Available online at http://scik.org

J. Math. Comput. Sci. 4 (2014), No. 2, 363-373

ISSN: 1927-5307

ON THE NUMERICAL SOLUTION OF INITIAL-BOUNDARY VALUE PROBLEM TO ONE NONLINEAR PARABOLIC EQUATION

MIKHEIL TUTBERIDZE

Ilia State University, 3/5 Kakutsa Cholokashvili ave., Tbilisi 0162, Georgia

Abstract: In this paper the difference scheme of the initial-boundary value problem to one nonlinear parabolic equation is considered. The convergence of the solution of the difference scheme to the solution of source problem is proved. The iteration process for finding of the solution of difference scheme is constructed and its convergence is obtained.

Keywords: difference scheme, partial differential equation, convergence.

2010 AMS Subject Classification: 65M06

1. Introduction

In some mathematical models of diffusion process the following problem occurs:

$$U_t = a(x,t,U)U_{xx} + b(x,t,U)[U_x]^2 + f(x,t), \quad (x,t) \in \Omega \times (0,T], \qquad (1.1)$$

$$U(x,0) = \varphi(x), \quad x \in \bar{\Omega}, \qquad (1.2)$$

$$U(0,t) = U(1,t) = 0, \quad t \in (0,T], \qquad (1.3)$$

where $U = U(x,t)$ is unknown function, a, b, f and φ are given functions, $T = const > 0$, $\Omega = (0,1)$.

Received November 25, 2013

Further in this article we assume that functions $a = a(x,t,u)$, $b = b(x,t,u)$, $f = f(x,t)$ are continuous and have continuous partial derivative with respect to argument u on $\overline{\Omega}\times[0,T]\times R$ and

$$a(x,t,u) > 0, \quad (x,t,u) \in \overline{\Omega}\times[0,T]\times R . \tag{1.4}$$

The initial-boundary problem for the equation like (1.1) was considered in article [6] by the author. The difference schemes to nonlinear parabolic equations were considered in a number of works (see, for example, [1], [2], [4], [5], [10]). Author received the same results as said above for problems containing another kind of parabolic equation in articles [8],[9].

In the present work we investigate questions of approximate solution of the problem (1.1) -(1.3) using the difference scheme for the problem (1.1)-(1.3). In certain conditions the convergence of the solution of the difference scheme to the solution of source problem is proved. The iteration process for finding of the solution of difference scheme is constructed and in certain conditions its convergence is obtained.

2. Preliminaries

Enter a grid $\omega_{h\tau}$ on the domain $\overline{\Omega}\times[0,T]$ as follows:

$$\omega_h = \{x_i = ih, h > 0, i = 0,1,\ldots,M; hM = 1\},$$

$$\omega_\tau = \{t_j = j\tau, \tau > 0, j = 0,1,\ldots,N; \tau N = T\},$$

$$\omega_{h\tau} = \omega_h \times \omega_\tau .$$

Let y_i^j be a function defined on the grid $\omega_{h\tau}$. Enter the following notations:

$$\left\| y^j \right\|_{C(\omega_h)} = \max_{i=0,1,\ldots,N} \left| y_i^j \right|,$$

$$\left\| y \right\|_{C(\omega_{h\tau})} = \max_{j=0,1,\ldots,N} \left\| y^j \right\|_{C(\omega_h)} .$$

Let

$$g(u) = \frac{1}{2}\left(1 - \frac{1}{1+u^2}\right). \tag{2.1}$$

Note that

$$0 \le g(u) \le \frac{1}{2}, \quad -1 < g'(u) < 1. \tag{2.2}$$

Make the proper approximation of the term $\left[\frac{\partial U}{\partial x}\right]^2$:

$$\left[\frac{\partial U}{\partial x}\right]^2 = \frac{2\lambda^2}{h^2} g\left(\frac{1}{2\lambda}\left[U\left(x_{i+1}, t^j\right) - U\left(x_{i-1}, t^j\right)\right]\right) + O\left(h^2\right)$$

where λ is arbitrary number.

Construct the difference scheme to the problem (1.1)-(1.3) as follows:

$$\frac{u_i^{j+1} - u_i^j}{\tau} = a\left(x_i, t^j, u_i^j\right)\frac{1}{h^2}\left[u_{i+1}^{j+1} - 2u_i^{j+1} + u_{i-1}^{j+1}\right] +$$

$$+ b\left(x_i, t^j, u_i^j\right)\frac{2\left(\lambda^j\right)^2}{h^2} g\left(\frac{1}{2\lambda^j}\left[u_{i+1}^{j+1} - u_{i-1}^{j+1}\right]\right) + f\left(x_i, t^{j+1}\right),$$

$$i = 1,2,\dots,M-1, \quad j = 0,1,\dots,N-1, \tag{2.3}$$

$$u_i^0 = \varphi_i, \quad i = 0,1,\dots,M, \tag{2.4}$$

$$u_0^j = u_M^j = 0, \quad j = 1,2,\dots,N, \tag{2.5}$$

where $\varphi_i = \varphi(x_i)$, $i = 0,1,\dots,M$, $j = 1,2,\dots,N$.

Further in this work we investigate the difference scheme (2.3)-(2.5). We prove the theorem of convergence of the solution of scheme (2.3)-(2.5) to the solution of the problem (1.1)-(1.3). Also, we construct the iteration process for finding of the solution of the scheme (2.3)-(2.5) and show its convergence to the solution of the scheme (2.3)-(2.5).

3. Main results

First we will refer to one important theorem from book [7], pages 15-17.

Consider the grid function y_i defined on the grid ω_h.

Define the maximum norm on the grid ω_h as follows

$$\|y\|_{C(\omega_h)} = \max_{i=0,1,\dots,M} |y_i|.$$

Let grid function y_i satisfies the following conditions:

$$\Lambda[y_i] = A_i y_{i-1} - C_i y_i + B_i y_{i+1} = -F_i, \quad i = 1,2,\dots,N-1, \tag{3.1}$$

$$y_0 = \mu_1, \quad y_N = \mu_2. \tag{3.2}$$

Theorem 1. Let the conditions

$$|A_i| > 0, \; |B_i| > 0, \; D_i = |C_i| - |A_i| - |B_i| > 0$$

hold for all $i = 1,2,...,M-1$ *and* $\mu_1 = \mu_2 = 0$. *A solution of the problem* (3.1),(3.2) *admits the estimate*

$$\|y\|_{C(\omega_h)} \le \left\|\frac{F}{D}\right\|_{C(\omega_h)}.$$

Basing on the theorem 1 we prove the following theorems:

Theorem 2. *Let* λ^j *satisfies the following condition*

$$0 < \lambda^j < \frac{\min\limits_{i\in\{1,2,...,n-1\}} \left\{a\left(x_i,t^j,u_i^j\right)\right\}}{\max\limits_{i\in\{1,2,...,n-1\}} \left\{\left|b\left(x_i,t^j,u_i^j\right)\right|\right\}} \tag{3.3}$$

and the solution u of the difference scheme (2.3)-(2.5) exists for small values of τ and h. Then the following estimate takes place:

$$\|u\|_{C(\omega_{h\tau})} \le \max_{x\in\Omega} |\varphi(x)| + T \max_{(x,t)\in\Omega\times[0,T]} |f(x,t)|.$$

Proof. Write equation in the following way:

$$a\left(x_i,t^j,u_i^j\right)\frac{1}{h^2}\left[u_{i+1}^{j+1} - 2u_i^{j+1} + u_{i-1}^{j+1}\right] +$$

$$+ b\left(x_i,t^j,u_i^j\right)\frac{2\left(\lambda^j\right)^2}{h^2} g\left(\frac{1}{2\lambda^j}\left[u_{i+1}^{j+1} - u_{i-1}^{j+1}\right]\right) - \frac{u_i^{j+1}}{\tau} =$$

$$= -\frac{u_i^j}{\tau} - f\left(x_i,t^{j+1}\right),$$

Transform the left side of the equation:

$$a\left(x_i,t^j,u_i^j\right)\frac{u_{i+1}^{j+1}}{h^2} + a\left(x_i,t^j,u_i^j\right)\frac{u_{i-1}^{j+1}}{h^2} - u_i^{j+1}\left[\frac{1}{\tau} + a\left(x_i,t^j,u_i^j\right)\frac{2}{h^2}\right] +$$

$$+ b\left(x_i,t^j,u_i^j\right)\frac{2\left(\lambda^j\right)^2}{h^2}\left[g\left(\frac{1}{2\lambda^j}\left[u_{i+1}^{j+1} - u_{i-1}^{j+1}\right]\right) - g(0) + g(0)\right] =$$

$$= a\left(x_i,t^j,u_i^j\right)\frac{u_{i+1}^{j+1}}{h^2} + a\left(x_i,t^j,u_i^j\right)\frac{u_{i-1}^{j+1}}{h^2} - u_i^{j+1}\left[\frac{1}{\tau} + a\left(x_i,t^j,u_i^j\right)\frac{2}{h^2}\right] +$$

$$+b\left(x_i,t^j,u_i^j\right)\frac{2\left(\lambda^j\right)^2}{h^2}\left[\frac{1}{2\lambda^j}\left[u_{i+1}^{j+1}-u_{i-1}^{j+1}\right]g'\left(\frac{\theta_i^j}{2\lambda^j}\left[u_{i+1}^{j+1}-u_{i-1}^{j+1}\right]\right)\right]=$$

$$=u_{i+1}^{j+1}\frac{1}{h^2}\left[a\left(x_i,t^j,u_i^j\right)+\lambda^j b\left(x_i,t^j,u_i^j\right)g'\left(\frac{\theta_i^j}{2\lambda^j}\left[u_{i+1}^{j+1}-u_{i-1}^{j+1}\right]\right)\right]-$$

$$-u_i^{j+1}\left[\frac{1}{\tau}+a\left(x_i,t^j,u_i^j\right)\frac{2}{h^2}\right]+$$

$$+u_{i-1}^{j+1}\frac{1}{h^2}\left[a\left(x_i,t^j,u_i^j\right)-\lambda^j b\left(x_i,t^j,u_i^j\right)g'\left(\frac{\theta_i^j}{2\lambda^j}\left[u_{i+1}^{j+1}-u_{i-1}^{j+1}\right]\right)\right]=$$

$$=-\frac{u_i^j}{\tau}-f\left(x_i,t^{j+1}\right),$$

for some numbers $\theta_i^j\in(0,1)$ according to Lagrange meanvalue theorem.

Taking into account (2.2), (3.3) we can apply threorem 1 to expression above and we obtain:

$$\left\|u^{j+1}\right\|_{C(\omega_h)}\le\left\|u^j\right\|_{C(\omega_h)}+\left\|f\left(x_i,t^{j+1}\right)\right\|_{C(\omega_h)}$$

from which easily can be obtained the required estimate.

Theorem 3. *Let exists solution* U *of the problem* (1.1)-(1.3) *such that* $\left|\frac{\partial^4 U}{\partial x^4}\right|\le V$ *,* $\left|\frac{\partial^2 U}{\partial t^2}\right|\le V$ *for some positive constant* V*. If* λ^j *satisfies the following condition*

$$0<\lambda^j<\frac{\min\limits_{i\in\{1,2,\dots,n-1\}}\left\{a\left(x_i,t^j,u_i^j\right)\right\}}{\max\limits_{i\in\{1,2,\dots,n-1\}}\left\{\left|b\left(x_i,t^j,u_i^j\right)\right|\right\}}\tag{3.4}$$

and the solution u *of the difference scheme* (2.3)-(2.5) *exists for small values of* τ *and* h *then the following estimate takes place*

$$\left\|u-U\right\|_{C(\omega_{h\tau})}\le TW\left(\tau+h^2\right),$$

where W *is positive constant independent from* τ *and* h*.*

Proof. For the difference equations (2.3) we have:

$$\frac{U_i^{j+1}-U_i^j}{\tau}=a\left(x_i,t^j,U_i^j\right)\frac{1}{h^2}\left[U_{i+1}^{j+1}-2U_i^{j+1}+U_{i-1}^{j+1}\right]+$$

$$+b\left(x_i,t^j,U_i^j\right)\frac{2\left(\lambda^j\right)^2}{h^2}g\left(\frac{1}{2\lambda^j}\left[U_{i+1}^{j+1}-U_{i-1}^{j+1}\right]\right)+f\left(x_i,t^{j+1}\right)+\underline{O}\left(\tau+h^2\right),$$

$$i=1,2,\ldots,M-1,\quad j=0,1,\ldots,N-1,$$

where $\underline{O}\left(\tau+h^2\right)$ is a quantity, which can be expressed as follows

$$\underline{O}\left(\tau+h^2\right)=\left(\tau+h^2\right)q\left(\tau,h\right)$$

for some bounded function q.

The grid function $Y_i^j=u_i^j-U_i^j$, is the solution of the following equations:

$$\frac{Y_i^{j+1}-Y_i^j}{\tau}+a\left(x_i,t^j,u_i^j\right)\frac{1}{h^2}\left[Y_{i+1}^{j+1}-2Y_i^{j+1}+Y_{i-1}^{j+1}\right]+\underline{O}\left(\tau+h^2\right)=$$

$$=b\left(x_i,t^j,u_i^j\right)\frac{2\left(\lambda^j\right)^2}{h^2}\left[g\left(\frac{1}{2\lambda}\left[u_{i+1}^{j+1}-u_{i-1}^{j+1}\right]\right)-g\left(\frac{1}{2\lambda}\left[U_{i+1}^{j+1}-U_{i-1}^{j+1}\right]\right)\right]+$$

$$+\left[a\left(x_i,t^j,u_i^j\right)-a\left(x_i,t^j,U_i^j\right)\right]\frac{1}{h^2}\left[U_{i+1}^{j+1}-2U_i^{j+1}+U_{i-1}^{j+1}\right]+$$

$$+\frac{2\left(\lambda^j\right)^2}{h^2}\left[b\left(x_i,t^j,u_i^j\right)-b\left(x_i,t^j,U_i^j\right)\right]g\left(\frac{1}{2\lambda^j}\left[U_{i+1}^{j+1}-U_{i-1}^{j+1}\right]\right),$$

$$i=1,2,\ldots,M-1,\quad j=0,1,\ldots,N-1. \tag{3.5}$$

Transform the right side of the inequality (3.5) using Lagrange Mean Value Theorem:

$$\frac{\lambda}{h^2}b\left(x_i,t^j,u_i^j\right)\left(Y_{i+1}^{j+1}-Y_{i-1}^{j+1}\right)\times$$

$$\times g'\left(\frac{1}{2\lambda}\left[\theta_i^{j+1}u_{i+1}^{j+1}+\left(1-\theta_i^{j+1}\right)U_{i+1}^{j+1}-\theta_i^{j+1}u_{i-1}^{j+1}-\left(1-\theta_i^{j+1}\right)U_{i-1}^{j+1}\right]\right)+$$

$$+a'\left(x_i,t^j,\theta_i^{j+1}u_i^{j+1}+\left(1-\theta_i^{j+1}\right)U_i^{j+1}\right)\times$$

$$\times\left[\left(U''\right)_i^{j+1}+O\left(h^2\right)\right]Y_i^j$$

$$+b'\left(x_i,t^j,\theta_i^{j+1}u_i^{j+1}+\left(1-\theta_i^{j+1}\right)U_i^{j+1}\right)\times$$

$$\left[\left(U'\right)_i^{j+1}+O\left(h^2\right)\right]^2Y_i^j,$$

for some numbers $\theta_i^{j+1}\in\left(0,1\right)$ according to the multidimensional analogue of Lagrange Mean Value Theorem.

Denote by A_i^j, and B_i^j the following quantities:

$$A_i^{j+1} = \frac{1}{h^2}\Big[a\left(x_i, t^j, u_i^j\right) + \lambda^j b\left(x_i, t^j, u_i^j\right) \times$$

$$\times g'\left(\frac{1}{2\lambda}\left[\theta_i^{j+1} u_{i+1}^{j+1} + \left(1-\theta_i^{j+1}\right) U_{i+1}^{j+1} - \theta_i^{j+1} u_{i-1}^{j+1} - \left(1-\theta_i^{j+1}\right) U_{i-1}^{j+1} \right] \right)\Big],$$

$$B_i^{j+1} = \frac{1}{h^2}\Big[a\left(x_i, t^j, u_i^j\right) - \lambda^j b\left(x_i, t^j, u_i^j\right) \times$$

$$\times g'\left(\frac{1}{2\lambda}\left[\theta_i^{j+1} u_{i+1}^{j+1} + \left(1-\theta_i^{j+1}\right) U_{i+1}^{j+1} - \theta_i^{j+1} u_{i-1}^{j+1} - \left(1-\theta_i^{j+1}\right) U_{i-1}^{j+1} \right] \right)\Big],$$

Then equation (3.5) can be written as follows

$$A_i^{j+1} Y_{i+1}^{j+1} - \left(A_i^{j+1} + B_i^{j+1} + \frac{1}{\tau} \right) Y_i^{j+1} + B_i^{j+1} Y_{i-1}^{j+1} =$$

$$= -Y_i^j \left[\frac{1}{\tau} + R\left(x_i, t^j\right) \right] + \underline{O}\left(\tau + h^2\right),$$

$$i = 0,1,...,M, \quad j = 0,1,...,N, \tag{3.6}$$

where

$$R\left(x_i, t^j\right) = a'\left(x_i, t^j, \theta_i^{j+1} u_i^{j+1} + \left(1-\theta_i^{j+1}\right) U_i^{j+1}\right) \left(U''\right)_i^{j+1} +$$

$$+ b'\left(x_i, t^j, \theta_i^{j+1} u_i^{j+1} + \left(1-\theta_i^{j+1}\right) U_i^{j+1}\right) \left[\left(U'\right)_i^{j+1} \right]^2$$

and according to theorem 1 there exists a positive number L such that

$$\left| R\left(x_i, t^j\right) \right| < L .$$

According to the conditions (1.4),(2.2),(3.4) $A_i^{j+1} \ge 0$, $B_i^{j+1} \ge 0$, $A_i^{j+1} + B_i^{j+1} + \frac{1}{\tau} > 0$, $\left| A_i^{j+1} + B_i^{j+1} + \frac{1}{\tau} \right| - \left| A_i^{j+1} \right| - \left| B_i^{j+1} \right| > 0$. Therefore we can apply the theorem 1 to (3.6). We obtain

$$\left\| Y^{j+1} \right\|_{C(\omega_h)} \le \tau \left(\left(\frac{1}{\tau} + L \right) \left\| Y^j \right\|_{C(\omega_h)} + W\left(\tau + h^2\right) \right), \quad j = 0,1,...,N-1.$$

This implies, that

$$\left\| Y^{j+1} \right\|_{C(\omega_h)} \le \sum_{r=1}^{j+1} \left(1 + \tau L\right) W\left(\tau + h^2\right)^{j\frac{T}{\tau}} < e^{TL} W\left(\tau + h^2\right), \quad j = 0,1,...,N-1$$

The theorem 3 is proved.

Construct the iteration process for finding of the solution of the difference scheme (2.3)-(2.5).

$$u_i^{j+1,l+1}=(1-\sigma)u_i^{j+1,l}+\sigma\left\{\tau\left[a\left(x_i,t^{j+1},u_i^j\right)\frac{1}{h^2}\left[u_{i+1}^{j+1,l}-2u_i^{j+1,l}+u_{i-1}^{j+1,l}\right]+\right.\right.$$

$$\left.\left.+b\left(x_i,t^j,u_i^j\right)\frac{2\lambda^2}{h^2}g\left(\frac{1}{2\lambda}\left[u_{i+1}^{j+1,l}-u_{i-1}^{j+1,l}\right]\right)+f\left(x_i,t^{j+1}\right)\right]+u_i^j\right\},$$

$$l=0,1,\ldots;\ i=1,2,\ldots,M-1,\ \ j=0,1,\ldots,N-1, \tag{3.7}$$

$$u_i^{j+1,0}=u_i^j,\ i=0,1,\ldots,M, \tag{3.8}$$

$$u_1^{j+1,l}=\phi_0^{j+1},\ u_M^{j+1,l}=\phi_1^{j+1},\ l=1,2,\ldots;\ \ j=0,1,\ldots,N-1. \tag{3.9}$$

Theorem 4. *Let* λ^j *satisfies the following condition*

$$0<\lambda^j<\frac{\min\limits_{i=1,2,\ldots,n-1}\left\{a\left(x_i,t^j,u_i^j\right)\right\}}{\max\limits_{i=1,2,\ldots,n-1}\left\{\left|b\left(x_i,t^j,u_i^j\right)\right|\right\}}, \tag{3.10}$$

and σ^j *satisfies the following condition*

$$0<\sigma^j\le\frac{h^2}{h^2+2\tau\max\limits_{i=1,2,\ldots,n-1}\left\{a\left(x_i,t^j,u_i^j\right)\right\}} \tag{3.11}$$

then iteration process (3.7)-(3.9) *converges for each* j *and the limit is the solution of the difference scheme* (2.3)-(2.5).

Proof. It is clear, that if $\lim\limits_{l\to\infty}\left(u_0^{j+1,l},u_1^{j+1,l},\ldots,u_M^{j+1,l}\right)$ exists then it will be the solution of the difference scheme (2.3)-(2.5). Consider $\overset{l+1}{\underset{i}{u}}-\overset{l}{\underset{i}{u}}$. Taking into account the method, used in the proof of theorem 3, we have:

$$u_i^{j+1,l+1}-u_i^{j+1,l}=\left(1-\sigma^j\right)\left(u_i^{j+1,l}-u_i^{j+1,l-1}\right)+$$

$$+\sigma^j\tau\left\{a\left(x_i,t^j,u_i^j\right)\frac{1}{h^2}\left[\left(u_{i+1}^{j+1,l}-u_{i+1}^{j+1,l-1}\right)-2\left(u_i^{j+1,l}-u_i^{j+1,l-1}\right)+\left(u_{i-1}^{j+1,l}-u_{i-1}^{j+1,l-1}\right)\right]+\right.$$

$$\left.+b\left(x_i,t^j,u_i^j\right)\frac{2\left(\lambda^j\right)^2}{h^2}\left[g\left(\frac{1}{2\lambda^j}\left[u_{i+1}^{j+1,l}-u_{i-1}^{j+1,l}\right]\right)-g\left(\frac{1}{2\lambda^j}\left[u_{i+1}^{j+1,l-1}-u_{i-1}^{j+1,l-1}\right]\right)\right]\right\}=$$

$$=\left(1-\sigma^j\right)\left(u_i^{j+1,l}-u_i^{j+1,l-1}\right)+$$

$$+\frac{\sigma^j\tau}{h^2}\Big\{a\left(x_i,t^j,u_i^j\right)\Big[\left(u_{i+1}^{j+1,l}-u_{i+1}^{j+1,l-1}\right)-2\left(u_i^{j+1,l}-u_i^{j+1,l-1}\right)+\left(u_{i-1}^{j+1,l}-u_{i-1}^{j+1,l-1}\right)\Big]+$$

$$+\lambda^j b\left(x_i,t^j,u_i^j\right)\Big[\left(u_{i+1}^{j+1,l}-u_{i+1}^{j+1,l-1}\right)-\left(u_{i-1}^{j+1,l}-u_{i-1}^{j+1,l-1}\right)\Big]\Big\}\times$$

$$\times g'\left(\frac{1}{2\lambda}\Big[\theta_i^{j+1,l}u_{i+1}^{j+1,l}+\left(1-\theta_i^{j+1,l}\right)u_{i+1}^{j+1,l-1}-\theta_i^{j+1,l}u_{i-1}^{j+1,l}-\left(1+\theta_i^{j+1,l}\right)u_{i-1}^{j+1,l-1}\Big]\right)=$$

$$=\left\{1-\sigma^j-\frac{\sigma^j\tau}{h^2}\Big[A_i^{j+1,l}+B_i^{j+1,l}\Big]\right\}\left(u_i^{j+1,l}-u_i^{j+1,l-1}\right)+$$

$$+\frac{\sigma^j\tau A_i^{j+1,l}}{h^2}\left(u_{i+1}^{j+1,l}-u_{i+1}^{j+1,l-1}\right)+\frac{\sigma^j\tau B_i^{j+1,l}}{h^2}\left(u_{i-1}^{j+1,l}-u_{i-1}^{j+1,l-1}\right),$$

where

$$A_i^{j+1,l}=a\left(x_i,t^j,u_i^j\right)+\lambda^j b\left(x_i,t^j,u_i^j\right)\times$$

$$\times g'\left(\frac{1}{2\lambda^j}\Big[\theta_i^{j+1,l}u_{i+1}^{j+1,l}+\left(1-\theta_i^{j+1,l}\right)u_{i+1}^{j+1,l-1}-\theta_i^{j+1,l}u_{i-1}^{j+1,l}-\left(1+\theta_i^{j+1,l}\right)u_{i-1}^{j+1,l-1}\Big]\right),$$

$$B_i^{j+1,l}=a\left(x_i,t^j,u_i^j\right)-\lambda^j b\left(x_i,t^j,u_i^j\right)\times$$

$$\times g'\left(\frac{1}{2\lambda^j}\Big[\theta_i^{j+1,l}u_{i+1}^{j+1,l}+\left(1-\theta_i^{j+1,l}\right)u_{i+1}^{j+1,l-1}-\theta_i^{j+1,l}u_{i-1}^{j+1,l}-\left(1+\theta_i^{j+1,l}\right)u_{i-1}^{j+1,l-1}\Big]\right),$$

for some numbers $\theta_i^{j+1,l}\in(0,1)$ according to the multidimensional analogue of Lagrange Mean Value Theorem.

Apply the maximum norm to the both sides of the last equation:

$$\left\|u^{j+1,l+1}-u^{j+1,l}\right\|_{C(\omega_h)}=\max_{i=1,2,\dots,M-1}\left|\frac{\sigma^j\tau A_i^{j+1,l}}{h^2}\left(u_{i+1}^{j+1,l}-u_{i+1}^{j+1,l-1}\right)+\right.$$

$$+\left\{1-\sigma^j-\frac{\sigma^j\tau}{h^2}\Big[A_i^{j+1,l}+B_i^{j+1,l}\Big]\right\}\left(u_i^{j+1,l}-u_i^{j+1,l-1}\right)+$$

$$\left.+\frac{\tau\sigma^j B_i^{j+1,l}}{h^2}\left(u_{i-1}^{j+1,l}-u_{i-1}^{j+1,l-1}\right)\right|\le$$

$$\le\max_{i=1,2,\dots,M-1}\left(\left|1-\sigma^j-\frac{\sigma^j\tau}{h^2}\Big[A_i^{j+1,l}+B_i^{j+1,l}\Big]\right|+\right.$$

$$+\left|\frac{\sigma^j\tau A_i^{j+1,l}}{h^2}\right|+\left|\frac{\sigma^j\tau B_i^{j+1,l}}{h^2}\right|\Bigg)\left\|u^{j+1,l}-u^{j+1,l-1}\right\|_{C(\omega_h)}. \qquad (3.12)$$

(1.4),(2.2),(3.10) imply that: $0 < A_i^{j+1,l} \le 2 \max\limits_{i=1,2,\dots,n-1}\left\{a\left(x_i,t^j,u_i^j\right)\right\}$, $0 < B_i^{j+1,l} \le 2 \max\limits_{i=1,2,\dots,n-1}\left\{a\left(x_i,t^j,u_i^j\right)\right\}$. According to (3.11) we have $1-\sigma^j-\frac{\sigma^j\tau}{h^2}\left[A_i^{j+1,l}+B_i^{j+1,l}\right]\ge 0$, therefore we can remove module in the right part of (3.12) and we have:

$$\left\|u^{j+1,l+1}-u^{j+1,l}\right\|_{C(\omega_h)} \le (1-\sigma)\left\|u^{j+1,l}-u^{j+1,l-1}\right\|_{C(\omega_h)}.$$

It is clear that $0<\left(1-\sigma^j\right)<1$, therefore the sequence $\overset{l}{u}$ converges and its limit is the solution of the difference scheme (2.3)-(2.5).

Conflict of Interests

The author declares that there is no conflict of interests.

REFERENCES

[1] R. Buckmire, K. McMurtry, R. E. Mickens, Numerical studies of a nonlinear heat equation with square root reaction term. Numer. Methods Partial Differ. Equations, 25 (2009), No. 3, 598-609.

[2] M.M. Butt, M.S.A. Taj, Numerical methods for heat equation with variable coefficients, Int. J. Comput. Math., 86 (2009), No. 9, 1612-1623.

[3] M. Gurtin, A. Pipkin, A general theory of heat conduction with finite wave speeds, Arch. Rational Mech. Anal., 31 (1968), 113-126.

[4] R. Horvath, On the sign-stability of finite difference solutions of semilinear parabolic problems, Lecture Notes in Computer Science, 5434 (2009), Springer, Berlin, 305-313.

[5] J. Luo, N.V. Krylov, On the rate of convergence of the finite-difference approximations for parabolic Bellman equations with constant coefficients, Appl. Math. Optim., 58 (2008), No. 3, 315-344.

[6] M. Tutberidze, Comparison And Convergence Theorems For Initial-Boundary Value Problem To One Nonlinear Parabolic Equation, International Journal Of Differential Equations and Applications, 12 (2013) No. 3, 159-170.

[7] A. Samarski, The Theory of Difference schemes, Marcell Dekker, New York (2001).

[8] M. Tutberidze, The comparison properties and convergence of the difference scheme for one nonlinear system of partial differential equations, Proc. I. Vekua Inst. Appl. Math., 54-55 (2004-2005), 74-84.

[9] M. Tutberidze, On the Numerical Integration of One Nonlinear Parabolic Equation, International Journal of Pure and Applied Mathematics, 66 (2011), No. 2, 209-220.

[10] C.-R. Ye, Z.-Z. Sun, A linearized compact difference scheme for an one-dimensional parabolic inverse problem, Appl. Math. Modelling, 33 (2009), No. 3, 1521-1528.