Available online at http://scik.org
J. Math. Comput. Sci. 4 (2014), No. 1, 35-49
ISSN: 1927-5307

L-APPROXIMATION OPERATORS AND ALEXANDROV L-TOPOLOGIES

YONG CHAN KIM

Department of Mathematics, Gangneung-Wonju National University,
Gangneung, Gangwondo 210-702, Korea

Abstract. In this paper, we investigate relations between L-upper (lower, join meet, meet join) approximation operators and Alexandrov L-topologies. We give their examples by various L-fuzzy relations.

Keywords: complete residuated lattices; L-upper (lower, join meet, meet join) approximation operators; Alexandrov L-topologies

2000 AMS Subject Classification: 03E72; 03G10; 06A15; 06F07

1. Introduction

Hájek [2] introduced a complete residuated lattice which is an algebraic structure for many valued logic. Pawlak [7,8] introduced rough set theory as a formal tool to deal with imprecision and uncertainty in data analysis. Radzikowska [9] developed fuzzy rough sets in complete residuated lattice. Bělohlávek [1] investigated information systems and decision rules in complete residuated lattices. Zhang [4,5] introduced Alexandrov L-topologies induced by fuzzy rough sets. Kim [3,4] investigate relations between lower approximation operators as a generalization

Received December 5, 2013

of fuzzy rough set and Alexandrov L-topologies. Algebraic structures of fuzzy rough sets are developed in many directions [3,9,10].

In this paper, we investigate relations between L-upper (lower, join meet, meet join) approximation operators and Alexandrov L-topologies. We give their examples by various L-fuzzy relations.

2. Preliminaries

Definition 2.1. [1,2] An algebra $(L,\wedge,\vee,\odot,\to,\bot,\top)$ is called a complete residuated lattice if it satisfies the following conditions:

(C1) $L=(L,\leq,\vee,\wedge,\bot,\top)$ is a complete lattice with the greatest element $\top$ and the least element $\bot$;

(C2) $(L,\odot,\top)$ is a commutative monoid;

(C3) $x\odot y\leq z$ iff $x\leq y\to z$ for $x,y,z\in L$.

In this paper, we assume $(L,\wedge,\vee,\odot,\to,^*\bot,\top)$ is a complete residuated lattice with the law of double negation;i.e. $x^{**}=x$. For $\alpha\in L, A,\top_x\in L^X$, $(\alpha\to A)(x)=\alpha\to A(x),\ \ (\alpha\odot A)(x)=\alpha\odot A(x)$ and $\top_x(x)=\top,\top_x(y)=\bot,$ otherwise.

Lemma 2.2. [1,2] For each $x,y,z,x_i,y_i\in L$, we have the following properties.

(1) If $y\leq z$, $(x\odot y)\leq(x\odot z)$, $x\to y\leq x\to z$ and $z\to x\leq y\to x$.

(2) $x\to(\bigwedge_{i\in\Gamma}y_i)=\bigwedge_{i\in\Gamma}(x\to y_i)$.

(3) $(\bigvee_{i\in\Gamma}x_i)\to y=\bigwedge_{i\in\Gamma}(x_i\to y)$.

(4) $\bigwedge_{i\in\Gamma}y_i^*=(\bigvee_{i\in\Gamma}y_i)^*$ and $\bigvee_{i\in\Gamma}y_i^*=(\bigwedge_{i\in\Gamma}y_i)^*$.

(5) $(x\odot y)\to z=x\to(y\to z)=y\to(x\to z)$.

(6) $x\odot y=(x\to y^*)^*$.

(7) $x\odot(x\to y)\leq y$.

(8) $(x\to y)\odot(y\to z)\leq x\to z$.

(9) $(x\to y)\to(x\to z)\geq y\to z$ and $(x\to z)\to(y\to z)\geq y\to x$.

(10) $x\odot y\to x\odot z\geq y\to z$.

Definition 2.3. [3,4] (1) A map $\mathbf{H} : L^X \to L^X$ is called an L-*upper approximation operator* iff it satisfies the following conditions

(H1) $A \leq \mathbf{H}(A)$,

(H2) $\mathbf{H}(\alpha \odot A) = \alpha \odot \mathbf{H}(A)$ where $\alpha(x) = \alpha$ for all $x \in X$,

(H3) $\mathbf{H}(\bigvee_{i \in I} A_i) = \bigvee_{i \in I} \mathbf{H}(A_i)$.

(2) A map $\mathbf{J} : L^X \to L^X$ is called an L-*lower approximation operator* iff it satisfies the following conditions

(J1) $\mathbf{J}(A) \leq A$,

(J2) $\mathbf{J}(\alpha \to A) = \alpha \to \mathbf{J}(A)$,

(J3) $\mathbf{J}(\bigwedge_{i \in I} A_i) = \bigwedge_{i \in I} \mathbf{J}(A_i)$.

(3) A map $\mathbf{K} : L^X \to L^X$ is called an L-*join meet approximation operator* iff it satisfies the following conditions

(K1) $\mathbf{K}(A) \leq A^*$,

(K2) $\mathbf{K}(\alpha \odot A) = \alpha \to \mathbf{K}(A)$,

(K3) $\mathbf{K}(\bigvee_{i \in I} A_i) = \bigwedge_{i \in I} \mathbf{K}(A_i)$.

(4) A map $\mathbf{M} : L^X \to L^X$ is called an L-*meet join approximation operator* iff it satisfies the following conditions

(M1) $A^* \leq \mathbf{M}(A)$,

(M2) $\mathbf{M}(\alpha \to A) = \alpha \odot \mathbf{M}(A)$,

(M3) $\mathbf{M}(\bigwedge_{i \in I} A_i) = \bigvee_{i \in I} \mathbf{M}(A_i)$.

Definition 2.4. [4,5] A subset $\tau \subset L^X$ is called an *Alexandrov L-topology* if it satisfies:

(T1) $\perp_X, \top_X \in \tau$ where $\top_X(x) = \top$ and $\perp_X(x) = \perp$ for $x \in X$.

(T2) If $A_i \in \tau$ for $i \in \Gamma$, $\bigvee_{i \in \Gamma} A_i, \bigwedge_{i \in \Gamma} A_i \in \tau$.

(T3) $\alpha \odot A \in \tau$ for all $\alpha \in L$ and $A \in \tau$.

(T4) $\alpha \to A \in \tau$ for all $\alpha \in L$ and $A \in \tau$.

Theorem 2.5. [4] (1) τ is an Alexandrov topology on X iff $\tau_* = \{A^* \in L^X \mid A \in \tau\}$ is an Alexandrov topology on X.

(2) If $\mathbf{H}$ is an L-upper approximation operator, then $\tau_{\mathbf{H}} = \{A \in L^X \mid \mathbf{H}(A) = A\}$ is an Alexandrov topology on X.

(3) If $\mathbf{J}$ is an L-lower approximation operator, then $\tau_{\mathbf{J}} = \{A \in L^X \mid \mathbf{J}(A) = A\}$ is an Alexandrov topology on X.

(4) If $\mathbf{K}$ is an L-join meet approximation operator, then $\tau_{\mathbf{K}} = \{A \in L^X \mid \mathbf{K}(A) = A^*\}$ is an Alexandrov topology on X.

(5) If $\mathbf{M}$ is an L-meet join operator, then $\tau_{\mathbf{M}} = \{A \in L^X \mid \mathbf{M}(A) = A^*\}$ is an Alexandrov topology on X.

3. L-approximation operators and Alexandrov L-topologies

Theorem 3.1. *Let $\mathbf{H} : L^X \to L^X$ be an L-upper approximation operator. Then the following properties hold.*

(1) *For $A \in L^X$, $\mathbf{H}(A)(y) = \bigvee_{x\in X}(A(x) \odot \mathbf{H}(\top_x)(y))$.*

(2) *Define $\mathbf{J}_H(B) = \bigvee\{A \mid \mathbf{H}(A) \le B\}$. Then $\mathbf{J}_H : L^X \to L^X$ with*

$$\mathbf{J}_H(B)(x) = \bigwedge_{y\in X}(\mathbf{H}(\top_x)(y) \to B(y))$$

is an L-lower approximation operator such that $(\mathbf{H}, \mathbf{J}_H)$ is a residuated connection;i.e.,

$$\mathbf{H}(A) \le B \ \textit{ iff } \ A \le \mathbf{J}_H(B).$$

Moreover, $\tau_{\mathbf{H}} = \tau_{\mathbf{J}_H}$.

(3) *If $\mathbf{H}(\mathbf{H}(A)) = \mathbf{H}(A)$ for $A \in L^X$, then $\mathbf{J}_H(\mathbf{J}_H(A)) = \mathbf{J}_H(A)$ for $A \in L^X$ such that $\tau_{\mathbf{H}} = \tau_{\mathbf{J}_H}$ with*

$$\tau_{\mathbf{H}} = \{\mathbf{H}(A) = \bigvee_{x\in X}(A(x) \odot \mathbf{H}(\top_x)) \mid A \in L^X\},$$

$$\tau_{\mathbf{J}_H} = \{\mathbf{J}_H(A)(x) = \bigwedge_{y\in X}(\mathbf{H}(\top_x)(y) \to A(y)) \mid A \in L^X\}.$$

(4) *If $\mathbf{H}(\mathbf{H}^*(A)) = \mathbf{H}^*(A)$ for $A \in L^X$, then $\mathbf{H}(\mathbf{H}(A)) = \mathbf{H}(A)$ such that*

$$\tau_{\mathbf{H}} = \{\mathbf{H}^*(A) = \bigwedge_{x\in X}(A(x) \to \mathbf{H}^*(\top_x)) \mid A \in L^X\} = (\tau_{\mathbf{H}})_*.$$

(5) *Define $\mathbf{J}_s(A) = \mathbf{H}(A^*)^*$. Then $\mathbf{J}_s : L^X \to L^X$ with*

$$\mathbf{J}_s(B)(x) = \bigwedge_{y\in X}(\mathbf{H}(\top_y)(x) \to B(y))$$

is an L-lower approximation operator. Moreover, $\tau_{\mathbf{J}_s} = (\tau_{\mathbf{H}})_*$.

(6) *If $\mathbf{H}(\mathbf{H}(A)) = \mathbf{H}(A)$ for $A \in L^X$, then $\mathbf{J}_s(\mathbf{J}_s(A)) = \mathbf{J}_s(A)$ for $A \in L^X$ such that $\tau_{\mathbf{J}_s} = (\tau_{\mathbf{H}})_* = (\tau_{\mathbf{J}_H})_*$. with*

$$\tau_{\mathbf{J}_s} = \{\mathbf{J}_s(A) = \bigwedge_{y\in X}(\mathbf{H}(\top_y) \to A(y)) \mid A \in L^X\}.$$

(7) *If $\mathbf{H}(\mathbf{H}^*(A)) = \mathbf{H}^*(A)$ for $A \in L^X$, then $\mathbf{J}_s(\mathbf{J}_s^*(A)) = \mathbf{J}_s^*(A)$ such that*

$$\tau_{\mathbf{J}_s} = \{\mathbf{J}_s^*(A) = \bigvee_{y\in X}(\mathbf{H}(\top_y) \odot A^*(y)) \mid A \in L^X\} = (\tau_{\mathbf{J}_s})_*.$$

(8) *Define $\mathbf{M}_H(A) = \mathbf{H}(A^*)$. Then $\mathbf{M}_H : L^X \to L^X$ with*

$$\mathbf{M}_H(A) = \bigvee_{y\in X}(\mathbf{H}(\top_y) \odot A^*(y))$$

is an L-meet join approximation operator.

(9) *If $\mathbf{H}(\mathbf{H}(A)) = \mathbf{H}(A)$ for $A \in L^X$, then $\mathbf{M}_H(\mathbf{M}_H^*(A)) = \mathbf{M}_H^*(A)$ for $A \in L^X$ such that $\tau_{\mathbf{M}_H} = (\tau_{\mathbf{H}})_*$ with*

$$\tau_{\mathbf{M}_H} = \{\mathbf{M}_H^*(A) = \bigwedge_{y\in X}(\mathbf{H}(\top_y) \to A(y)) \mid A \in L^X\}.$$

(10) *If $\mathbf{H}(\mathbf{H}^*(A)) = \mathbf{H}^*(A)$ for $A \in L^X$, then $\mathbf{M}_H(\mathbf{M}_H(A)) = \mathbf{M}_H^*(A)$ such that*

$$\tau_{\mathbf{M}_H} = \{\mathbf{M}_H(A) = \bigvee_{y\in X}(\mathbf{H}(\top_y) \odot A^*(y)) \mid A \in L^X\} = (\tau_{\mathbf{M}_H})_*.$$

(11) *Define $\mathbf{K}_H(A) = (\mathbf{H}(A))^*$. Then $\mathbf{K}_H : L^X \to L^X$ with*

$$\mathbf{K}_H(A)(y) = \bigwedge_{x\in X}(A(x) \to \mathbf{H}^*(\top_x)(y))$$

is an L-join meet approximation operator. Moreover, $\tau_{\mathbf{K}_H} = \tau_{\mathbf{H}}$.

(12) *If $\mathbf{H}(\mathbf{H}(A)) = \mathbf{H}(A)$ for $A \in L^X$, then $\mathbf{K}_H(\mathbf{K}_H^*(A)) = \mathbf{K}_H(A)$ for $A \in L^X$ such that $\tau_{\mathbf{K}_H} = (\tau_{\mathbf{H}})_*$ with*

$$\tau_{\mathbf{K}_H} = \{\mathbf{K}_H(A)(y) = \bigvee_{x\in X}(A(x) \odot \mathbf{H}(\top_x)(y)) \mid A \in L^X\}.$$

(13) *If $\mathbf{H}(\mathbf{H}^*(A)) = \mathbf{H}^*(A)$ for $A \in L^X$, then $\mathbf{K}_H(\mathbf{K}_H(A)) = \mathbf{K}_H^*(A)$ such that*

$$\tau_{\mathbf{H}_K} = \{\mathbf{K}_H(A)(y) = \bigwedge_{x\in X}(A(x) \to \mathbf{H}^*(\top_x)(y)) \mid A \in L^X\} = (\tau_{\mathbf{H}_K})_*.$$

(14) *Define* $\mathbf{M}_{J_H}(A) = (\mathbf{J}_H(A))^*$. *Then* $\mathbf{M}_{J_H} : L^X \to L^X$ *with*

$$\mathbf{M}_{J_H}(A)(y) = \bigvee_{x \in X} (A^*(x) \odot \mathbf{H}(\top_y)(x))$$

is an L*-meet join approximation operator. Moreover,* $\tau_{\mathbf{M}_{J_H}} = \tau_{\mathbf{H}}$.

(15) *If* $\mathbf{H}(\mathbf{H}(A)) = \mathbf{H}(A)$ *for* $A \in L^X$, *then* $\mathbf{M}_{J_H}(\mathbf{M}^*_{J_H}(A)) = \mathbf{M}_{J_H}(A)$ *for* $A \in L^X$ *such that* $\tau_{\mathbf{M}_{J_H}} = (\tau_{\mathbf{H}})_*$ *with*

$$\tau_{\mathbf{M}_{J_H}} = \{\mathbf{M}^*_{J_H}(A)(y) = \bigwedge_{x \in X} (\mathbf{H}(\top_y)(x) \to A(x)) \mid A \in L^X\}.$$

(16) *If* $\mathbf{J}_H(\mathbf{J}^*_H(A)) = \mathbf{J}^*_H(A)$ *for* $A \in L^X$, *then* $\mathbf{M}_{J_H}(\mathbf{M}_{J_H}(A)) = \mathbf{M}^*_{J_H}(A)$ *such that*

$$\tau_{\mathbf{M}_{J_H}} = \{\mathbf{M}_{J_H}(A)(y) = \bigvee_{x \in X} (A^*(x) \odot \mathbf{H}(\top_y)(x)) \mid A \in L^X\} = (\tau_{\mathbf{M}_{J_H}})_*.$$

(17) *Define* $\mathbf{K}_{J_H}(A) = \mathbf{J}_H(A^*)$. *Then* $\mathbf{K}_{J_H} : L^X \to L^X$ *with*

$$\mathbf{K}_{J_H}(A)(y) = \bigwedge_{x \in X} (A(x) \to \mathbf{H}^*(\top_y)(x))$$

is an L*-join meet approximation operator. Moreover,* $\tau_{\mathbf{K}_{J_H}} = (\tau_{\mathbf{H}})_*$.

(18) *If* $\mathbf{H}(\mathbf{H}(A)) = \mathbf{H}(A)$ *for* $A \in L^X$, *then* $\mathbf{K}_{J_H}(\mathbf{K}^*_{J_H}(A)) = \mathbf{K}_{J_H}(A)$ *for* $A \in L^X$ *such that* $\tau_{\mathbf{K}_{J_H}} = (\tau_{\mathbf{H}})_*$ *with*

$$\tau_{\mathbf{K}_{J_H}} = \{\mathbf{K}^*_{J_H}(A)(y) = \bigvee_{x \in X} (\mathbf{H}(\top_y)(x) \odot A(x)) \mid A \in L^X\}.$$

(19) *If* $\mathbf{J}_H(\mathbf{J}^*_H(A)) = \mathbf{J}^*_H(A)$ *for* $A \in L^X$, *then* $\mathbf{K}_{J_H}(\mathbf{K}_{J_H}) = \mathbf{K}^*_{J_H}(A)$ *such that*

$$\tau_{\mathbf{K}_{J_H}} = \{\mathbf{K}_{J_H}(y) = \bigwedge_{x \in X} (A(x) \to \mathbf{H}^*(\top_y)(x)) \mid A \in L^X\} = (\tau_{\mathbf{K}_{J_H}})_*.$$

(20) $(\mathbf{K}_{J_H}, \mathbf{K}_H)$ *is a Galois connection;i.e,*

$$A \leq \mathbf{K}_{J_H}(B) \ \text{ iff } \ B \leq \mathbf{K}_H(A).$$

Moreover, $\tau_{\mathbf{K}_H} = (\tau_{\mathbf{K}_{J_H}})_*$.

(21) $(\mathbf{M}_H, \mathbf{M}_{J_H})$ *is a dual Galois connection;i.e,*

$$\mathbf{M}_{J_H}(A) \leq B \ \text{ iff } \ \mathbf{M}_H(B) \leq A.$$

Moreover, $\tau_{\mathbf{M}_H} = (\tau_{\mathbf{J}_{H_M}})_*$.

Proof. (1) Since $A = \bigvee_{x\in X}(A(x)\odot \top_x)$, by (H2) and (H3), $\mathbf{H}(A)(y) = \bigvee_{x\in X}(A(x)\odot \mathbf{H}(\top_x)(y))$.

(2) Since $\mathbf{H}(A)(y) = \bigvee_{x\in X}(A(x)\odot \mathbf{H}(\top_x)(y)) \le B(y)$ iff $A(x) \le \mathbf{H}(\top_x)(y) \to B(y)$, we have

$$\mathbf{J}_H(B)(x) = \bigwedge_{y\in Y}(\mathbf{H}(\top_x)(y) \to B(y)).$$

(J1) Since $\mathbf{H}(\mathbf{J}_H(B)) \le B$, we have $\mathbf{J}_H(B) \le \mathbf{H}(\mathbf{J}_H(B)) \le B$.

(J2) Since $\mathbf{H}(a\odot \mathbf{J}_H(a\to B)) = a\odot \mathbf{H}(\mathbf{J}_H(a\to B)) \le a\odot(a\to B) \le B$, by the definition of $\mathbf{J}_H$, then $a\odot \mathbf{J}_H(a\to B) \le \mathbf{J}_H(B))$. We have

$$\mathbf{J}_H(a\to B) \le a \to \mathbf{J}_H(B)).$$

Since $a\odot \mathbf{H}(a \to \mathbf{J}_H(B)) = \mathbf{H}(a\odot(a\to \mathbf{J}_H(B))) \le \mathbf{H}(\mathbf{J}_H(B)) \le B$, then $\mathbf{H}(a\to \mathbf{J}_H(B)) \le a\to B$. By the definition of $\mathbf{J}_H$, we have

$$a\to \mathbf{J}_H(B) \le \mathbf{J}_H(a\to B).$$

(J3) By the definition of $\mathbf{J}_H$, since $\mathbf{J}_H(A) \le \mathbf{J}_H(B)$ for $B \le A$, we have

$$\mathbf{J}_H(\bigwedge_{i\in\Gamma} A_i) \le \bigwedge_{i\in\Gamma}\mathbf{J}_H(A_i).$$

Since $\mathbf{H}(\bigwedge_{i\in\Gamma}\mathbf{J}_H(A_i)) \le \mathbf{H}(\mathbf{J}_H(A_i)) \le A_i$, then $\mathbf{H}(\bigwedge_{i\in\Gamma}\mathbf{J}_H(A_i)) \le \bigwedge_{i\in\Gamma} A_i$. Thus

$$\mathbf{J}_H(\bigwedge_{i\in\Gamma} A_i) \ge \bigwedge_{i\in\Gamma}\mathbf{J}_H(A_i).$$

Thus $\mathbf{J}_H : L^X \to L^X$ is an L-lower approximation operator. By the definition of $\mathbf{J}_H$, we have

$$A \le \mathbf{J}_H(B) \text{ iff } B \le \mathbf{H}(A).$$

Since $A \le \mathbf{J}_H(A)$ iff $A \le \mathbf{H}(A)$, we have $\tau_{\mathbf{J}_H} = \tau_{\mathbf{H}}$.

(3) Let $\mathbf{H}(\mathbf{H}(A)) = \mathbf{H}(A)$ for $A \in L^X$. Since $\mathbf{H}(B) \le \mathbf{J}_H(A)$ iff $\mathbf{H}(\mathbf{H}(B)) = \mathbf{H}(B) \le A$ from the definition of $\mathbf{J}_H$, we have

$$\begin{aligned}\mathbf{J}_H(\mathbf{J}_H(A)) &= \bigvee\{B \mid \mathbf{H}(B) \le \mathbf{J}_H(A)\}\\ &= \bigvee\{B \mid \mathbf{H}(\mathbf{H}(B)) = \mathbf{H}(B) \le A\}\\ &= \mathbf{J}_H(A).\end{aligned}$$

(4) Let $\mathbf{H}^*(A) \in \tau_{\mathbf{H}}$. Since $\mathbf{H}(\mathbf{H}^*(A)) = \mathbf{H}^*(A)$, $\mathbf{H}(\mathbf{H}(A)) = \mathbf{H}(\mathbf{H}^*(\mathbf{H}^*(A))) = (\mathbf{H}(\mathbf{H}^*(A)))^* = \mathbf{H}(A)$. Hence $\mathbf{H}(A) \in \tau_{\mathbf{H}}$; i.e. $\mathbf{H}^*(A) \in (\tau_{\mathbf{H}})_*$. Thus, $\tau_{\mathbf{H}} \subset (\tau_{\mathbf{H}})_*$.

Let $A \in (\tau_{\mathbf{H}})_*$. Then $A^* = \mathbf{H}(A^*)$. Since $\mathbf{H}(A) = \mathbf{H}(\mathbf{H}^*(A^*)) = \mathbf{H}^*(A^*) = A$, then $A \in\in \tau_{\mathbf{H}}$. Thus, $(\tau_{\mathbf{H}})_* \subset \tau_{\mathbf{H}}$.

(5) (J1) Since $A^* \le \mathbf{H}(A^*)$, $\mathbf{J}_s(A) = \mathbf{H}(A^*)^* \le A$.

(J2)

$$\begin{aligned}\mathbf{J}_s(\alpha \to A) &= (\mathbf{H}((\alpha \to A)^*)^* = (\mathbf{H}(\alpha \odot A^*))^* \\ &= (\alpha \odot \mathbf{H}(A^*))^* = \alpha \to \mathbf{H}(A^*)^* \\ &= \alpha \to \mathbf{J}_s(A).\end{aligned}$$

(J3)

$$\begin{aligned}\mathbf{J}_s(\textstyle\bigwedge_{i\in\Gamma} A_i) &= (\mathbf{H}(\textstyle\bigwedge_{i\in\Gamma} A_i)^*)^* = (\mathbf{H}(\textstyle\bigvee_{i\in\Gamma} A_i^*))^* \\ &= (\textstyle\bigvee_{i\in\Gamma} \mathbf{H}(A_i^*))^* = \textstyle\bigwedge_{i\in\Gamma}(\mathbf{H}(A_i^*))^* \\ &= \textstyle\bigwedge_{i\in\Gamma} \mathbf{J}_s(A_i).\end{aligned}$$

Hence $\mathbf{J}_s$ is an L-lower approximation operator such that

$$\mathbf{J}_s(B)(x) = (\mathbf{H}(B^*)(x))^* = \bigwedge_{y\in X}(\mathbf{H}(\top_y)(x) \to B(y)).$$

Moreover, $\tau_{\mathbf{J}_s} = (\tau_{\mathbf{H}})_*$ from:

$$A = \mathbf{J}_s(A) \ \text{ iff } \ A = \mathbf{H}(A^*)^* \ \text{ iff } \ A^* = \mathbf{H}(A^*).$$

(6) Let $\mathbf{H}(\mathbf{H}(A)) = \mathbf{H}(A)$ for $A \in L^X$. Then

$$\begin{aligned}\mathbf{J}_s(\mathbf{J}_s(A)) &= \mathbf{H}^*(\mathbf{J}_s^*(A)) = (\mathbf{H}(\mathbf{H}(A^*)))^* \\ &= \mathbf{H}^*(A^*) = \mathbf{J}_s(A).\end{aligned}$$

Hence $\tau_{\mathbf{J}_s} = \{\mathbf{J}_s(A) = \bigwedge_{y\in X}(\mathbf{H}(\top_y) \to A(y)) \mid A \in L^X\}$.

(7) Let $\mathbf{H}(\mathbf{H}^*(A)) = \mathbf{H}^*(A)$ for $A \in L^X$. Then

$$\begin{aligned}\mathbf{J}_s(\mathbf{J}_s^*(A)) &= \mathbf{H}^*(\mathbf{J}_s(A)) = (\mathbf{H}(\mathbf{H}^*(A^*)))^* \\ &= (\mathbf{H}^*(A^*))^* = \mathbf{J}_s^*(A).\end{aligned}$$

Hence $\tau_{\mathbf{J}_s} = \{\mathbf{J}_s^*(A) = \bigvee_{y\in X}(\mathbf{H}(\top_y) \odot A^*(y)) \mid A \in L^X\}$.

$$\begin{aligned}\mathbf{J}_s(\mathbf{J}_s(A)) &= \mathbf{J}_s(\mathbf{J}_s^*(\mathbf{J}_s^*(A))) \\ &= \mathbf{J}_s^*(\mathbf{J}_s^*(A)) = \mathbf{J}_s(A).\end{aligned}$$

By a similar method in (4), $\tau_{\mathbf{J}_s} = (\tau_{\mathbf{J}_s})_*$.

(8) It is similarly proved as (4).

(9) If $\mathbf{H}(\mathbf{H}(A)) = \mathbf{H}(A)$ for $A \in L^X$, then $\mathbf{M}_H(\mathbf{M}^*_H(A)) = \mathbf{M}_H(A)$

$$\begin{aligned}\mathbf{M}_H(\mathbf{M}^*_H(A)) &= \mathbf{M}_H(\mathbf{H}^*(A^*)) = \mathbf{H}(\mathbf{H}(A^*)) \\ &= \mathbf{H}(A^*) = \mathbf{M}_H(A).\end{aligned}$$

(10) If $\mathbf{H}(\mathbf{H}^*(A)) = \mathbf{H}^*(A)$ for $A \in L^X$, then $\mathbf{M}_H(\mathbf{M}_H(A)) = \mathbf{M}^*_H(A)$

$$\begin{aligned}\mathbf{M}_H(\mathbf{M}_H(A)) &= \mathbf{H}(\mathbf{M}^*_H(A)) = \mathbf{H}(\mathbf{H}^*(A^*)) \\ &= \mathbf{H}^*(A^*) = \mathbf{M}^*_H(A).\end{aligned}$$

Since $\mathbf{M}_H(\mathbf{M}_H(A)) = \mathbf{M}^*_H(A)$,

$$\begin{aligned}\mathbf{M}_H(\mathbf{M}^*_H(A)) &= \mathbf{M}_H(\mathbf{M}_H(\mathbf{M}_H(A))) \\ &= \mathbf{M}^*_H(\mathbf{M}_H(A)) = \mathbf{M}_H(A).\end{aligned}$$

Hence $\tau_{\mathbf{M}_H} = \{\mathbf{M}_H(A) \mid A \in L^X\} = (\tau_{\mathbf{M}_H})_*$.

(11), (12), (13) and (14) are similarly proved as (5), (9), (10) and (5), respectively.

(15) If $\mathbf{H}(\mathbf{H}(A)) = \mathbf{H}(A)$ for $A \in L^X$, then $\mathbf{J}_H(\mathbf{J}_H(A)) = \mathbf{J}_H(A)$. Thus, $\mathbf{M}_{J_H}(\mathbf{M}^*_{J_H}(A)) = \mathbf{M}_{J_H}(A)$

$$\begin{aligned}\mathbf{M}_{J_H}(\mathbf{M}^*_{J_H}(A)) &= \mathbf{M}_{J_H}(\mathbf{J}_H(A)) \\ &= (J_H(\mathbf{J}_H(A)))^* = (\mathbf{J}_H(A))^* = \mathbf{M}_{J_H}(A).\end{aligned}$$

Since $\mathbf{H}(A) = A$ iff $\mathbf{J}_H(A) = A$ iff $\mathbf{M}_{J_H}(A) = A^*$, $\tau_{\mathbf{M}_{J_H}} = (\tau_{\mathbf{H}})_*$ with

$$\tau_{\mathbf{M}_{J_H}} = \{\mathbf{M}^*_{J_H}(A)(y) = \bigwedge_{x \in X} (\mathbf{H}(\top_y)(x) \to A(x)) \mid A \in L^X\}.$$

(16) If $\mathbf{J}_H(\mathbf{J}^*_H(A)) = \mathbf{J}^*_H(A)$ for $A \in L^X$, then $\mathbf{M}_{J_H}(\mathbf{M}_{J_H}(A)) = \mathbf{M}^*_{J_H}(A)$

$$\begin{aligned}\mathbf{M}_{J_H}(\mathbf{M}_{J_H}(A)) &= \mathbf{M}_{J_H}(\mathbf{J}^*_H(A)) = \mathbf{J}^*_H(\mathbf{J}^*_H(A)) \\ &= \mathbf{J}_H(A) = \mathbf{M}^*_{J_H}(A).\end{aligned}$$

(17), (18) and (19) are similarly proved as (14), (15) and (16), respectively.

(20) $(\mathbf{K}_{J_H}, \mathbf{K}_H)$ is a Galois connection; i.e,

$$A \leq \mathbf{K}_{J_H}(B) \ \text{ iff } \ A \leq \mathbf{J}_H(B^*)$$

$$\text{iff } \ \mathbf{H}(A) \leq B^* \ \text{ iff } \ B \leq \mathbf{H}^*(A) = \mathbf{K}_H(A)$$

Moreover, since $A^* \leq \mathbf{K}_H(A)$ iff $A \leq \mathbf{K}_{J_H}(A^*)$, $\tau_{\mathbf{K}_H} = (\tau_{\mathbf{K}_{J_H}})_*$.

(21) $(\mathbf{M}_H, \mathbf{M}_{J_H})$ is a dual Galois connection; i.e,

$$\mathbf{M}_{J_H}(A) \leq B \text{ iff } \mathbf{J}_H(A) \geq B^*$$

$$\text{iff } \mathbf{H}(B^*) \leq A \text{ iff } \mathbf{M}_H(B) \leq A.$$

Since $\mathbf{M}_{J_H}(A^*) \leq A$ iff $\mathbf{M}_H(A) \leq A^*$, $\tau_{\mathbf{M}_H} = (\tau_{\mathbf{M}_{J_H}})_*$.

Definition 3.2. [3,4] Let X be a set. A function $R : X \times X \to L$ is called:

(R1) *reflexive* if $R(x,x) = \top$ for all $x \in X$.

(R2) *symmetric* if $R(x,y) = R(y,x)$ for all $x, y \in X$.

(R3) *transitive* if $R(x,y) \odot R(y,z) \leq R(x,z)$, for all $x, y, z \in X$.

(R4) *Euclidean* if $R(x,z) \odot R(y,z) \leq R(x,y)$, for all $x, y, z \in X$.

If R satisfies (R1) and (R3), R is called an *L-fuzzy preorder*.

If R satisfies (R1), (R2) and (R3), R is called an *L-fuzzy equivalence relation*.

Let $R \in L^{X \times X}$ be an L-fuzzy relation. Define operators as follows

$$\begin{aligned}
\mathbf{H}_R(A)(y) &= \bigvee_{x\in X}(A(x) \odot R(x,y)), \\
\mathbf{J}_R(A)(y) &= \bigwedge_{x\in X}(R(x,y) \to A(x)), \\
\mathbf{K}_R(A)(y) &= \bigwedge_{x\in X}(A(x) \to R(x,y)) \\
\mathbf{M}_R(A)(y) &= \bigvee_{x\in X}(A^*(x) \odot R(x,y)).
\end{aligned}$$

Example 3.3. Let R be a reflexive L-fuzzy relation. Define $\mathbf{H}_R : L^X \to L^X$ as follows:

$$\mathbf{H}_R(A)(y) = \bigvee_{x\in X}(A(x) \odot R(x,y)).$$

(1) (H1) $\mathbf{H}_R(A)(y) \geq A(y) \odot R(y,y) = A(y)$. $\mathbf{H}_R$ satisfies the conditions (H1) and (H2). Hence $\mathbf{H}_R$ is an L-upper approximation operator.

(2) Define $\mathbf{J}_{H_R}(B) = \bigvee\{A \mid \mathbf{H}_R(A) \leq B\}$. Since $\mathbf{H}_R(A)(y) \leq B(y)$ iff $A(x) \leq \bigwedge_{y\in X}(R(x,y) \to B(y))$, then

$$\mathbf{J}_{H_R}(B)(x) = \bigwedge_{y\in X}(R(x,y) \to B(y)) = \mathbf{J}_{R^{-1}}(B)(x).$$

By Theorem 3.1(2), $\mathbf{J}_{H_R} = \mathbf{J}_{R^{-1}}$ is an L-lower approximation operator such that $(\mathbf{H}_R, \mathbf{J}_{H_R})$ is a residuated connection; i.e.,

$$\mathbf{H}_R(A) \leq B \text{ iff } A \leq \mathbf{J}_{H_R}(B).$$

Moreover, $\tau_{\mathbf{J}_{H_R}} = \tau_{\mathbf{H}_R}$.

(3) If R is an L-fuzzy preorder, then R^{-1} is an L-fuzzy preorder. Since

$$\begin{aligned}\mathbf{H}_R(\mathbf{H}_R(A))(z) &= \bigvee_{y\in X}(\mathbf{H}_R(A)(y) \odot R(y,z)) \\ &= \bigvee_{y\in X}(\bigvee_{x\in X}(A(x) \odot R(x,y)) \odot R(y,z)) \\ &= \bigvee_{x\in X}(A(x) \odot \bigvee_{x\in X}(R(x,y) \odot R(y,z))) \\ &= \bigvee_{x\in X}(A(x) \odot R(x,z)) = \mathbf{H}_R(A)(z).\end{aligned}$$

By Theorem 3.1(3), $\mathbf{J}_{H_R}(\mathbf{J}_{H_R}(A)) = \mathbf{J}_{H_R}(A)$. By Theorem 3.1(3), $\tau_{\mathbf{J}_{H_R}} = \tau_{\mathbf{H}_R}$ with

$$\tau_{\mathbf{J}_{H_R}} = \tau_{\mathbf{J}_{R^{-1}}} = \{\mathbf{J}_{R^{-1}}(A) = \bigwedge_{x\in X}(R(-,x) \to A(x)) \mid A \in L^X\},$$

$$\tau_{\mathbf{H}_R} = \{\mathbf{H}_R(A) = \bigvee_{x\in X}(A(x) \odot R(x,-)) \mid A \in L^X\}.$$

(4) Let R be a reflexive and Euclidean L-fuzzy relation. Since $(R(x,y) \to A(x)) \odot R(y,z) \odot R(x,z) \leq (R(x,y) \to A(x)) \odot R(x,y) \leq A(x)$, then $(R(x,y) \to A(x)) \odot R(y,z) \leq R(x,z) \to A(x)$. Thus, $\mathbf{H}_R(\mathbf{H}^*_R(A)) = \mathbf{H}^*_R(A)$ from:

$$\begin{aligned}\mathbf{H}_R(\mathbf{H}^*_R(A))(z) &= \bigvee_{y\in X}(\mathbf{H}^*_R(A)(y) \odot R(y,z)) \\ &= \bigvee_{y\in X}(\bigwedge_{x\in X}(R(x,y) \to A(x)) \odot R(y,z)) \\ &\leq \bigwedge_{x\in X}(R(x,z) \to A(x)) = \mathbf{H}^*_R(A)(z).\end{aligned}$$

By Theorem 3.1(4), $\mathbf{H}_R(\mathbf{H}_R(A)) = \mathbf{H}_R(A)$ for $A \in L^X$. Thus, $\tau_{\mathbf{H}_R} = (\tau_{\mathbf{H}_R})_*$ with

$$\tau_{\mathbf{H}_R} = \{\mathbf{H}^*_R(A) = \bigwedge_{x\in X}(R(x,-) \to A(x)) = \mathbf{J}_R(A) \mid A \in L^X\}.$$

(5) Define $\mathbf{J}_s(A) = \mathbf{H}_R(A^*)^*$. By Theorem 3.1(5), $\mathbf{J}_s = \mathbf{J}_R$ is an L-lower approximation operator such that

$$\mathbf{J}_s(A)(y) = (\bigvee_{x\in X} A^*(x) \odot R(x,y))^* = \bigwedge_{x\in X}(R(x,y) \to A(x)).$$

Moreover, $\tau_{\mathbf{J}_s} = (\tau_{\mathbf{H}_R})_* = (\tau_{\mathbf{J}_{H_R}})_*$.

(6) If R is an L-fuzzy preorder, then $\mathbf{H}_R(\mathbf{H}_R(A)) = \mathbf{H}_R(A)$ for $A \in L^X$. By Theorem 3.1(6), then $\mathbf{J}_s(\mathbf{J}_s(A)) = \mathbf{J}_s(A)$ for $A \in L^X$ such that $\tau_{\mathbf{J}_s} = (\tau_{\mathbf{H}_R})_* = (\tau_{\mathbf{J}_{H_R}})_*$ with

$$\tau_{\mathbf{J}_s} = \{\mathbf{J}_s(A) = \bigwedge_{y\in X}(R(y,-) \to A(y)) \mid A \in L^X\}.$$

(7) If R is a reflexive and Euclidean L-fuzzy relation, then $\mathbf{H}_R(\mathbf{H}^*_R(A)) = \mathbf{H}^*_R(A)$ for $A \in L^X$. By Theorem 3.1(7), $\mathbf{J}_s(\mathbf{J}^*_s(A)) = \mathbf{J}^*_s(A)$ such that

$$\tau_{\mathbf{J}_s} = \{\mathbf{J}^*_s(A) = \bigvee_{y\in X}(R(y,-) \odot A^*(y)) = \mathbf{M}_R(A) \mid A \in L^X\} = (\tau_{\mathbf{J}_s})_*.$$

(8) Define $\mathbf{M}_{H_R}(A) = \mathbf{H}_R(A^*)$. Then $\mathbf{M}_{H_R} : L^X \to L^X$ with

$$\mathbf{M}_{H_R}(A)(y) = \bigvee_{x\in X}(R(x,y) \odot A^*(x)) = \mathbf{M}_R(y)$$

is an L-meet join approximation operator. Moreover, $\tau_{\mathbf{M}_{H_R}} = (\tau_{\mathbf{H}_R})_*$.

(9) R is an L-fuzzy preorder, then $\mathbf{H}_R(\mathbf{H}_R(A)) = \mathbf{H}_R(A)$ for $A \in L^X$. By Theorem 3.1(9), $\mathbf{M}_{H_R}(\mathbf{M}^*_{H_R}(A)) = \mathbf{M}_{H_R}(A)$ for $A \in L^X$ such that $\tau_{\mathbf{M}_{H_R}} = (\tau_{\mathbf{H}_R})_*$ with

$$\tau_{\mathbf{M}_{H_R}} = \{\mathbf{M}^*_{H_R}(A) = \bigwedge_{x\in X}(R(x,-) \to A(x)) = \mathbf{J}_R(A) \mid A \in L^X\}.$$

(10) If R is a reflexive and Euclidean L-fuzzy relation, then $\mathbf{H}_R(\mathbf{H}^*_R(A)) = \mathbf{H}^*_R(A)$ for $A \in L^X$. By Theorem 3.1(10), $\mathbf{M}_{H_R}(\mathbf{M}_{H_R}(A)) = \mathbf{M}^*_{H_R}(A)$ such that

$$\tau_{\mathbf{M}_{H_R}} = \{\mathbf{M}_{H_R}(A) = \bigvee_{x\in X}(R(x,-) \odot A^*(x)) \mid A \in L^X\} = (\tau_{\mathbf{M}_{H_R}})_*.$$

(11) Define $\mathbf{K}_{H_R}(A) = (\mathbf{H}_R(A))^*$. Then $\mathbf{K}_{H_R} : L^X \to L^X$ with

$$\mathbf{K}_{H_R}(A)(y) = \bigwedge_{x\in X}(A(x) \to R^*(x,y)) = \mathbf{K}_{R^*}(A)(y)$$

is an L-join meet approximation operator. Moreover, $\tau_{\mathbf{K}_{H_R}} = \tau_{\mathbf{H}_R}$.

(12) If R is an L-fuzzy preorder, then $\mathbf{H}_R(\mathbf{H}_R(A)) = \mathbf{H}_R(A)$ for $A \in L^X$. By Theorem 3.1(12), $\mathbf{K}_{H_R}(\mathbf{K}^*_{H_R}(A)) = \mathbf{K}_{H_R}(A)$ for $A \in L^X$ such that $\tau_{\mathbf{K}_{H_R}} = \tau_{\mathbf{H}_R}$ with

$$\tau_{\mathbf{K}_{H_R}} = \{\mathbf{K}^*_{H_R}(A) = \bigvee_{x\in X}(A(x) \odot R(x,-)) \mid A \in L^X\}.$$

(13) If R is a reflexive and Euclidean L-fuzzy relation, then $\mathbf{H}_R(\mathbf{H}^*_R(A)) = \mathbf{H}^*_R(A)$ for $A \in L^X$. By Theorem 3.1(13), $\mathbf{K}_{H_R}(\mathbf{K}_{H_R}(A)) = \mathbf{K}^*_{H_R}(A)$ such that

$$\tau_{\mathbf{K}_{H_R}} = \{\mathbf{K}_{H_R}(A) = \bigwedge_{x\in X}(A(x) \to R^*(x,-)) \mid A \in L^X\} = (\tau_{\mathbf{K}_{H_R}})_*.$$

(14) Define $\mathbf{M}_{J_{H_R}}(A) = (\mathbf{J}_{H_R}(A))^*$. Then $\mathbf{M}_{J_{H_R}} : L^X \to L^X$ with

$$\mathbf{M}_{J_H}(A)(y) = \bigvee_{x\in X}(A^*(x) \odot R(y,x)) = \mathbf{M}_{R^{-1}}(A)(y)$$

is an L-join meet approximation operator. Moreover, $\tau_{\mathbf{M}_{R^{-1}}} = \tau_{\mathbf{H}_R} = \tau_{\mathbf{J}_{R^{-1}}}$.

(15) If R is an L-fuzzy preorder, then $\mathbf{H}_R(\mathbf{H}_R(A)) = \mathbf{H}_R(A)$ for $A \in L^X$. By Theorem 3.1(15), $\mathbf{M}_{R^{-1}}(\mathbf{M}^*_{R^{-1}}(A)) = \mathbf{M}_{R^{-1}}(A)$ for $A \in L^X$ such that $\tau_{\mathbf{M}_{R^{-1}}} = \tau_{\mathbf{H}_R} = \tau_{\mathbf{J}_{R^{-1}}}$ with

$$\tau_{\mathbf{M}_{R^{-1}}} = \{\mathbf{M}^*_{R^{-1}}(A)(y) = \bigwedge_{x\in X}(R(y,x) \to A(x)) \mid A \in L^X\}.$$

(16) Let R^{-1} be a reflexive and Euclidean L-fuzzy relation. Since $(R(y,x) \to A(x)) \odot R(z,y) \odot R(z,x) \le R(y,x) \to A(x)) \odot R(y,x) \le A(x)$, then $(R(y,x) \to A(x)) \odot R(z,y) \le R(z,x) \to A(x)$. Thus,

$$\begin{aligned}\mathbf{M}_{R^{-1}}(\mathbf{M}_{R^{-1}}(A))(z) &= \bigvee_{y\in X}(\mathbf{M}_{R^{-1}}(A)(y) \odot R(z,y)) \\ &= \bigvee_{y\in X}(\bigwedge_{x\in X}(R(y,x) \to A(x)) \odot R(z,y)) \\ &\le \bigwedge_{x\in X}(R(z,x) \to A(x)) = \mathbf{M}_{R^{-1}}(A)(z).\end{aligned}$$

By (M1), $\mathbf{M}_{R^{-1}}(\mathbf{M}_{R^{-1}}(A)) = \mathbf{M}^*_{R^{-1}}(A)$ such that

$$\tau_{\mathbf{M}_{R^{-1}}} = \{\mathbf{M}_{R^{-1}}(A) = \bigvee_{x\in X}(A^*(x) \odot R(-,x)) \mid A \in L^X\} = (\tau_{\mathbf{M}_{R^{-1}}})_*.$$

(17) Define $\mathbf{K}_{J_{H_R}}(A) = \mathbf{J}_{H_R}(A^*)$. Then $\mathbf{K}_{J_{H_R}} : L^X \to L^X$ is an L-join meet approximation operator as follows:

$$\begin{aligned}\mathbf{K}_{J_{H_R}}(A)(y) &= \bigwedge_{x\in X}(R(y,x) \to A^*(x)) \\ &= \bigwedge_{x\in X}(A(x) \to R^*(y,x)) \\ &= \mathbf{K}_{R^{-1*}}(A)(y).\end{aligned}$$

Moreover, $\tau_{\mathbf{K}_{J_{H_R}}} = (\tau_{\mathbf{H}_R})_*$.

(18) If R is an L-fuzzy preorder, then $\mathbf{H}_R(\mathbf{H}_R(A)) = \mathbf{H}_R(A)$ for $A \in L^X$. By Theorem 3.1(18), $\mathbf{K}_{J_{H_R}}(\mathbf{K}^*_{J_{H_R}}(A)) = \mathbf{K}_{J_{H_R}}(A)$ for $A \in L^X$ such that $\tau_{\mathbf{K}_{J_H}} = (\tau_{\mathbf{H}})_*$ with

$$\tau_{\mathbf{K}_{J_{H_R}}} = \{\mathbf{K}^*_{J_{H_R}}(A)(y) = \bigvee_{x\in X}(R(y,x)\odot A(x)) = \mathbf{H}_{R^{-1}}(A)(y) \mid A \in L^X\}.$$

(19) Let R be a reflexive and Euclidean L-fuzzy relation. Since $R(z,y)\odot R(z,x) \le R(y,x)$ iff $R(z,y) \le R(z,x) \to R(y,x)$ iff $R(z,x)\odot R^*(y,x) \le R^*(z,y)$, we have

$$R(z,x)\odot A(x)\odot (A(x)\to R^*(y,x)) \le R(z,x)\odot R^*(y,x) \le R^*(z,y).$$

Thus,

$$\begin{aligned}\mathbf{K}_{R^{-1*}}(\mathbf{K}_{R^{-1*}}(A))(z) &= \bigwedge_{y\in X}(\mathbf{K}_{R^{-1*}}(A)(y) \to R^*(z,y))\\ &= \bigwedge_{y\in X}(\bigwedge_{x\in X}(A(x)\to R^*(y,x)) \to R^*(z,y))\\ &\ge \bigvee_{x\in X}(R(z,x)\odot A(x)) = \mathbf{K}_{R^{-1*}}(A)(z).\end{aligned}$$

Moreover,

$$\tau_{\mathbf{K}_{J_{H_R}}} = \{\mathbf{K}_{J_{H_R}}(y) = \bigwedge_{x\in X}(A(x)\to R(y,x)^*) \mid A\in L^X\} = (\tau_{\mathbf{K}_{J_{H_R}}})_*.$$

(20) $(\mathbf{K}_{J_{H_R}} = \mathbf{K}_{R^{-1*}}, \mathbf{K}_{H_R} = \mathbf{K}_{R^*})$ is a Galois connection;i.e,

$$A \le \mathbf{K}_{J_{H_R}}(B) \ \text{ iff } \ B \le \mathbf{K}_{H_R}(A).$$

Moreover, $\tau_{\mathbf{K}_{H_R}} = (\tau_{\mathbf{K}_{J_{H_R}}})_*$.

(21) $(\mathbf{M}_{H_R} = \mathbf{M}_R, \mathbf{M}_{J_{H_R}} = \mathbf{M}_{R^{-1}})$ is a dual Galois connection;i.e,

$$\mathbf{M}_{J_{H_R}}(A) \le B \ \text{ iff } \ \mathbf{M}_{H_R}(B) \le A.$$

Moreover, $\tau_{\mathbf{M}_{H_R}} = (\tau_{\mathbf{J}_{H_{M_R}}})_*$.

Conflict of Interests

The author declares that there is no conflict of interests.

REFERENCES

[1] R. Bělohlávek, Fuzzy Relational Systems, Kluwer Academic Publishers, New York , (2002).

[2] P. Hájek, Metamathematices of Fuzzy Logic, Kluwer Academic Publishers, Dordrecht (1998).

[3] Y.C. Kim, L-approximatins and join preserving maps, J. Math. Comput. Sci. 3 (2013), 1193-1210.

[4] Y.C. Kim, Alexandrov L-topologies, to appear Int. J. Pure Appl. Math.

[5] H. Lai, D. Zhang, Fuzzy preorder and fuzzy topology, Fuzzy Sets Sys. 157 (2006), 1865-1885.

[6] H. Lai, D. Zhang, Concept lattices of fuzzy contexts: Formal concept analysis vs. rough set theory, Int. J. Approx. Reasoning, 50 (2009), 695-707.

[7] Z. Pawlak, Rough sets, Int. J. Comput. Inf. Sci. 11 (1982), 341-356.

[8] Z. Pawlak, Rough probability, Bull. Pol. Acad. Sci. Math. 32 (1984), 607-615.

[9] A. M. Radzikowska, E.E. Kerre, A comparative study of fuzy rough sets, Fuzzy Sets Sys. 126 (2002), 137-155.

[10] Y.H. She, G.J. Wang, An axiomatic approach of fuzzy rough sets based on residuated lattices, Comput. Math. Appl. 58 (2009), 189-201.

[11] Z.M. Ma, B.Q. Hu, Topological and lattice structures of L-fuzzy rough set determined by lower and upper sets, Info. Sci. 218 (2013), 194-204.