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ASYMPTOTIC EXPANSION FORMULAS FOR EIGENVALUES AND EIGENFUNCTIONS OF THE DISCONTINUOUS BOUNDARY VALUE PROBLEM WITH TRANSMISSION CONDITION

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Abstract: In this paper we investigate the boundary value problem according to a special Hilbert space and define an operator which has the same eigenvalue with the problem defined in this Hilbert space. Therefore the boundary value problem is expressed in the form of operator equation. After than some basic properties of the eigenvalues and eigenfunctions of differential equations are given and asymptotic expansion formulas of eigenvalues and eigenfunctions are obtained.

Keywords: Discontinuous boundary value problem, eigenvalue and eigenfunction, asymptotic expansion, transmission condition.

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1. Introduction

It is well known that Sturm-Liouville Theory is an important aid in solving many problems in mathematical physics. The literature is voluminous and we refer to [2,5,13,21]. In particular [5,9,13,15,17] and [19] contains many references to problems in physics and mechanics. The theory of discontinuous Sturm-Liouville type problems mainly has been developed by Muhtarov and his students [1,4,7,10,11,18]. Particularly, there has been an increasing interest in the spectral analysis of boundary value problems with eigenvalue-dependent boundary

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conditions [1,2,4,5,6,7,10,11,13,14,18,20].

In this paper we investigate the Sturm-Liouville equation

(1) $$\tau u := -u'' + q(x)u = \lambda u ,$$

to hold in finite interval (-1,1) except at one inner point $x = 0$, subject to the eigenparameter-dependent boundary conditions

(2) $$L_1(u) := u(-1) + u'(-1) = 0 ,$$

(3) $$L_2(u) := \lambda\alpha u(1) + (\lambda - \beta)u'(1) = 0 ,$$

and transmission conditions at the inner point $x = 0$

(4) $$L_3(u) := u(0+) - u(0-) = 0 ,$$

(5) $$L_4(u) := \gamma_1 u'(0-) - \gamma_2 u'(0+) = (\lambda\delta_1 + \delta_2)u(0) ,$$

where λ is a complex eigenvalue parameter; the function $q(x)$ is real-valued and continuous in each of the intervals [-1,0) and (0,1] and has finite limits $q(\pm) := \lim_{x\to\pm 0} q(x); \alpha, \beta, \gamma_i, \delta_i$ (i=1,2) are real numbers. Throughout this study we assume that $\alpha\beta > 0$.

It must be noted that some special cases of this problem arises after an application of the method of separation of variables to the varied assortment of physical problems, heat and mass transfer problems (see, for example, [9]), vibrating string problems when the string loaded additionally with point masses (see, for example, [15]), diffraction problems (see, for example, [19]).

2. Operator-Theoretic Formulation of the Problem in the Adequate Hilbert Space

Throughout this paper we shall assume that the coefficients $\gamma_1, \gamma_2, \delta_1$ and δ_2 are positive.

For a suitable operator-theoretic formulation of the considered problem (1)-(5), we introduce a new equivalent inner product on H=$\mathbf{L_2}[-1,1]\oplus\mathbf{C}\oplus\mathbf{C}$ by

$$(6)\qquad <F,G>=\gamma_1\int_{-1}^{0} f(x)\overline{g(x)}dx+\gamma_2\int_{0}^{1} f(x)\overline{g(x)}dx+\frac{1}{\delta_1}f_1\overline{g_1}+\frac{\gamma_2}{\alpha\beta}f_2\overline{g_2}$$

for $F=\begin{pmatrix} f(x)\\ f_1\\ f_2\end{pmatrix}$ ve $G=\begin{pmatrix} g(x)\\ g_1\\ g_2\end{pmatrix}\in H$.

In this Hilbert space we construct the lineer operator $A:H\rightarrow H$ with domain

$$D(A)=\left\{\begin{pmatrix} f(x)\\ f_1\\ f_2\end{pmatrix}\left|\begin{array}{c} f \text{ and } f' \text{ absolutely continuous in } [-1,0) \text{ and } (0,1];\\ \text{and has finite limits } f(0\pm),\ f'(0\pm);\ \tau f\in[-1,1];\\ f(-1)+f'(-1)=0,\ f(0+)=f(0-)=f(0),\\ f_1=\delta_1 f(0),\ f_2=\alpha f(1)+f'(1)\end{array}\right.\right\}$$

which acts by the rule

$$AF=\begin{pmatrix} -f''+q(x)f\\ \gamma_1 f'(0-)+\gamma_2 f'(0+)-\delta_2 f(0)\\ \beta f'(1)\end{pmatrix}\ \text{with}\ F=\begin{pmatrix} f(x)\\ \delta_1 f(0)\\ \alpha f(1)+f'(1)\end{pmatrix}\in D(A).$$

So we can pose the considered problem (1)-(5) in the operator-equation form as

$$(7)\qquad AU=\lambda U,\ U=\begin{pmatrix} u(x)\\ \delta_1 u(0)\\ \alpha u(1)+u'(1)\end{pmatrix}\in D(A)$$

Naturally, by eigenvalues and eigenfunctions of the problem (1)-(5) we mean eigenvalues and first components of corresponding eigenelements of the operator A, respectively.

Teorem 2.1. *The lineer operator A is symmetric.*

Proof. Let $F,G\in D(A)$. Twice integrating by parts we find

$$(8)\qquad <AF,G>-<F,AG>=\gamma_1 W(f,\overline{g};0-)-\gamma_1 W(f,\overline{g};-1)+\gamma_2 W(f,\overline{g};1)$$

$$-\gamma_2 W(f,\bar{g};0+)+\gamma_2\left(f'(1)\overline{g(1)}-f(1)\overline{g'(1)}\right)$$

$$+\gamma_1\left(f'(0-)\overline{g(0)}-f(0)\overline{g'(0-)}\right)$$

$$+\gamma_2\left(f(0)\overline{g'(0+)}-f'(0+)\overline{g(0)}\right)$$

where, as usual. $W(f,g;x)$ denotes the Wronskian of f and g;

$$W(f,g;x):=f(x)g'(x)-f'(x)g(x).$$

Since $F,G\in D(A)$

$$f(-1)+f'(-1)=0\,,\quad g(-1)+g'(-1)=0$$

$$f(0-)=f(0+)=f(0)\ \ ,\ \ g(0-)=g(0+)=g(0)\,.$$

Substituing into (8) we have

$$<AF,G>=<F,AG>\qquad (F,G\in D(A))$$

so A is symmetric.

Corollary 2.2. *All eigenvalue of (1)-(5) are real.*

We can now assume that all eigenfunctions of (1)-(5) are real-valued.

Corollary 2.3. *Let λ ve μ be two different eigenvalues of (1)-(5). Then the corresponding $\phi(x, \lambda)$ and $\psi(x, \mu)$ of this problem are orthogonal in the sense of*

$$\gamma_1\int_{-1}^{0}\phi(x,\lambda)\psi(x,\mu)dx+\gamma_2\int_{0}^{1}\phi(x,\lambda)\psi(x,\mu)d\mu=-\left[\frac{\gamma_2\alpha}{\beta}\phi(1,\lambda)\psi(1,\mu)+\frac{\gamma_2}{\beta}\left(\phi(1,\lambda)\psi'(1,\mu)+\phi'(1,\lambda)\psi(1,\mu)\right)\right.$$
$$\left.+\frac{\gamma_2}{\alpha\beta}\phi'(1,\lambda)\psi'(1,\mu)+\delta_1\phi(0,\lambda)\psi(0,\mu)\right]$$

3. Construction and Asymptotic Approximations of Fundamental Solutions

Since the function q(x) is continuous on [-1,0) and has a finite left-hand-side limit q(-0), the Cauchy problem

$$\tau u := -u'' + q(x)u = \lambda u\,,\quad x\in[-1,0)$$

(9) $\qquad u(-1)=1\,,\ u'(-1)=-1$

has a unique solution $\phi_{1\lambda}(x) := \phi_1(x,\lambda)$ which is an entire function of λ for each $x\in[-1,0)$. (See ,for example, [16, Theorem 1.5].)

Now we can define the solution $\phi_{2\lambda}(x) := \phi_2(x,\lambda)$ of equation (1) on (0,1] by the initial conditions

(10) $\qquad u(0)=\phi_1(0,\lambda)\quad ,\qquad u'(0)=\dfrac{\gamma_1}{\gamma_2}\phi_1{}'(0,\lambda)-\dfrac{(\lambda\delta_1+\delta_2)}{\gamma_2}\,\phi_1(0,\lambda)\,.$

Consequently, the function $\phi_\lambda(x) := \phi(x,\lambda)$ defined by

$$\phi_\lambda(x)=\begin{cases}\phi_{1\lambda}(x),\ x\in[-1,0)\\ \phi_{2\lambda}(x),\ x\in(0,1]\end{cases}$$

is a solution of equation (1) on [-1,0) $\cup$ (0,1], which satisfies the boundary condition (2) and both transmission conditions (4) and (5).

To construct the another $\chi_\lambda(x) := \chi(x,\lambda)$ fundamental solution

$$\chi_\lambda(x) = \begin{cases} \chi_{1\lambda}(x), \, x \in [-1,0) \\ \chi_{2\lambda}(x), \, x \in (0,1] \end{cases}$$

of the problem (1)-(5), we first define the solution $\chi_{2\lambda}(x) := \chi_2(x,\lambda)$ of equation (1) on (0,1] by initial conditions

(11) $\quad u(1) = \beta - \lambda \quad , \quad u'(1) = \lambda\alpha$

and then the solution $\chi_{1\lambda}(x) := \chi_1(x,\lambda)$ of equation (1) on [-1,0) by initial conditions

(12) $\quad u(0) = \chi_2(0,\lambda) \quad , \quad u'(0) = \dfrac{\gamma_2}{\gamma_1}\chi_2'(0,\lambda) + \dfrac{\lambda\delta_1 + \delta_2}{\gamma_1}\chi_2(0,\lambda)$

Consequently, the function $\chi_\lambda(x) := \chi(x,\lambda)$ is a solution of equation on [-1,0) $\cup$ (0,1], which satisfies the other boundary condition (3) and both transmission conditions (4) and (5).

Let us consider the Wronskians

$$\omega_j(\lambda) := W(\phi_{j\lambda}, \chi_{j\lambda}; x) = \phi_{j\lambda}(x)\chi'_{j\lambda}(x) - \phi'_{j\lambda}(x)\chi_{j\lambda}(x)$$

which are independent of x and entire functions. The short calculation gives

$$\gamma_1\omega_1(\lambda) = \gamma_2\omega_2(\lambda).$$

Now we may introduce to the consideration the characteristic function $\omega(\lambda)$ as

$$\omega(\lambda) := \gamma_1\omega_1(\lambda) = \gamma_2\omega_2(\lambda)$$

Theorem 3.1. *The eigenvalues of the problem (1)-(5) are consist of the zeros of the functions* $\omega(\lambda)$ *and* $\Delta(\lambda):=\omega(\lambda)+(\lambda\delta_1+\delta_2)\phi(0,\lambda)\chi(0,\lambda)$.

Proof. Let λ_0 be an eigenvalue and $u_0(x)$ be any corresponding eigenfunction. Show that $\omega(\lambda_0)=0$ or $\Delta(\lambda_0)=0$. Let us assume the contrary, that $\omega(\lambda_0)\neq 0$ and

$\Delta(\lambda_0)\neq 0$. Then this implies that $W(\phi_j(x,\lambda_0),\chi_j(x,\lambda_0))\neq 0$ (j=1,2). Consequently, each pair of functions $\phi_j(x,\lambda_0)$ and $\chi_j(x,\lambda_0)$ and linearly independent. Therefore eigenfunction $u_0(x)$ can be written in the form

$$u_0(x)=\begin{cases} c_1\phi_1(x,\lambda_0)+c_2\chi_1(x,\lambda_0)\ , & x\in[-1,0) \\ c_3\phi_2(x,\lambda_0)+c_4\chi_2(x,\lambda_0)\ , & x\in(0,1] \end{cases}$$

where at least one of the constants c_j (1,2,3,4) is not zero. By substituting this reprensentation in the conditions (2)-(5) we obtain a system of linear, homogenous equations for the determination of the constants c_j (1,2,3,4). By routine calculation we see that the determinant of this system is equal to $\omega(\lambda)\Delta(\lambda)$, which is not zero by assumption. Hence this system of equations has only trivial solution c_j=0 (1,2,3,4)

and so we a contradiction. Thus it is shown that each eigenvalue is zero of the functions $\omega(\lambda)$ or $\Delta(\lambda)$. Now we must show that if $\omega(\lambda_0)\Delta(\lambda_0)=0$, then λ_0 is an eigenvalue. Consider the possible cases $\omega(\lambda_0)=0$ and $\omega(\lambda_0)\neq 0$, $\Delta(\lambda_0)=0$ separately. If $\omega(\lambda_0)=0$ then $\omega_1(\lambda_0)=W(\phi_1(x,\lambda_0),\chi_1(x,\lambda_0))=0$, and consequently the functions $\phi_1(x,\lambda_0),\chi_1(x,\lambda_0)$ are linearly dependent solutions of equation (1) in the [-1,0), $\phi_1(x,\lambda_0)=k_1\chi_1(x,\lambda_0)$ for some $k_1\neq 0$. In this case, since $\phi_1(x,\lambda_0)$ satisfies the boundary condition (2), $\chi_1(x,\lambda_0)$ also satisfies boundary condition (2). Thus, the solution $\chi_j(x,\lambda_0)$ (j=1,2) satisfies all boundary and transmission conditions (2)-(5) and would be any eigenfunction for eigenvalue λ_0.

Finally if $\omega(\lambda_0)\neq 0$ and $\Delta(\lambda_0)=0$ then, since $\omega_i(\lambda_0)=W(\phi_i(x,\lambda_0),\chi_i(x,\lambda_0))\neq 0$ each of pair functions $\phi_i(x,\lambda_0)$ and $\chi_i(x,\lambda_0)$ (j=1,2) are linearly independent. As a result , the general solution of equation (1) may be expressed as

$$u(x,\lambda_0)=\begin{cases} c_1\phi_1(x,\lambda_0)+c_2\chi_1(x,\lambda_0)\ , & x\in[-1,0) \\ c_3\phi_2(x,\lambda_0)+c_4\chi_2(x,\lambda_0)\ , & x\in(0,1] \end{cases} \tag{13}$$

where c_1,c_2,c_3,c_4 are arbitrary constants. Again, substituting (13) in the conditions (2)-(5) we obtain a system linear, homogenous equations for determination of the constants c_1,c_2,c_3,c_4, determinant of which is equal to $\omega(\lambda_0)\Delta(\lambda_0)$, which is equal to zero by assumption. Hence there is nontrivial solution $(c_1,c_2,c_3,c_4)\neq(0,0,0,0)$ of these system of equations. So the corresponding solution $u(x,\lambda_0)$ would be an eigenfunction for the eigenvalue λ_0.

Lemma 3.2. *Let* $\lambda=s^2, s=\sigma+it$. *Then the solution* $\phi_{j\lambda}(x)=\phi_j(x,\lambda)$ $(j=1,2)$ *satisfies the following integral equations for* k = 0 *and* k = 1:

$$\phi_{1\lambda}^{(k)}(x)=\left(\cos[s(x+1)]\right)^{(k)}-\frac{1}{s}\left(\sin[s(x+1)]\right)^{(k)}+\frac{1}{s}\int_{-1}^{x}\left(\sin[s(x-y)]\right)^{(k)}q(y)\phi_{1\lambda}(y)dy \tag{14}$$

$$\phi_{2\lambda}^{(k)}(x)=\phi_{1\lambda}(0)\left(\cos(sx)\right)^{(k)}+\frac{1}{\gamma_2 s}\left(\gamma_1\phi_{1\lambda}'(0)-\left(s^2\delta_1+\delta_2\right)\phi_{1\lambda}(0)\right)\left(\sin(sx)\right)^{(k)} \tag{15}$$

$$+\frac{1}{s}\int_{0}^{x}\left(\sin[s(x-y)]\right)^{(k)}q(y)\phi_{2\lambda}(y)dy$$

Proof. For proving it is enough substitute $\phi''_{1\lambda}(y)+s^2\phi_{1\lambda}(y)$ and $\phi''_{2\lambda}(y)+s^2\phi_{2\lambda}(y)$ instead of $q(y)\phi_{1\lambda}(y)$ and $q(y)\phi_{2\lambda}(y)$ in the integral ters of the (14) and (15) respectively and integrate by parts twice.

Lemma 3.3. *Let* $\lambda = s^2, s = \sigma + it$.*Then the solution* $\chi_{j\lambda}(x) = \chi_j(x,\lambda)\ (j=1,2)$ *satisfies the following integral equations for* k = 0 *and* k = 1:

(16) $$\chi_{1\lambda}^{(k)}(x) = \chi_{2\lambda}(0)(\cos(sx))^{(k)} + \frac{1}{\gamma_1 s}\left[\gamma_2\chi'_{2\lambda}(0) + (s^2\delta_1+\delta_2)\chi_{2\lambda}(0)\right](\sin(sx))^{(k)}$$

$$+\frac{1}{s}\int_0^x(\sin[s(x-y)])^{(k)}q(y)\chi_{1\lambda}(y)dy$$

(17) $$\chi_{2\lambda}^{(k)}(x) = s\alpha(\sin[s(x-1)])^{(k)} + (\cos[s(x-1)])^{(k)}(\beta - s^2)$$

$$-\frac{1}{s}\int_x^1(\sin[s(x-y)])^{(k)}q(y)\chi_{2\lambda}(y)dy$$

Proof. For proving it is enough substitute $\chi''_{1\lambda}(y)+s^2\chi_{1\lambda}(y)$ and $\chi''_{2\lambda}(y)+s^2\chi_{2\lambda}(y)$ instead of $q(y)\chi_{1\lambda}(y)$ and $q(y)\chi_{2\lambda}(y)$ in the integral ters of the (16) and (17) respectively and integrate by parts twice.

Lemma 3.4. *Let* $\lambda = s^2, s = \sigma + it$. *The following asymptotic formulas are satisfied as* $|\lambda| \to \infty$ (k=0,1):

(18) $$\phi_{1\lambda}^{(k)}(x) = O\left(|s|^k e^{|t|(x+1)}\right)$$

(19) $$\phi_{1\lambda}^{(k)}(x) = (\cos[s(x+1)])^{(k)} + O\left(|s|^{k-1}e^{|t|(x+1)}\right)$$

(20) $$\phi_{2\lambda}^{(k)}(x) = O\left(|s|^{k+1}e^{|t|(x+1)}\right)$$

(21) $$\phi_{2\lambda}^{(k)}(x) = -s\frac{\delta_1}{\gamma_2}\cos s(\sin(sx))^{(k)} + O\left(|s|^{-k} e^{|t|(x+1)}\right)$$

Proof. The asymptotic formulas for $\phi_{1\lambda}(x)$ can be found easily by applying the Titchmarsh's lemma [16, lemma 1.7]. But the proof of the formulas (20) and (21) need special consideration since the function $\phi_{2\lambda}(x)$ is defined by the special type initial conditions of the form (10).

Substituting (18) in (15) (for k=0) we get

$$\phi_{2\lambda}(x) = \cos s\cos(sx) - \frac{\gamma_1}{\gamma_2}\sin s\sin(sx) - \left(\frac{(s^2\delta_1+\delta_2)}{\gamma_2 s}\cos s\sin(sx)\right) + \frac{1}{s}\int_0^x \sin[s(x-y)]q(y)\phi_{2\lambda}(y)dy$$

$$+\cos(sx)O\left(\frac{1}{|s|}e^{|t|}\right) + \frac{\gamma_1}{\gamma_2 s}\sin(sx)O\left(e^{|t|}\right) - \frac{\left(s^2\delta_1+\delta_2\right)}{\gamma_2 s}\sin(sx)O\left(\frac{1}{|s|}e^{|t|}\right).$$

Taking into account that $\sin(sx) = O\left(e^{|t|x}\right)$, $\cos(sx) = O\left(e^{|t|x}\right)$ and denoting $F_{2\lambda}(x) = s^{-1}e^{-|t|(x+1)}\phi_{2\lambda}(x)$ we have

$$F_{2\lambda}(x) = \frac{1}{s}e^{-|t|(x+1)}\left\{\cos s\cos(sx) - \frac{\gamma_1}{\gamma_2}\sin s\sin(sx) - \left(\frac{(s^2\delta_1+\delta_2)}{\gamma_2 s}\cos s\sin(sx)\right)\right\}$$

$$+\left|\frac{1}{s}\right|\int_0^x \sin[s(x-y)]q(y)e^{-|t|(x-y)}F_{2\lambda}(y)dy + O\left(\frac{1}{|s|}\right).$$

Now, denoting $F_2(\lambda) = \max_{0\le x\le 1}|F_{2\lambda}(x)|$, from the last equation we can derive that

$$F_2(\lambda) \le \frac{1}{|s|} + \left|\frac{\gamma_1}{\gamma_2}\right|\frac{1}{|s|} + \frac{\delta_1}{|\gamma_2|} + \frac{\delta_2}{|\gamma_2|}\frac{1}{|s|^2} + \frac{1}{|s|^2}\int_0^1 |q(y)|F_2(\lambda)dy + \frac{M}{|s|}$$

for some M > 0. Consequently $F_2(\lambda) = O(1)$ as $|\lambda| \to \infty$, so

$$\phi_{2\lambda}(x) = O\left(|s|\, e^{|t|(x+1)}\right) \text{ as } |\lambda| \to \infty.$$

The case k=1 of the (20) follows by applying the same procedure as in the case k=0. The proof of (21) is similar to that of (20) and hence omitted.

Lemma 3.5. *Let* $\lambda = s^2$, $\operatorname{Im} s = t$. *The following asymptotic formulas are satisfied as*

$|\lambda| \to \infty$ (k=0,1):

(22) $$\chi_{1\lambda}^{(k)}(x) = O\left(|s|^{k+3} e^{|t|(1-x)}\right)$$

(23) $$\chi_{1\lambda}^{(k)}(x) = -s^3 \frac{\delta_1}{\gamma_1} \cos(s)(\sin(sx))^{(k)} + O\left(|s|^{k+2} e^{|t|(1-x)}\right)$$

(24) $$\chi_{2\lambda}^{(k)}(x) = O\left(|s|^{k+2} e^{|t|(1-x)}\right)$$

(25) $$\chi_{2\lambda}^{(k)}(x) = -s^2 (\cos s(x-1))^{(k)} + O\left(|s|^{k+1} e^{|t|(1-x)}\right)$$

Theorem 3.6. $\lambda = s^2$, $\mathrm{Im} s = t$. *Then the functions* $\omega(\lambda)$ *and* $\Delta(\lambda)$ *have the following asymtotic reprensentations:*

(26) $$\omega(\lambda) = -s^4 \delta_1 \cos^2 s + O\left(|s|^3 e^{2|t|}\right), \ |\lambda| \to \infty$$

(27) $$\Delta(\lambda) = -2s^4 \delta_1 \cos^2 s + O\left(|s|^3 e^{2|t|}\right), \ |\lambda| \to \infty$$

4. Asymptotic Formulas for Eigenvalues

Theorem 4.1. *Let* $\lambda = s^2$**,** $s = \sigma + it$. *Then the following asymptotic formulas hold for the eigenvalues of the boundary-value-transmission problem (1)-(5).*

(28) $$S_n^{'} = \pi\left(n + \frac{1}{2}\right) + O\left(\frac{1}{\sqrt{n}}\right)$$

(29) $$S_n^{''} = \pi\left(n - \frac{1}{2}\right) + O\left(\frac{1}{\sqrt{n}}\right)$$

Proof. Denoting by $\omega_1(\lambda)$ and $\omega_2(\lambda)$ the first and O-term of the right of (26) respectively. It is readily showed that $|\omega_1(\lambda)| > |\omega_2(\lambda)|$ on the contours

$$C_n^{'} = \left\{ s^{'} \in \mathbf{C} \ \middle| \ |s^{'}| = (\pi(n+1)) \right\}$$

for sufficiently large n.

Let $\lambda_0^{'} \le \lambda_1^{'} \le \lambda_2^{'} \le \ldots$ are zeros of $w(\lambda)$, and $\lambda_n^{'} = S_n^{'2}$.

Then, by applying well-known Rouche's theorem which assert that if f(s) and g(s) are analytic inside and on a closed contour C, and $|g(s)| < |f(s)|$ on C, then f(s) and $f(s)+g(s)$ have the same number zeros inside C provided that each zeros is counted according to their multiplicity, we have

$$S_n' = \pi\left(n+\frac{1}{2}\right)+\delta_n', \quad |\delta_n'| \le \frac{\pi}{2} \quad \text{for sufficiently large n.}$$

By putting this in (26) we get

$$-|S_n'|^4 \delta_1 \cos^2\left(\pi\left(n+\frac{1}{2}\right)+\delta_n'\right)+O\left(|S_n'|^3\right)=0$$

and consequently $\cos^2\left(\pi\left(n+\frac{1}{2}\right)+\delta_n'\right)=O\left(\frac{1}{|S_n'|}\right)$.

Since $\cos^2\left(\pi\left(n+\frac{1}{2}\right)+\delta_n'\right)=\sin^2\delta_n'$ and $O\left(\frac{1}{|S_n'|}\right)=O\left(\frac{1}{n}\right)$, we have $\sin^2\delta_n' = O\left(\frac{1}{n}\right)$.

From this, since $|\delta_n'| \le \frac{\pi}{2}$, follows that $\delta_n' = O\left(\frac{1}{\sqrt{n}}\right)$, which completes the proof for the first formula. The proof of the other formula is similar.

5. Asymptotic Formulas for Eigenfunctions

Theorem 5.1. *Let* S_n' *and* S_n'' *be eigenvalues of the problem (1)-(5). Then the following asymptotic formulas hold for the corresponding eigenfunctions* $\phi_n^1(x)$ *and* $\phi_n^2(x)$ *of this problem.*

(30)
$$\phi_n^1(x)=\begin{cases}(-1)^{n+1}\sin\left(\pi\left(n+\frac{1}{2}\right)x\right)+O\left(\frac{1}{\sqrt{n}}\right) & ,x\in[-1,0)\\ O\left(\frac{n}{\sqrt{n}}\right) & ,x\in(0,1]\end{cases}$$

(31) $$\phi_n^2(x)=\begin{cases}(-1)^n\sin\left(\pi\left(n-\frac{1}{2}\right)x\right)+O\left(\frac{1}{\sqrt{n}}\right) & , \quad x\in[-1,0)\\ O\left(\frac{n}{\sqrt{n}}\right) & , x\in(0,1]\end{cases}$$

Proof. By putting $\lambda=\left(S_n'\right)^2$ in (19) and (21) gives

(32) $$\phi_{1n}^1(x):=\phi_1\left(x,\left(S_n'\right)^2\right)=\cos\left[S_n'(x+1)\right]+O\left(\frac{1}{\left|S_n'\right|}\right)$$

and

(33) $$\phi_{2n}^1(x):=\phi_2\left(x,\left(S_n'\right)^2\right)=-S_n'\frac{\delta_1}{\gamma_2}\cos S_n'\sin\left(S_n'x\right)+O(1)$$

respectively.

From (28) we obtain the equalities

(34) $$\frac{1}{S_n'}=O\left(\frac{1}{n}\right)$$

and

$$\sin\left(S_n'x\right)=\sin\left(\pi\left(n+\frac{1}{2}\right)x+O\left(\frac{1}{\sqrt{n}}\right)\right)$$

$$=\sin\left(\pi\left(n+\frac{1}{2}\right)x\right)\cos O\left(\frac{1}{\sqrt{n}}\right)+\sin O\left(\frac{1}{\sqrt{n}}\right)\cos\left(\pi\left(n+\frac{1}{2}\right)x\right).$$

Since

$$\cos O\left(\frac{1}{\sqrt{n}}\right)=1+O\left(\frac{1}{n}\right) \quad , \quad \sin O\left(\frac{1}{\sqrt{n}}\right)=O\left(\frac{1}{\sqrt{n}}\right)$$

we get

(35) $$\sin\left(S_n'x\right)=\sin\left(\pi\left(n+\frac{1}{2}\right)x\right)+O\left(\frac{1}{\sqrt{n}}\right)$$

Putting x=1 in (35) yields

(36) $$\sin\left(S_n'\right)=\sin\left(\pi\left(n+\frac{1}{2}\right)+O\left(\frac{1}{\sqrt{n}}\right)\right)$$

$$=(-1)^n+O\left(\frac{1}{\sqrt{n}}\right)$$

Similary, we obtain

(37) $$\cos S_n' = O\left(\frac{1}{\sqrt{n}}\right).$$

In addition,

(38) $$\cos\left[S_n'(x+1)\right] = \cos\left(S_n' x + S_n'\right) = \cos(S_n' x)\cos S_n' - \sin(S_n' x)\sin S_n' \quad .$$

Substituing (28), (35), (36) ve (37) asymtotic equalities in (38)yields

(39) $$\cos\left[S_n'(x+1)\right] = (-1)^{n+1}\sin\left(\pi\left(n+\frac{1}{2}\right)x\right) + O\left(\frac{1}{\sqrt{n}}\right)$$

Substituing (34) ve (39) formulas in (32) yields

(40) $$\phi_{1n}{}^{1}(x) = (-1)^{n+1}\sin\left(\pi\left(n+\frac{1}{2}\right)x\right) + O\left(\frac{1}{\sqrt{n}}\right)$$

Substituing (28), (35) ve (37) equalities in (33) yields

(41) $$\phi_{2n}{}^{1}(x) = O\left(\frac{n}{\sqrt{n}}\right)$$

Consequently , we have

$$\phi_n^1(x) = \begin{cases} (-1)^{n+1}\sin\left(\pi\left(n+\frac{1}{2}\right)x\right) + O\left(\frac{1}{\sqrt{n}}\right) & , x \in [-1,0) \\ O\left(\frac{n}{\sqrt{n}}\right) & , x \in (0,1] \end{cases}.$$

The proof of (31) formula is similar.

Conflict of Interests

The author declares that there is no conflict of interests.

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