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# NEW ITERATIVE METHOD FOR CAUCHY PROBLEMS

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**Abstract.** In this paper, the new iterative method is used to solve the Cauchy problem. Some examples are given to elucidate the solution procedure and reliability of the obtained results. The new iterative method algorithm leads to exact solutions in the present study.

**Keywords**: New iterative method; Cauchy problems.
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## 1. Introduction

In 2006, Daftardar-Gejji and Jafari [1] proposed a new technique for solving linear and nonlinear functional equations via new iterative method (NIM). This method has proven useful for solving a variety of linear and nonlinear equations such as algebraic equations, integral equations, ordinary and partial differential equations and systems of equations of integer and fractional order as well and the obtained results are of surprising accuracy (sometimes exact solutions can be obtained), the method can be easily understood by non-mathematical students and applied to various nonlinear problems. NIM is simple to understand and easy to implement using computer packages and yields better results Adomian decomposition method (ADM) [2], homotopy perturbation method [3], variational iteration method [4].

The main property of the method is its flexibility and ability to solve nonlinear equations accurately and conveniently, for example, in [5, 6,7] applied the method to various nonlinear systems partial differential equations, and concluded that the NIM is a reliable analytical tool for solving linear and nonlinear.

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Recently, [8, 9, 10, 11] summarized various NIM algorithms for various nonlinear equations including fractional differential equation. Though much achievement has been achieved, application of the NIM to Cauchy problems has not yet been dealt with. In this paper, we use the NIM to discuss the first-order partial differential equation in the form [12]:

$$u_t(x,t)+a(x,t)u_x(x,t)=\phi(x), \quad x\in R,\ t>0, \tag{1.1}$$

with the initial condition

$$u(x,0)=\psi(x), \quad x\in R. \tag{1.2}$$

when $a(x,t)=a$ a is a constant and $\phi(x)=0$. Eq.(1.1) is a linear equation called the transport equation which can describe many interesting phenomena such as the spread of AIDS, the moving of wind. When $a(x,t)=u(x,t)$, the equation is called the inviscid Burgers' equation arising in one dimensional stream of particles or fluid having zero viscosity.

## 2. Basic idea of NIM

To describe the idea of the NIM, consider the following general functional equation [1]:

$$u(x)=f(x)+N(u(x)), \tag{2.1}$$

where N is a nonlinear operator from a Banach space B→B and f is a known function. We are looking for a solution u of (2.1) having the series form

$$u(x)=\sum_{i=0}^{\infty}u_i(x). \tag{2.2}$$

The nonlinear operator N can be decomposed as follows

$$N\left(\sum_{i=0}^{\infty}u_i\right)=N(u_0)+\sum_{i=1}^{\infty}\left\{N\left(\sum_{j=0}^{i}u_j\right)-N\left(\sum_{j=0}^{i-1}u_j\right)\right\}. \tag{2.3}$$

From Eqs. (2.2) and (2.3), Eq. (2.1) is equivalent to

$$\sum_{i=0}^{\infty}u_i = f+N(u_0)+\sum_{i=1}^{\infty}\left\{N\left(\sum_{j=0}^{i}u_j\right)-N\left(\sum_{j=0}^{i-1}u_j\right)\right\}. \tag{2.4}$$

We define the recurrence relation:

$$u_0 = f, \tag{2.5a}$$

$$u_1 = N(u_0), \tag{2.5b}$$

$$u_{n+1} = N(u_0+u_1+\ldots+u_n)-N(u_0+u_1+\ldots+u_{n-1}), \quad n=1,2,3,\ldots. \tag{2.5c}$$

Then:

$(u_1 + \ldots + u_{n+1}) = N(u_0 + u_1 + \ldots + u_n), \qquad n = 1,2,3,\ldots\ ,$

$$u = \sum_{i=0}^{\infty} u_i = f + N\left(\sum_{i=0}^{\infty} u_i\right). \tag{2.6}$$

If N is a contraction, that is

$\|N(x) - N(y)\| \le k\|x - y\|, \quad 0 < k < 1,$

then:

$$\|u_{n+1}\| = \|N(u_0 + u_1 + \ldots + u_n) - N(u_0 + u_1 + \ldots + u_{n-1})\| \tag{2.7}$$

$\le k\|u_n\| \le \ldots \le k^n \|u_0\| \quad n = 0,1,2,\ldots,$

and the series $\sum_{i=0}^{\infty} u_i$ absolutely and uniformly converges to a solution of (2.1) [13], which is unique, in view of the Banach fixed point theorem [14]. The n-term approximate solution of (2.1) and (2.2) is given by $\sum_{i=0}^{n-1} u_i$ .

## 2.1 Reliable algorithm

After the above presentation of the NIM, we introduce a reliable algorithm for solving nonlinear partial differential equations using the NIM. Consider the following nonlinear partial differential equation of arbitrary order:

$$D_t^n u(x,t) = A(u, \partial u) + B(x,t), \qquad n \in N, \tag{2.8a}$$

with the initial conditions

$$\frac{\partial^m}{\partial t^m} u(x,0) = h_m(x), \quad m = 0,1,2,\ldots,n-1, \tag{2.8b}$$

where A is a nonlinear function of u and $\partial$ u (partial derivatives of u with respect to x and t) and B is the source function. In view of the integral operators, the initial value problem (2.8a) and (2.8b) is equivalent to the following integral equation

$$u(x,t) = \sum_{m=0}^{n-1} h_m(x) \frac{t^m}{m!} + I_t^n B(x,t) + I_t^n A = f + N(u), \tag{2.9}$$

where

$$f = \sum_{m=0}^{n-1} h_m(x)\frac{t^m}{m!} + I_t^n B(x,t), \tag{2.10}$$

and

$$N(u) = I_t^n A, \tag{2.11}$$

where $I_t^n$ t is an integral operator of n fold. We get the solution of (2.9) by employing the algorithm (2.5).

## 2.2 Convergence analysis of the NIM

Now, we introduce the condition of convergence of the NIM, which is proposed by Daftardar-Gejji and Jafari in (2006) [1], also called (DJM) [15]. From (2.3), the nonlinear operator $N$ is decomposed as follows [15]:

$N(u) = N(u_0) + [N(u_0+u_1) - N(u_0)] + [N(u_0+u_1+u_2) - N(u_0+u_1)] + \dots$ .

Let $G_0 = N(u_0)$ and

$$G_n = N\left(\sum_{i=0}^{n} u_i\right) - N\left(\sum_{i=0}^{n-1} u_i\right), \quad n = 1,2,3,\dots \ . \tag{2.12}$$

Then $N(u) = \sum_{i=0}^{\infty} G_i$ .

Set

$$u_0 = f, \tag{2.13}$$

$$u_n = G_{n-1}, \quad n = 1,2,3,\dots \ . \tag{2.14}$$

Then

$$u = \sum_{i=0}^{\infty} u_i \tag{2.15}$$

is a solution of the general functional equation (2.1). Also, the recurrence relation (2.5) becomes

$$\begin{aligned} u_0 &= f, \\ u_n &= G_{n-1}, \quad n = 1,2,\dots . \end{aligned} \tag{2.16}$$

Using Taylor series expansion for $G_i, i = 1,2,\dots,n$, we have

$$G_1 = N(u_0 + u_1) - N(u_0)$$

$$= N(u_0) + N'(u_0)u_1 + N''(u_0)\frac{u_1{}^2}{2!} + \ldots - N(u_0) \tag{2.17}$$

$$= \sum_{k=1}^{\infty} N^k(u_0)\frac{u_1{}^k}{k!},$$

$$G_2 = N(u_0 + u_1 + u_2) - N(u_0 + u_1)$$

$$= N'(u_0 + u_1)u_2 + N''(u_0 + u_1)\frac{u_2{}^2}{2!} + \ldots$$

$$= \sum_{j=1}^{\infty}\left[\sum_{i=0}^{\infty} N^{(i+j)}(u_0)\frac{u_1{}^i}{i!}\right]\frac{u_2{}^j}{j!} \tag{2.18}$$

$$G_3 = \sum_{i_3=1}^{\infty}\sum_{i_2=0}^{\infty}\sum_{i_1=0}^{\infty} N^{(i_1+i_2+i_3)}(u_0)\frac{u_3{}^{i_3}}{i_3!}\frac{u_2{}^{i_2}}{i_2!}\frac{u_1{}^{i_1}}{i_1!}. \tag{2.19}$$

In general:

$$G_n = \sum_{i_n=1}^{\infty}\sum_{i_{n-1}=0}^{\infty}\cdots\sum_{i_1=0}^{\infty}\left[N^{(\sum_{k=1}^{n} i_k)}(u_0)\left(\prod_{j=1}^{n}\frac{u_j^{i_j}}{i_j!}\right)\right]. \tag{2.20}$$

In the following theorem we state and prove the condition of convergence of the method.

**Theorem 2.1** If N is $C^{(\infty)}$ in a neighborhood of $u_0$ and

$$\left\|N^{(n)}(u_0)\right\| = \sup\left\{N^{(n)}(u_0)(h_1,\ldots,h_n) : \|h_i\| \le 1,\ 1 \le i \le n\right\} \le L,$$

(2.21)

for any n and for some real $L > 0$ and $\|u_i\| \le M < \frac{1}{e}$, $i = 1,2,\ldots,$ then the series $\sum_{n=0}^{\infty} G_n$ is absolutely convergent, and moreover,

$$\|G_n\| \le LM^n e^{n-1}(e-1),\ n = 1,2,\ldots\ . \tag{2.22}$$

*Proof.* In view of (2.20)

$$\|G_n\| \le LM^n \sum_{i_n=1}^{\infty}\sum_{i_{n-1}=0}^{\infty}\cdots\sum_{i_1=0}^{\infty}\left[\left(\prod_{j=1}^{n}\frac{u_j^{i_j}}{i_j!}\right)\right] = LM^n e^{n-1}(e-1). \tag{2.23}$$

Thus, the series $\sum_{n=1}^{\infty}\|G_n\|$ is dominated by the convergent series $LM(e-1)\sum_{n=1}^{\infty}(Me)^{n-1}$ ,where $M<\frac{1}{e}$. Hence, $\sum_{n=0}^{\infty}G_n$ is absolutely convergent, due to the comparison test.

For more details, see [15].

## 3. Numerical Applications

**Example 1.**

Consider the transport equation [12, 16]

$$u_t(x,t)+au_x(x,t)=0, \qquad x\in R, \quad t>0, \tag{3.1a}$$

with the initial condition

$$u(x,0)=x^2, \quad x\in R. \tag{3.1b}$$

From (2.5a) and (2.10), we have

$u_0(x,t)=x^2$.

Therefore, from (2.9), the initial value problem (3.1) is equivalent to the following integral equation:

$u(x,t)=x^2-I_t(au_x)$.

Taking

$N(u)=-I_t(au_x)$.

Therefore, from (2.5), we can obtain easily the following first few components of the new iterative solution for the equation (3.1):

$u_0(x,t)=x^2$,

$u_1(x,t)=-2atx$,

$u_2(x,t)=a^2t^2$,

$u_n(x,t)=0, \quad n\geq 3$,

which in closed form gives exact solution

$$u(x,t)=\sum_{i=0}^{\infty}u_i(x,t)=x^2-2atx+a^2t^2.$$

**Example 2.**

Consider the nonlinear Cauchy problem [17]

$$u_t(x,t) + xu_x(x,t) = 0, \qquad x \in R, \quad t > 0, \tag{3.2a}$$

with the initial condition

$$u(x,0) = x^2, \quad x \in R. \tag{3.2b}$$

From (2.5a) and (2.10), we have

$u_0(x,t) = x^2.$

Therefore, from (2.9), the initial value problem (3.2) is equivalent to the following integral equation:

$u(x,t) = x^2 - I_t(xu_x).$

Taking

$N(u) = -I_t(xu_x).$

Therefore, from (2.5), we can obtain easily the following first few components of the new iterative solution for the equation (3.2):

$u_0(x,t) = x^2,$

$u_1(x,t) = -2x^2 t,$

$u_2(x,t) = 2x^2 t^2,$

$u_3(x,t) = -\frac{4}{3}x^2 t^3,$

$\vdots.$

which in closed form gives exact solution

$$u(x,t) = \sum_{i=0}^{\infty} u_i(x,t) = x^2\left(1 - 2t + \frac{(2t)^2}{2!} - \frac{(2t)^3}{3!} + \frac{(2t)^4}{4!} - \frac{(2t)^5}{5!} + \ldots\right) = x^2 e^{-2t}.$$

**Example 3**

Consider the following non-homogeneous Cauchy problem [17]

$$u_t(x,t) + u_x(x,t) = x, \qquad x \in R, \quad t > 0, \tag{3.3a}$$

with the initial condition

$$u(x,0) = e^x, \quad x \in R. \tag{3.3b}$$

From (2.5a) and (2.10), we have

$u_0(x,t) = e^x + xt.$

Therefore, from (2.9), the initial value problem (3.3) is equivalent to the following integral equation:

$u(x,t) = e^x + xt - I_t(u_x).$

Taking

$N(u) = -I_t(u_x).$

Therefore, from (2.5), we can obtain easily the following first few components of the new iterative solution for the equation (3.3):

$u_0(x,t) = e^x + xt,$

$u_1(x,t) = -te^x - \frac{t^2}{2},$

$u_2(x,t) = e^x \frac{t^2}{2},$

$u_3(x,t) = -e^x \frac{t^3}{6},$

$\vdots$

which in closed form gives exact solution

$$u(x,t) = \sum_{i=0}^{\infty} u_i(x,t) = t\left(x - \frac{t}{2}\right) + e^x\left(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \frac{t^4}{4!} - \frac{t^5}{5!} + \ldots\right) = t\left(x - \frac{t}{2}\right) + e^{x-t}.$$

**Example 4**

Consider the inviscid Burgers' equation [16, 17]

$$u_t(x,t) + u(x,t)u_x(x,t) = 0, \qquad x \in R, \quad t > 0, \tag{3.4a}$$

with the initial condition

$$u(x,0) = x, \quad x \in R. \tag{3.4b}$$

From (2.5a) and (2.10), we have

$u_0(x,t) = x.$

Therefore, from (2.9), the initial value problem (3.4) is equivalent to the following integral equation:

$u(x,t) = x - I_t(uu_x).$

Taking

$N(u) = -I_t(uu_x)$.

Therefore, from (2.5), we can obtain easily the following first few components of the new iterative solution for the equation (3.4):

$u_0(x,t) = x$,

$u_1(x,t) = -xt$,

$u_2(x,t) = xt^2 - \frac{xt^3}{3}$,

$u_3(x,t) = -\frac{2xt^3}{3} + \frac{2xt^4}{3} - \frac{2xt^5}{3} + \frac{xt^6}{9} - \frac{xt^7}{63}$,

$\vdots$

which in closed form gives exact solution

$$u(x,t) = \sum_{i=0}^{\infty} u_i(x,t) = \left(x - xt + xt^2 - xt^3 + xt^4 - xt^5 + \ldots + (-1)^n xt^n + \ldots\right) = \frac{x}{1-t} .$$

## 4. Conclusion

In this paper, the NIM was successfully applied to solve the Cauchy problems with initial conditions. The fact that the NIM solves nonlinear problems without using Adomian's polynomials or He's polynomials is a clear advantage of this technique. The results show that the NIM is powerful and efficient technique in finding exact and approximate solutions for nonlinear differential equations.

**Conflict of Interests**

The authors declare that there is no conflict of interests.

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