_0. \tag{3.9}$$

If $x(t)$ be a solution of (1.1) and (1.2)such that $x(t-\rho) > 0$ for $t \ge t_0$, then $z(t) > 0$ for $t \ge t_0$.

Proof. Firstly we claim that $z(t_k) \ge 0$ for $k = 1, 2, \dots$. If this is not the case, then there exists some $m \ge 1$ such that $z(t_m) = -\mu < 0$. By Lemma 3.1, $z(t)$ is decreasing on $[t_0, \infty)$, therefore $z(t) \le -\mu < 0$ for $t \ge t_m$. From (3.3) we have

$$x(t) \le -\mu + R(t)x(t-\gamma) + \sum_{i=1}^{p}\sum_{l\in J_i}\int_{t-\tau_i+\sigma_l}^{t} Q_l(s)x(s-\sigma_l)ds \tag{3.10}$$

We consider the following two possible cases.

Case (1):

If $\limsup_{t\to\infty} x(t) = +\infty$. Then there exists a sequence of points $\{a_n\}_{n=1}^{\infty}$ such that $a_n \ge t_m + \rho$, $\lim_{n\to\infty} x(a_n) = +\infty$ and $x(a_n) = \max\{x(t), t_m \le t \le a_n\}$. From (3.3) and (3.10) we obtain

$$\begin{aligned} x(a_n) &\leq -\mu + R(a_n)x(a_n-\gamma) + \sum_{i=1}^{p}\sum_{l\in J_i}\int_{a_n-\tau_i+\sigma_l}^{a_n} Q_l(s)x(s-\sigma_l)ds \\ &\leq -\mu + \left[R(a_n) + \sum_{i=1}^{p}\sum_{l\in J_i}\int_{a_n-\tau_i+\sigma_l}^{a_n} Q_l(s)ds\right]x(a_n) \\ &\leq -\mu + x(a_n), \quad \text{which is a contradiction.} \end{aligned}$$

Case (2):

If $\limsup_{t\to\infty} x(t) = L < +\infty$. Choose a sequence of points $\{a_n\}_{n=1}^{\infty}$ such that $\lim_{n\to\infty} x(a_n) = L$ and $x(\xi_n) = \max\{x(s) : a_n - \rho \leq s \leq a_n - \delta\}$. Then $\xi_n \to \infty$ as $n \to \infty$ and $\limsup_{n\to\infty} x(\xi_n) \leq L$. Thus we have,

$$\begin{aligned} x(a_n) &\leq -\mu + \left[R(a_n) + \sum_{i=1}^{p}\sum_{l\in J_i}\int_{a_n-\tau_i+\sigma_l}^{a_n} Q_l(s)ds\right]x(\xi_n) \\ &\leq -\mu + x(\xi_n) \end{aligned}$$

taking the superior limit as $n \to \infty$, we get $L \leq -\mu + L$, which is also a contradiction. Combining case (1) and case (2), we see that $z(t_k) \geq 0$ for $k \geq 1$. Therefore, from (3.5), $z(t_0) \geq 0$.

To prove $z(t) > 0$ for $t \geq t_0$, we first prove that $z(t_k) > 0$, $(k \geq 0)$. If it is not true, then there exists some $\overline{m} \geq 0$ such that $z(t_{\overline{m}}) = 0$. Thus from (3.5) we have

$$\begin{aligned} z(t_{\overline{m}+1}) &= z(t_{\overline{m}}^{+}) - \int_{t_m}^{t_{\overline{m}+1}} \sum_{i=1}^{m} H_i(s)x(s-\tau_i)ds \\ &\leq z(t_{\overline{m}}) - \int_{t_{\overline{m}}}^{t_{\overline{m}+1}} \sum_{i=1}^{m} H_i(s)x(s-\tau_i)ds < 0. \end{aligned}$$

This contradiction shows that $z(t_k) > 0$ $(k \geq 0)$. Therefore, from (3.5), we have $z(t) \geq z(t_{k+1}) > 0, \quad t \in (t_k, t_{k+1}], (k \geq 0)$. So, $z(t) > 0$ for $t \geq t_0$. The proof is complete.

Lemma 3.3. *Let all the assumptions of Lemma 3.1 hold. Suppose that*

$$R(t) + \sum_{i=1}^{p}\sum_{l\in J_i}\int_{t-\tau_i+\sigma_l}^{t} Q_l(s)ds \geq 1, t \geq t_0.$$

Furthermore, assume that the impulsive differential inequality

$$y''(t) + \rho^{-1}\sum_{i=1}^{m} H_i(t)y(t) \le 0, \quad t \ge T+\rho, t \ne t_k$$

$$y(t_k^+) = y(t_k), \quad k = 1, 2, \dots \tag{3.12}$$

$$y(t_k^+) = b_k y'(t_k), \quad k = 1, 2, \dots$$

has no eventually positive solution. If $x(t)$ is a solution of (1.1) and (1.2) such that $x(t-\rho) > 0$ for $t \ge t_0$, then $z(t)$ eventually negative.

Proof. By Lemma 3.1, $z(t)$ is decreasing for $t \ge t_0$. If $z(t)$ is not eventually negative, then $z(t)$ is eventually positive. Let $t_1 > t_0 + \rho$ be such that $x(t-\rho) > 0$, $z(t) > 0$ for $t \ge t_1$. Set $M = 2^{-1}min\{x(t) : t_1 - \rho \le t \le t_1\}$, then $M > 0$ for $t_1 - \rho \le t \le t_1$. We claim that

$$x(t) > M, \quad t \ge t_1. \tag{3.13}$$

If (3.13) does not hold, then there exists a $t^* > t_1$ such that $x(t^*) = M$ and $x(t) > M$ for $t_1 - \rho \le t < t^*$. From (3.3), we have

$$\begin{aligned} M = x(t^*) &= z(t^*) + R(t^*)x(t^*-\gamma) + \sum_{i=1}^{p}\sum_{l\in J_i}\int_{t^*-\tau_i+\sigma_l}^{t^*} Q_l(s)x(s-\sigma_l)ds \\ &> \left[R(t^*) + \sum_{i=1}^{p}\sum_{l\in J_i}\int_{t^*-\tau_i+\sigma_l}^{t^*} Q_l(s)ds\right] M \ge M \end{aligned}$$

which is contradiction and so (3.13) holds. Noting that $z(t_1^+) \ge z(t_2) > 0$ and from (3.7) and (3.8) it follows that

$$\begin{aligned} x(t_1^+) &> z(t_1^+) + R(t_1^+)x(t_1-\gamma)^+ + \sum_{i=1}^{p}\sum_{l\in J_i}\int_{t_1-\tau_i+\sigma_l}^{t_1} Q_l(s)x(s-\sigma_l)ds \\ &> R(t_1^+)x(t_1-\gamma)^+ + \sum_{i=1}^{p}\sum_{l\in J_i}\int_{t_1-\tau_i+\sigma_l}^{t_1} Q_l(s)x(s-\sigma_l)ds \\ &> \left(R(t_1) + \sum_{i=1}^{p}\sum_{l\in J_i}\int_{t_1-\tau_i+\sigma_l}^{t_1} Q_l(s)ds\right) M \ge M. \end{aligned}$$

Repeating the above argument, by induction, we obtain

$$x(t) > M, \quad t \ge t_0 - \rho$$

$$x(t_k^+) \geq M, \quad k = 1, 2, \ldots$$

Because $z(t) > 0$ and $z(t)$ is decreasing, $\lim_{t\to\infty} z(t)$ exists.

Let $\lim_{t\to\infty} z(t) = a$. There are two possible cases.

Case (1):

$a = 0$. Let $T_1 > t_1$ such that $z(t) \leq \frac{M}{2}$ for $t \geq T_1$. Then for any $\bar{t} > T_1$ we have

$$\frac{1}{\rho}\int_{\bar{t}}^{t+\rho} z(u)du \leq M < x(t), \quad t \in \left[\bar{t}, \bar{t}+\rho\right].$$

Case (2):

$a > 0$, then $z(t) \geq a$ for $t \geq t_0$. From (3.3) and (3.11) we get

$$\begin{aligned} x(t) &\geq a + R(t)x(t-\gamma) + \sum_{i=1}^{p}\sum_{l\in J_i}\int_{t-\tau_i+\sigma_l}^{t} Q_l(s)x(s-\sigma_l)ds \\ &\geq a + \left(R(t) + \sum_{i=1}^{p}\sum_{l\in J_i}\int_{t-\tau_i+\sigma_l}^{t} Q_l(s)ds\right)M \geq a + M, \quad t \geq t_0 \end{aligned}$$

By induction, it is easy to see that $x(t) \geq na + M$ for $t \geq t_0 + (n-1)\rho$ $(n = 1, 2, 3, \ldots)$ and so $\lim_{t\to\infty} x(t) = \infty$, which implies that there exists a $T > T_1$ such that

$$\frac{1}{\rho}\int_{T}^{t+\rho} z(u)du \leq 2z(t) < x(t), \quad t \in [T, T+\rho].$$

Combining case 1 and case 2, we see that

$$x(t) > \frac{1}{\rho}\int_{T}^{t+\rho} z(u)du, \quad t \in [T, T+\rho].$$

Let $k^* = \min\{k > 0, t_k > T+\rho\}$, we claim that

$$x(t) > \frac{1}{\rho}\int_{T}^{t+\rho} z(u)du, \quad t \in [T+\rho, t_{k^*}]. \tag{3.14}$$

Otherwise, there exists a $t^* \in (T+\rho, t_{k^*}]$ such that

$$x(t^*) = \frac{1}{\rho}\int_{T}^{t^*+\rho} z(u)du,$$

$$x(t) > \frac{1}{\rho}\int_{T}^{t+\rho} z(u)du, \quad t \in [T+\rho, t_{k^*}].$$

Then from (3.3) we have

$$\begin{aligned}\frac{1}{\rho}\int_T^{t^*+\rho} z(u)du = x(t^*) &= z(t^*) + R(t^*)x(t^*-\gamma) + \sum_{i=1}^{p}\sum_{l\in J_i}\int_{t^*-\tau_i+\sigma_l}^{t^*} Q_l(s)x(s-\sigma_l)ds \\ &\geq \frac{1}{\rho}\int_{t^*}^{t^*+\rho} z(u)du + \left(R(t^*) + \sum_{i=1}^{p}\sum_{l\in J_i}\int_{t^*-\tau_i+\sigma_l}^{t^*} Q_l(s)ds\right)\frac{1}{\rho}\int_T^{t^*} z(u)du \\ &\geq \frac{1}{\rho}\int_T^{t^*+\rho} z(u)du.\end{aligned}$$

This is a contradiction and so (3.14) holds. Thus, if $k^* \in E_{1k}$,we have

$$\begin{aligned}x(t_{k^*}^+) &\geq z(t_{k^*}^+) + R(t_{k^*}^+)x(t_{k^*}^+-\gamma) + \sum_{i=1}^{p}\sum_{l\in J_i}\int_{t_k^*-\tau_i+\sigma_l}^{t_k^*} Q_l(s)x(s-\sigma_l)ds \\ &\geq \frac{1}{\rho}\int_{t_{k^*}}^{t_{k^*}+\rho} z(u)du + \left(R(t_{k^*}) + \sum_{i=1}^{p}\sum_{l\in J_i}\int_{t_{k^*}-\tau_i+\sigma_l}^{t_{k^*}} Q_l(s)ds\right)\frac{1}{\rho}\int_T^{t_{k^*}} z(u)du \\ &\geq \frac{1}{\rho}\int_T^{t_{k^*}+\rho} z(u)du.\end{aligned}$$

Similarly,when $k^* \in E_{2k}$, we have also

$$x(t_{k^*}^+) \geq \frac{1}{\rho}\int_T^{t_{k^*}+\rho} z(u)du.$$

Repeating the above procedure, by induction, we can see that

$$x(t) > \frac{1}{\rho}\int_T^{t+\rho} z(u)du, \quad t \geq T. \tag{3.15}$$

Thus, for $t > T + \rho$, we obtain

$$x(t-\tau_i) > \frac{1}{\rho}\int_T^{t+\rho-\tau_i} z(u)du \geq \frac{1}{\rho}\int_T^{t} z(u)du, \quad i = 1, 2, \ldots, m.$$

By (3.5) and (3.15) we get

$$z'(t) \leq -\frac{1}{\rho}\sum_{i=1}^{m} H_i(t)\int_T^t z(u)du \leq 0, \quad t > T + \rho, \quad t \neq t_k$$

Let $y(t) = \displaystyle\int_T^t z(u)du$ then $y(t_k^+) = y(t_k)$

$$y'(t_k^+) = \frac{1}{\rho}z(t_k^+) \leq \frac{1}{\rho}b_k z(t_k) = b_k y'(t_k), \quad \text{for} \quad k = 1, 2, 3, \ldots.$$

Thus $y(t) > 0$ for $t > T + \rho$ and $y(t)$ satisfies (3.12) which contradicts the assumption that (3.12) has no eventually positive solution. So $z(t)$ is eventually negative. The proof is complete.

Lemma 3.4. *Consider the impulsive differential inequality*

$$y''(t) + G(t)y(t) \leq 0, \quad t \geq t_0, t \neq t_k$$

$$y(t_k^+) \geq y(t_k), \quad k = 1, 2, 3, \dots$$

$$y'(t_k^+) \leq c_k y'(t_k), \quad k = 1, 2, 3, \dots, \tag{3.16}$$

where $0 \leq t_0 < t_1 < t_2 < \dots < t_k < \dots$ *are fixed points with* $\lim_{k\to\infty} t_k = \infty$,

$$G(t) \in PC([t_0, \infty), R^+] \quad \text{and } c_k > 0. \text{ If}$$

$$\sum_{i=0}^{\infty} \int_{t_i}^{t_{i+1}} \frac{1}{c_0 c_1 \dots c_i} G(t)dt = \infty, \quad \text{where } c_o = 1.$$

Then inequality (3.16) has no solution $y(t)$ *such that* $y(t) > 0$ *for* $t \geq t_0$.

Proof. Proof of this Lemma follows form the similar arguments to that of Theorem 1 in [5] by letting $\phi(x) = x$.We omit the details.

Theorem 3.5. *Assume that all the conditions of Lemma 3.1. hold and*

$$R(t) + \sum_{i=1}^{p} \sum_{l \in J_i} \int_{t-\tau_i+\sigma_l}^{t} Q_l(s)ds \equiv 1, t \geq t_0. \tag{3.17}$$

Further assume that (3.12) has no eventually positive solution, then every solution of (1.1) and (1.2) oscillates.

Proof. Suppose that (1.1) and (1.2) has a non oscillatory solution $x(t)$. Without the loss of generality, we assume that $x(t - \rho) > 0$ for $t \geq t_0$.Then by Lemma 3.2, $z(t) > 0$ for $t \geq t_0$, while Lemma 3.3 implies $z(t) < 0$. This is a contradiction and hence every solution of (1.1) and (1.2) oscillates.

From Lemma 3.4. and Theorem 3.5 it is easy to see that the following Theorem 3.6 is true.

Theorem 3.6. *Let all the conditions of Lemma 3.1 and (3.17) hold.If*

$$\int_{t_0+\rho}^{t_r} G(t)dt + \sum_{j=0}^{\infty} \frac{1}{b_r b_{r+1} \dots b_{r+j}} \int_{t_{r+j}}^{t_{r+j+1}} G(t)dt = \infty, \tag{3.18}$$

where $G(t) = \rho^{-1} \sum_{i=1}^{m} H_i(t), \quad r = min\{k \geq 1, t_k > t_0 + \rho\}$, *then every solution of (1.1) and (1.2) oscillates.*

Proof. By Lemma 3.4. and condition (3.18), the second order impulsive differential inequality (3.16) has no positive solution. Therefore by theorem 3.5, every solution of (1.1) and (1.2) oscillates.

Corollary 3.7. *Assume that there exists a constant* $\beta > 0$ *such that*

$$\frac{1}{b_k} \geq \left(\frac{t_{k+1}}{t_k}\right)^{\beta}, \quad k = 1, 2, \dots \tag{3.19}$$

and

$$\int_{t_0}^{\infty} t^{\beta} G(t)dt = +\infty \tag{3.20}$$

then every solution of (1.1) and (1.2) oscillates.

Proof. From (3.19), we have

$$\begin{aligned}
&\int_{t_0+\rho}^{t_r} G(t)dt + \sum_{j=0}^{\infty} \frac{1}{b_r b_{r+1} \dots b_{r+j}} \int_{t_{r+j}}^{t_{r+j+1}} G(t)dt \\
&\geq \frac{1}{b_r} \int_{t_r}^{t_{r+1}} G(t)dt + \dots + \frac{1}{b_r b_{r+1} \dots b_{r+n}} \int_{t_{r+n}}^{t_{r+n+1}} G(t)dt \\
&\geq \frac{1}{t_r^{\beta}} \left(\int_{t_r}^{t_{r+1}} t_{r+1}^{\beta} G(t)dt + \dots + \int_{t_{r+n}}^{t_{r+n+1}} t_{r+n+1}^{\beta} G(t)dt \right) \\
&\geq \frac{1}{t_r^{\beta}} \left(\int_{t_r}^{t_{r+1}} t^{\beta} G(t)dt + \dots + \int_{t_{r+n}}^{t_{r+n+1}} t^{\beta} G(t)dt \right) \\
&= \frac{1}{t_r^{\beta}} \int_{t_r}^{t_{r+n+1}} t^{\beta} G(t)dt.
\end{aligned}$$

Let $n \to +\infty$.It follows from (3.20) we see that (3.18) holds.By Theorem 3.6. we get that all solutions of (1.1)- (1.2) oscillate.

Remark 3.8. When $m = 1$ and $n = 1$ the results of this paper reduce to the results of [8].

Example 4.1. *Consider the impulsive neutral differential equation*

$$[x(t)-0.5x(t-2)]'+\left(\frac{3}{8}+\frac{1}{t}\right)x(t-3)-\frac{1}{4}x(t-2)-\frac{1}{8}x(t-1)=0,\quad t\geq 4, t\neq k. \quad (4.1)$$

$$x(k^+)=\frac{k}{k+1}x(k),\quad k=5,6,\ldots. \quad (4.2)$$

Here $t_k=k$, $m=1, n=2, p=1$, $J_1=\{1,2\}$, $\gamma=\frac{1}{2}, \tau_1=3, \sigma_1=2, \sigma_2=1$, $R(t)=\frac{1}{2}$, $P_1(t)=\frac{3}{8}+\frac{1}{t}$, $Q_1(t)=\frac{1}{4}$ and $Q_2(t)=\frac{1}{8}$.Clearly (A1)-(A3)hold.

$$\begin{aligned}H_1(t)&=P_1(t)-\sum_{l\in\{1,2\}}Q_l(t-\tau_1+\sigma_l)\\&=P_1(t)-Q_l(t-\tau_1+\sigma_l)-Q_2(t-\tau_1+\sigma_2)\\&=\frac{3}{8}+\frac{1}{t}-\frac{1}{4}-\frac{1}{8}=\frac{1}{t}\geq 0,\quad t\geq 4.\end{aligned}$$

Therefore,$H_i(t)=P_i(t)-\sum_{l\in J_i}Q_l(t-\tau_i+\sigma_l)\geq 0$, for $i=1,2\ldots,p$ holds.

$$\begin{aligned}R(t)+\sum_{i=1}^{p}\sum_{l\in J_i}\int_{t-\tau_i+\sigma_l}^{t}Q_l(s)ds&=0.5+\sum_{i=1}^{1}\sum_{l\in J_1}\int_{t-\tau_i+\sigma_l}^{t}Q_l(s)ds\\&=0.5+\sum_{l\in\{1,2\}}\int_{t-\tau_1+\sigma_l}^{t}Q_l(s)ds\\&=0.5+\int_{t-\tau_1+\sigma_1}^{t}Q_1(s)ds+\int_{t-\tau_1+\sigma_2}^{t}Q_2(s)ds\\&=0.5+\int_{t-1}^{t}\frac{1}{4}ds+\int_{t-2}^{t}\frac{1}{8}ds=1.\end{aligned}$$

Take $\beta=1, \frac{1}{b_k}=\frac{k+1}{k}=\frac{t_{k+1}}{t_k}=\left(\frac{t_{k+1}}{t_k}\right)^{\beta}$.

$$\begin{aligned}\int_{t_r}^{\infty}t^{\beta}G(t)dt&=\int_{4}^{\infty}t\frac{1}{\rho}\sum_{i=1}^{m}H_i(t)dt\\&=\frac{1}{\rho}\int_{4}^{\infty}t\frac{1}{t}dt=\frac{1}{\rho}\int_{4}^{\infty}dt=\infty.\end{aligned}$$

By Corollary 3.7, all solutions of (4.1) and (4.2) oscillate.

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