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SOME PROPERTIES OF FUZZY PRESEPARATION AXIOMS

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Abstract. We introduce r-fuzzy preopen (closed) sets in Šostak's fuzzy topological spaces. We investigate some their properties. Moreover, we investigate the relationship between fuzzy irresolutity and fuzzy (supra) continuity. We define preseparation axioms and investigate some their properties.

Keywords: fuzzy (supra) topological spaces; fuzzy interior operators; r-fuzzy preopen (closed) sets; r-PR_i spaces; fuzzy (supra)continuity.

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1. Introduction and preliminaries

Šostak [8] introduced the fuzzy topology as an extension of Chang's fuzzy topology [2]. Mashhour et al.[5,6] defined preopen and preclosed sets in topological spaces and investigated properties of them. Singal and Prakash [7] extended the notion of preopen sets to fuzzy sets and defined the separation axioms and properties of them.

In this paper, we introduce the notion of r-fuzzy preopen(closed) sets in Šostak's fuzzy topological space. In particular, we can obtain the fuzzy supra topology induced by the family of

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r-fuzzy preopen sets of a fuzzy topological space. We show the relationship among fuzzy continuity, fuzzy supra continuity and fuzzy irresolutity. Moreover, we define the separation axioms of fuzzy topological spaces using r-fuzzy preopen (closed) sets.

Throughout this paper, let X be a nonempty set, $I=[0,1]$ and $I_0=(0,1]$. For $\alpha\in I$, $\overline{\alpha}(x)=\alpha$ for all $x\in X$. The family of all fuzzy sets on X denoted by I^X. For $x\in X$ and $t\in I$, a fuzzy point x_t is defined by

$$x_t(y)=\begin{cases} t & \text{if } y=x,\\ 0 & \text{if } y\neq x.\end{cases}$$

Let $Pt(X)$ be the family of all fuzzy points in X. For $\lambda,\mu\in I^X$, λ is called *quasi-coincident* with μ, denoted by $\lambda\ q\ \mu$, if there exists $x\in X$ such that $\lambda(x)+\mu(x)>1$. Otherwise we denote $\lambda\overline{q}\mu$.

Definition 1.1. [8] A function $\tau: I^X\to I$ is called a *fuzzy supra topology* on X if it satisfies the following conditions:

(S1) $\tau(\overline{0})=\tau(\overline{1})=1$,

(S2) $\tau(\bigvee_{i\in\Gamma}\mu_i)\geq\bigwedge_{i\in\Gamma}\tau(\mu_i)$, for any $\{\mu\}_{i\in\Gamma}\subset I^X$.

The pair (X,τ) is called a *fuzzy supra topological space*.

A fuzzy supra topology on X is called *a fuzzy topology* on X if

(O) $\tau(\mu_1\wedge\mu_2)\geq\tau(\mu_1)\wedge\tau(\mu_2)$, for any $\mu_1,\mu_2\in I^X$.

The pair (X,τ) is called a *fuzzy topological space* (for short, fts).

Definition 1.2. ([3,4]) A function $int: I^X\times I\to I^X$ is called a *fuzzy interior operator* if it satisfies the following conditions: for $\lambda,\mu\in I^X$ and $r,s\in I$,

(I1) $int(\overline{1})=\overline{1}$, $int(\lambda,0)=\lambda$,

(I2) $int(\lambda,r)\leq\lambda$,

(I3) $int(\lambda,r)\wedge int(\mu,r)=int(\lambda\wedge\mu,r)$,

(I4) $int(\lambda,r)\leq int(\lambda,s)$, if $r\geq s$,

(I5) $int(int(\lambda,r),r)=int(\lambda,r)$.

Theorem 1.3 ([3,4]) Let $int: I^X\times I\to I^X$ be a fuzzy interior operator. Define a function $\tau_{int}: I^X\to I$ on X by

$$\tau_{int}(\lambda)=\bigvee\{r\in I\mid int(\lambda,r)=\lambda\}.$$

Then τ_{int} is a fuzzy topology on X.

Theorem 1.4 ([3,4]) Let (X,τ) be a fts. Define functions $int_\tau, cl_\tau : I^X \times I \to I^X$ as follows:

$$int_\tau(\lambda,r) = \bigvee\{\mu \in I^X \mid \mu \leq \lambda, \tau(\mu) \geq r\},$$

$$cl_\tau(\lambda,r) = \bigwedge\{\mu \in I^X \mid \lambda \leq \mu, \tau(\overline{1}-\mu) \geq r\}.$$

The following results hold:

(1) int_τ is a fuzzy interior operator.

(2) $\tau_{int_\tau} = \tau$.

(3) $int_\tau(\overline{1}-\lambda,r) = \overline{1} - cl_\tau(\lambda,r)$ for each $r \in I, \lambda \in I^X$.

2. Some properties of r-fuzzy preopen sets

Definition 2.1. Let (X,τ) be a fts. For $\lambda \in I^X$ and $r \in I$,

(1) λ is called *r-fuzzy preopen* (for short, r-fpo) if for $0 \leq s \leq r$,

$$\lambda \leq int_\tau(cl_\tau(\lambda,s),s),$$

(2) λ is called *r-fuzzy preclosed* (for short, r-fpc) if for $0 \leq s \leq r$,

$$\lambda \geq cl_\tau(int_\tau(\lambda,s),s).$$

Theorem 2.2. Let (X,τ) be a fts. Let $\lambda \in I^X$ and $r,s \in I$.

(1) λ is r-fpo iff $\overline{1}-\lambda$ is r-fpc.

(2) Any union of r-fpo sets is r-fpo.

(3) Any intersection of r-fpc sets is r-fpc.

(4) If $\tau(\lambda) \geq r$, then λ is r-fpo.

(5) $int_\tau(\lambda,r)$ is r-fpo and $cl_\tau(\lambda,r)$ is r-fpc.

(6) If λ is r-fpo and $0 \leq s \leq r$, then λ is s-fpo.

Proof. (1) By Theorem 1.4 (3), it easily proved from

$$\lambda \leq int_\tau(cl_\tau(\lambda,s),s) \text{ iff } \overline{1}-\lambda \geq cl_\tau(int_\tau(\overline{1}-\lambda,s),s).$$

(2) Let λ_i be r-fpo for $i \in \Gamma$. Then $\bigvee_{i\in\Gamma} \lambda_i$ is r-fpo from , for $0 \leq s \leq r$,

$$\bigvee_{i\in\Gamma} \lambda_i \leq \bigvee_{i\in\Gamma} int_\tau(cl_\tau(\lambda_i, s), s) \leq cl_\tau(int_\tau(\bigvee_{i\in\Gamma} \lambda_i, s), s).$$

Other cases are easily proved from Definition 2.1.

Theorem 2.3. Let (X,τ) be a fts. Define a function $Pint_\tau : I^X \times I \to I^X$ as follows:

$$Pint_\tau(\lambda, r) = \bigvee\{\mu \in I^X \mid \mu \leq \lambda,\ \mu \text{ is r-fpo }\}.$$

For $\lambda, \mu \in I^X$ and $r \in I$, it holds the following properties.

(1) $Pint_\tau(\overline{1}, r) = \overline{1}$ and $Pint_\tau(\lambda, 0) = \lambda$.

(2) $Pint_\tau(\lambda, r) \leq \lambda$.

(3) $Pint_\tau(\lambda \wedge \mu, r) \leq Pint_\tau(\lambda, r) \wedge Pint_\tau(\mu, r)$.

(4) If $r \leq s$ for $r, s \in I$, then $Pint_\tau(\lambda, r) \geq Pint_\tau(\lambda, s)$.

(5) $Pint_\tau(Pint_\tau(\lambda, r), r) = Pint_\tau(\lambda, r)$.

(6) $Pint_\tau(\lambda, r) = \lambda$ iff λ is r-fpo.

Proof. (1), (2) and (6) are easily proved from the definition of $Pint_\tau$.

(3) Since $\lambda \wedge \mu \leq \lambda, \mu$, we have

$$Pint_\tau(\lambda \wedge \mu, r) \leq Pint_\tau(\lambda, r) \wedge Pint_\tau(\mu, r).$$

(4) By Theorem 2.2(6), it is trivial.

(5) From (2), we only show $Pint_\tau(\lambda, r) \leq Pint_\tau(Pint_\tau(\lambda, r), r)$. Suppose

$$Pint_\tau(\lambda, r) \not\leq Pint_\tau(Pint_\tau(\lambda, r), r).$$

There exist $x \in X$ and $t \in (0,1)$ such that

$$Pint_\tau(\lambda, r)(x) > t > Pcl_\tau(Pcl_\tau(\lambda, r), r)(x). \qquad (A)$$

Since $Pint_\tau(\lambda, r)(x) > t$, by the definition of $Pint_\tau$, there exists r-fpo set λ_1 with $\lambda_1 \leq \lambda$ such that

$$Pint_\tau(\lambda, r)(x) \geq \lambda_1(x) > t.$$

Again, since $\lambda_1 \leq \lambda$ and $Pcl_\tau(\lambda, r) \geq \lambda_1$, by the definition of $Pint_\tau$, $Pint_\tau(Pint_\tau(\lambda, r), r) \geq \lambda_1$. Hence

$$Pint_\tau(Pint_\tau(\lambda, r), r)(x) \geq \lambda_1(x) > t.$$

It is a contradiction for (A). Thus, $Pint_\tau(\lambda, r) = Pint_\tau(Pint_\tau(\lambda, r), r)$.

Theorem 2.4. Let (X, τ) be a fts. Define a function $Pcl_\tau : I^X \times I \to I^X$ as follows:

$$Pcl_\tau(\lambda, r) = \bigwedge\{\mu \in I^X \mid \lambda \leq \mu,\ \ \mu \text{ is r-fpc }\}.$$

Then:

(1) $Pint_\tau(\overline{1} - \lambda, r) = \overline{1} - Pcl_\tau(\lambda, r)$.

(2) $int_\tau(\lambda, r) \leq Pint_\tau(\lambda, r) \leq \lambda \leq Pcl_\tau(\lambda, r) \leq cl_\tau(\lambda, r)$.

Proof. (1) For each $\lambda \in I^X$ and $r \in I$, we have

$$\begin{aligned} Pint_\tau(\overline{1} - \lambda, r) &= \bigvee\{\mu \in I^X \mid \mu \leq \overline{1} - \lambda,\ \ \mu \text{ is r-fpo }\} \\ &= \overline{1} - \bigwedge\{\overline{1} - \mu \in I^X \mid \overline{1} - \mu \geq \lambda,\ \ \overline{1} - \mu \text{ is r-fpc }\} \\ &= \overline{1} - Pcl_\tau(\lambda, r). \end{aligned}$$

(2) It is easy from Theorem 2.2 (5).

Theorem 2.5. Let (X, τ) be a fts and $Pint_\tau$ be a function provided with the properties (1)-(4) in Theorem 2.3. Define the function $\tau_P : I^X \to I$ on X by

$$\begin{aligned} \tau_P(\lambda) &= \bigvee\{r \in I \mid Pint_\tau(\lambda, r) = \lambda\} \\ &= \bigvee\{r \in I \mid \quad \lambda \text{ is r-fpo}\} \\ &= \bigvee\{r \in I \mid Pcl_\tau(\overline{1} - \lambda, r) = \overline{1} - \lambda\}. \end{aligned}$$

Then τ_P is a fuzzy supra topology on X with $\tau_P(\lambda) \geq \tau(\lambda)$ for all $\lambda \in I^X$.

Proof. The map τ_P is well defined from

$$Pint_\tau(\lambda, r) = \lambda \text{ iff } \lambda \text{ is r-fpo iff } Pcl_\tau(\overline{1} - \lambda, r) = \overline{1} - \lambda.$$

(S1) Since $Pint_\tau(\overline{0}, r) \leq \overline{0}$ from Theorem 2.3(2), then $Pint_\tau(\overline{0}, r) = \overline{0}$. Furthermore, $Pint_\tau(\overline{1}, r) = \overline{1}$, for all $r \in I$. Thus $\tau_P(\overline{0}) = \tau_P(\overline{1}) = 1$.

(S2) Suppose that there exists a family $\{\lambda_i \in I^X \mid i \in \Gamma\}$ such that

$$\tau_P(\bigvee_{i\in\Gamma} \lambda_i) < \bigwedge_{i\in\Gamma} \tau_P(\lambda_i).$$

Then there exists $r_0 \in I$ such that

$$\tau_P(\bigvee_{i\in\Gamma} \lambda_i) < r_0 < \bigwedge_{i\in\Gamma} \tau_P(\lambda_i). \qquad (B)$$

Since $\tau_P(\lambda_i) > r_0$, for all $i \in \Gamma$, there exist $r_i \in I$ with $Pint_\tau(\lambda_i, r_i) = \lambda_i$ such that

$$r_0 < r_i \leq \tau_P(\lambda_i).$$

On the other hand, since $Pint_\tau(\lambda_i, r_0) \geq Pint_\tau(\lambda_i, r_i) = \lambda_i$, by Theorem 2.3 (2), we have

$$Pint_\tau(\lambda_i, r_0) = \lambda_i$$

It implies for all $i \in \Gamma$,

$$Pint_\tau(\bigvee_{i\in\Gamma} \lambda_i, r_0) \geq Pint_\tau(\lambda_i, r_0) = \lambda_i.$$

It follows

$$Pint_\tau(\bigvee_{i\in\Gamma} \lambda_i, r_0) \geq \bigvee_{i\in\Gamma} \lambda_i.$$

Thus, $Pint_\tau(\bigvee_{i\in\Gamma} \lambda_i, r_0) = \bigvee_{i\in\Gamma} \lambda_i$, that is, $\tau_P(\lambda) \geq r_0$. It is a contradiction for (B). Thus, $\tau_P(\bigvee_{i\in\Gamma} \lambda_i) \geq \bigwedge_{i\in\Gamma} \tau_P(\lambda_i)$. Hence τ_P is a fuzzy supra topology on X. From Theorem 2.4(2) and Theorem 1.4(2), $int_\tau(\lambda, r) = \lambda$ implies $Pint_\tau(\lambda, r) = \lambda$. Thus, $\tau_P(\lambda) \geq \tau(\lambda)$ for all $\lambda \in I^X$.

Example 2.6. Let $X = \{a, b\}$ be a set. We define a fuzzy topology $\tau : I^X \to I$ as follows:

$$\tau(\lambda) = \begin{cases} 1, & \text{if } \lambda = \overline{0} \text{ or } \overline{1}, \\ \frac{1}{2}, & \text{if } \lambda = \overline{0.4}, \\ 0, & \text{otherwise.} \end{cases}$$

(a) If $\overline{0} \neq \lambda \leq \overline{0.4}$, then $int_\tau(cl_\tau(\lambda, r), r) = \overline{0.4}$ for $0 < r \leq \frac{1}{2}$. Thus, λ is $\frac{1}{2}$-fpo. Similarly, if $\lambda \not\leq \overline{0.6}$, then $int_\tau(cl_\tau(\lambda, r), r) = \overline{1}$ for $r \in I_0$. Thus, λ is r-fpo, for all $r \in I$.

(1) Let $\lambda = a_{0.7} \vee b_{0.5}$ and $\mu = a_{0.5} \vee b_{0.7}$. Then $\lambda \wedge \mu = \overline{0.5}$. By (a), λ and μ are $\frac{1}{2}$-fpo. But $\lambda \wedge \mu$ is not $\frac{1}{2}$-fpo because

$$\overline{0.5} > \Big(int_\tau(cl_\tau(\overline{0.5}, \frac{1}{2}), \frac{1}{2}) = \overline{0.4}\Big).$$

(2) From (1), $Pint(\lambda, r) = \lambda$ and $Pint(\mu, r) = \mu$ for $0 \le r \le \frac{1}{2}$. But $Pint(\lambda \wedge \mu, r) = \overline{0.4}$ for $0 < r \le \frac{1}{2}$. Thus,

$$\left(\overline{0.4} = Pint_\tau(\lambda \wedge \mu, \frac{1}{2})\right) \neq \left(Pint_\tau(\lambda, \frac{1}{2}) \wedge Pint_\tau(\mu, \frac{1}{2}) = \overline{0.5}\right).$$

Therefore $Pint_\tau$ is not a fuzzy interior operator which it does not satisfy the condition (I3) of Definition 1.2.

(3) We can obtain a fuzzy supra topology $\tau_P : I^X \to I$ as follows:

$$\tau_P(\lambda) = \begin{cases} 1, & \text{if } \lambda = \overline{0} \text{ or } \overline{1}, \\ \frac{1}{2}, & \text{if } \overline{0} \neq \lambda \le \overline{0.4}, \\ 1, & \text{if } \lambda \not\le \overline{0.6}, \\ 0, & \text{otherwise.} \end{cases}$$

By (1) and (2), for $\lambda = a_{0.7} \vee b_{0.5}$ and $\mu = a_{0.5} \vee b_{0.7}$,

$$0 = \tau_P(\lambda \wedge \mu) \not\ge \left(\tau_P(\lambda) \wedge \tau_P(\mu) = 1\right).$$

Hence τ_P is not a fuzzy topology.

Example 2.7. Let $X = \{a, b\}$ be a set. We define fuzzy topologies $\eta, \gamma : I^X \to I$ as follows:

$$\eta(\lambda) = \begin{cases} 1, & \text{if } \lambda = \overline{0} \text{ or } \overline{1}, \\ 0, & \text{otherwise,} \end{cases} \qquad \gamma(\lambda) = 1, \forall \lambda \in I^X.$$

(1) Since $int_\eta(cl_\eta(\lambda, r), r) = \overline{1}$ for $\overline{0} \neq \lambda \in I^X$ and $r \in I_0$, then every fuzzy set $\lambda \in I^X$ is r-fpo and r-fpc for $r \in I$.

(2) Since $int_\gamma(cl_\gamma(\lambda, r), r) = \lambda$ for $\lambda \in I^X$ and $r \in I$, then every fuzzy set $\lambda \in I^X$ is r-fpo and r-fpc for $r \in I$.

By (1) and (2), we obtain fuzzy supra-topologies

$$\left(\eta_P = \gamma_P\right)(\lambda) = 1, \forall \lambda \in I^X.$$

Definition 2.8. Let (X, τ) and (Y, η) be fuzzy (resp. supra) topological spaces. A function $f : (X, \tau) \to (Y, \eta)$ is called *fuzzy (resp. supra) continuous* iff $\tau(f^{-1}(\mu)) \ge \eta(\mu)$ for each $\mu \in I^Y$.

Example 2.9. We define fuzzy topologies $\tau,\eta : I^X \to I$ as follows:

$$\tau(\lambda)=\begin{cases} 1, & \text{if } \lambda \in \{\overline{0},\overline{1}\},\\ \frac{1}{2}, & \text{if } \lambda \in \{\overline{0.2},\overline{0.3}\},\\ 0, & \text{otherwise,}\end{cases} \qquad \eta(\lambda)=\begin{cases} 1, & \text{if } \lambda \in \{\overline{0},\overline{1}\},\\ \frac{1}{2}, & \text{if } \lambda \in \{\overline{0.3}\},\\ 0, & \text{otherwise.}\end{cases}$$

We can obtain fuzzy topologies $\tau_P,\eta_P : I^X \to I$ as follows:

$$\tau_P(\lambda)=\begin{cases} 1, & \text{if } \lambda \in \{\overline{0},\overline{1}\},\\ \frac{1}{2}, & \text{if } \overline{0}\neq\lambda \leq \overline{0.3},\\ 1, & \text{if } \lambda \not\leq \overline{0.8},\\ 0, & \text{otherwise,}\end{cases} \qquad \eta_P(\lambda)=\begin{cases} 1, & \text{if } \lambda \in \{\overline{0},\overline{1}\},\\ \frac{1}{2}, & \text{if } \overline{0}\neq\lambda \leq \overline{0.3},\\ 1, & \text{if} \lambda \not\leq \overline{0.7},\\ 0, & \text{otherwise,}\end{cases}$$

(1) The identity function $id_X : (X,\tau) \to (X,\eta)$ is fuzzy continuous. But $id_X : (X,\tau_P) \to (X,\eta_P)$ is not fuzzy supra continuous because

$$0=\tau_P(\overline{0.75}) \not\geq \eta_P(\overline{0.75})=1.$$

(2) The identity function $id_X : (X,\eta_P) \to (X,\tau_P)$ is fuzzy supra continuous. But $id_X : (X,\eta) \to (X,\tau)$ is not fuzzy continuous.

Definition 2.10. Let (X,τ) and (Y,η) be fts's. Let $f : (X,\tau) \to (Y,\eta)$ be a function.

(1) f is called *fuzzy irresolute map* iff $f^{-1}(\mu)$ is r-fpo for each r-fpo $\mu \in I^Y$ and $r \in I$.

(2) f is called *fuzzy irresolute open* iff $f(\lambda)$ is r-fpo for each r-fpo $\lambda \in I^Y$ and $r \in I$.

(3) f is called *fuzzy irresolute closed* iff $f(\lambda)$ is r-fpc for each r-fpc $\lambda \in I^Y$ and $r \in I$.

Theorem 2.11. Let (X,τ) and (Y,η) be fts's satisfying the condition:

(T) $\tau_P(\overline{1}-\lambda) \geq r$ implies $Pcl_\tau(\lambda,r)=\lambda$.

Then the following statements are equivalent.

(1) $f : (X,\tau_P) \to (Y,\eta_P)$ is fuzzy supra continuous.

(2) $f(Pcl_\tau(\lambda,r)) \leq Pcl_\eta(f(\lambda),r)$, for each $\lambda \in I^X$ and $r \in I$.

(3) $Pcl_\tau(f^{-1}(\mu),r) \leq f^{-1}(Pcl_\eta(\mu,r))$, for each $\mu \in I^Y$ and $r \in I$.

(4) $Pint_\tau(f^{-1}(\mu),r) \geq f^{-1}(Pint_\eta(\mu,r))$, for each $\mu \in I^Y$ and $r \in I$.

Proof. (1) $\Rightarrow$ (2). Suppose there exist $\lambda \in I^X$ and $r \in I_0$ such that

$$f(Pcl_\tau(\lambda,r)) \not\leq Pcl_\eta(f(\lambda),r).$$

Then there exist $y \in Y$ and $t \in I_0$ such that

$$f(Pcl_\tau(\lambda,r))(y) > t > Pcl_\eta(f(\lambda),r)(y).$$

If $f^{-1}(\{y\}) = \emptyset$, it is a contradiction since $f(Pcl_\tau(\lambda,r))(y) = 0$.
If $f^{-1}(\{y\}) \neq \emptyset$, there exists $x \in f^{-1}(\{y\})$ such that

$$f(Pcl_\tau(\lambda,r))(y) \geq Pcl_\tau(\lambda,r)(x) > t > Pcl_\eta(f(\lambda),r)(f(x)). \qquad (C)$$

Since $Pcl_\eta(f(\lambda),r)(f(x)) < t$, by the definition of Pcl_η, there exists r-fpc $\mu \in I^Y$ with $f(\lambda) \leq \mu$ such that

$$Pcl_\eta(f(\lambda),r)(f(x)) \leq \mu(f(x)) < t. \qquad (D)$$

Since $\lambda \leq f^{-1}(\mu)$, $Pcl_\tau(f^{-1}(\mu),r) \geq Pcl_\tau(\lambda,r)$. By (C) and (D),

$$Pcl_\tau(f^{-1}(\mu),r)(x) \geq Pcl_\tau(\lambda,r)(x) > t > \mu(f(x)) = f^{-1}(\mu)(x).$$

By **(T)**, $Pcl_\tau(f^{-1}(\mu),r) \neq f^{-1}(\mu)$ implies $\tau_P(\overline{1} - f^{-1}(\mu)) < r$. Moreover, $\eta_P(\overline{1}-\mu) \geq r$ because $Pcl_\eta(\mu,r) = \mu$. So, $\eta_P(\overline{1}-\mu) \geq r > \tau_P(f^{-1}(\overline{1}-\mu))$. Hence $f : (X,\tau_P) \to (Y,\eta_P)$ is not fuzzy supra continuous.

(2) $\Rightarrow$ (3). By (2), put $\lambda = f^{-1}(\mu)$. Since $f(f^{-1}(\mu)) \leq \mu$, then

$$Pcl_\tau(f^{-1}(\mu),r) \leq f^{-1}(f(Pcl_\tau(f^{-1}(\mu),r))) \leq f^{-1}(Pcl_\eta(\mu,r)).$$

(3) $\Rightarrow$ (4). It is easy from Theorem 1.4(3).

(4) $\Rightarrow$ (1). If $Pint_\eta(\mu,r) = \mu$, then $Pint_\tau(f^{-1}(\mu),r) = f^{-1}(\mu)$. It implies $\tau_P(f^{-1}(\mu)) \geq \eta_P(\mu)$ for all $\mu \in I^Y$.

Theorem 2.12. Let (X,τ) and (Y,η) be fts's. If $f : (X,\tau) \to (Y,\eta)$ is fuzzy irresolute, then $f : (X,\tau_P) \to (Y,\eta_P)$ is fuzzy supra continuous.

Proof. Suppose there exists $\rho \in I^Y$ such that

$$\tau_P(f^{-1}(\rho)) < \eta_P(\rho).$$

Then there exists $r \in I_0$ with $Pint_\eta(\rho,r) = \rho$ such that

$$\tau_P(f^{-1}(\rho)) < r \leq \eta_P(\rho). \qquad (E)$$

Since f is fuzzy irresolute and $Pint_\eta(\rho,r)=\rho$ is r-fpo, then $f^{-1}(\rho)$ is r-fpo. Thus $Pint_\tau(f^{-1}(\rho),r)=f^{-1}(\rho)$. So, $\tau_P(f^{-1}(\rho))\geq r$. It is a contradiction for (E). Hence f is fuzzy supra continuous.

3. Some properties of fuzzy preseparation axioms

Definition 3.1. A fts (X,τ) is said to be:

(1) r-FP_s if $\tau(\mu)\geq r$ for each r-fpo $\mu\in I^X$.

(2) r-PR_0 iff for each $x_t\overline{q}y_s$, there exists r-fpo $\mu_1\in I^X$ such that $x_t\in\mu_1$ and $y_s\overline{q}\mu_1$, or there exists r-fpo $\mu_2\in I^X$ such that $y_s\in\mu_2$ and $x_t\overline{q}\mu_2$.

(3) r-PR_1 iff for each $x_t\overline{q}y_s$, there exist r-fpo sets $\mu_1,\mu_2\in I^X$ such that $x_t\in\mu_1, y_s\overline{q}\mu_1, y_s\in\mu_2$ and $x_t\overline{q}\mu_2$.

(4) r-PR_2 iff for each $x_t\overline{q}y_s$, there exist r-fpo sets $\mu_1,\mu_2\in I^X$ such that $x_t\in\mu_1, y_s\in\mu_2$ and $\mu_1\overline{q}\mu_2$.

(5) r-$PR_{2\frac{1}{2}}$ iff for each $x_t\overline{q}y_s$, there exist r-fpo sets $\mu_1,\mu_2\in I^X$ such that $x_t\in\mu_1, y_s\in\mu_2$ and $Pcl_\tau(\mu_1,r)\overline{q}Pcl_\tau(\mu_2,r)$.

(6) r-PR_3 iff $x_t\overline{q}\lambda$ for each r-fpc λ implies there exist there exist r-fpo sets $\mu_1,\mu_2\in I^X$ such that $x_t\in\mu_1,\lambda\leq\mu_2$ and $\mu_1\overline{q}\mu_2$.

(7) r-PR_4 iff $\lambda_1\overline{q}\lambda_2$ for each r-fpc sets λ_i and $i\in\{1,2\}$ implies there exist r-fpo sets $\mu_i\in I^X$ such that $\lambda_i\leq\mu_i$ and $\mu_1\overline{q}\mu_2$.

Theorem 3.2. Let (X,τ) be a fts. Then the following statements are equivalent:

(1) (X,τ) is r-FP_s for all $r\in I$.

(2) $Pint_\tau(\lambda,r)=int_\tau(\lambda,r)$ for each $\lambda\in I^X$ and $r\in I$.

(3) $\tau=\tau_P$.

Proof. (1)$\Rightarrow$ (2). Suppose there exist $\lambda\in I^X$ and $r\in I$ such that

$$Pint_\tau(\lambda,r)\not\leq int_\tau(\lambda,r).$$

Then there exist $x\in X$ and $t\in(0,1)$ such that

$$Pint_\tau(\lambda,r)(x)>t>int_\tau(\lambda,r)(x).$$

By the definition of $Pint_\tau(\lambda, r)$, there exists a r-fpo set $\rho \in I^X$ with $\rho \leq \lambda$ such that

$$Pint_\tau(\lambda, r)(x) \geq \rho(x) > t > int_\tau(\lambda, r)(x). \qquad (F)$$

By (1), $\tau(\rho) \geq r$ with $\rho \leq \lambda$. Then

$$int_\tau(\lambda, r)(x) \geq \rho(x) > t.$$

It is a contradiction for (F). Hence $Pint_\tau(\lambda, r) \leq int_\tau(\lambda, r)$. Furthermore, by Theorem 2.4(2),

$$Pint_\tau(\lambda, r) = int_\tau(\lambda, r).$$

(2)$\Rightarrow$ (3). Since $Pint_\tau(\lambda, r) = int_\tau(\lambda, r)$, by Theorems 1.3, 1.4 and 2.5, $\tau_P = \tau_{int_\tau} = \tau$.

(3)$\Rightarrow$ (1). Suppose (X, τ) is not r-FP_s. Then there exists a r-fpo set $\rho \in I^X$ with $\tau(\rho) < r$. Thus $\tau_P(\rho) \geq r > \tau(\rho)$.

Theorem 3.3. A fts (X, τ) is r-PR_0 iff for each $x_t\overline{q}y_s$, we have either $x_t\overline{q}Pcl_\tau(y_s, r)$ or $y_s\overline{q}Pcl_\tau(x_t, r)$.

Proof. Let (X, τ) be r-PR_0. For each $x_t\overline{q}y_s$, if there exists r-fpo $\mu_1 \in I^X$ such that $x_t \in \mu_1$ and $y_s\overline{q}\mu_1$, then $y_s \in \overline{1} - \mu_1 \leq \overline{1} - x_t$. By the definition of Pcl_τ, $Pcl_\tau(y_s, r) \leq \overline{1} - \mu_1 \leq \overline{1} - x_t$. Thus, $x_t\overline{q}Pcl_\tau(y_s, r)$. Other case, similarly, $y_s\overline{q}Pcl_\tau(x_t, r)$.

Conversely, for each $x_t\overline{q}y_s$, $x_t\overline{q}Pcl_\tau(y_s, r)$ implies

$$x_t \in Pint_\tau(\overline{1} - y_s, r) = \Big(\overline{1} - Pcl_\tau(y_s, r)\Big), \ \ y_s\overline{q}\Big(\overline{1} - Pcl_\tau(y_s, r)\Big),$$

or $y_s \in Pint_\tau(\overline{1} - x_t, r)$ and $x_t\overline{q}Pint_\tau(\overline{1} - x_t, r)$. Hence (X, τ) is r-PR_0.

Theorem 3.4. A fts (X, τ) is r-PR_1 iff each $x_t \in Pt(X)$ is r-fpc.

Proof. Let (X, τ) be r-PR_1. For each $y_s \in \overline{1} - x_t$, that is, $x_t\overline{q}y_s$, there exists r-fpo $\mu_{y_s} \in I^X$ such that

$$y_s \in \mu_{y_s} \leq \overline{1} - x_t.$$

Hence $\overline{1} - x_t = \bigvee_{y_s \in \overline{1} - x_t} \mu_{y_s}$. Thus, x_t is a r-fpc set.

Conversely, it is easily proved.

Example 3.5. Let $X=\{a,b\}$ be a set. We define a fuzzy topology $\tau : I^X \to I$ as follows:

$$\tau(\lambda)=\begin{cases} 1, & \text{if } \lambda\in\{\overline{0},\overline{1}\},\\ \frac{1}{2}, & \text{if } \lambda\in\{a_1,a_t \mid 0\le t\le 0.5\},\\ \frac{1}{2}, & \text{if } \lambda\in\{b_1,b_s \mid 0\le s\le 0.5\},\\ \frac{1}{2}, & \text{if } \lambda\in\{a_1\vee b_s, a_t\vee b_1, a_t\vee b_s \mid 0<t,s\le 0.5\},\\ 0, & \text{otherwise.}\end{cases}$$

If a_t for $0.5<t<1$ and $0<r\le\frac{1}{2}$, then $a_t > int_\tau(cl_\tau(a_t,r),r)=a_{0.5}$. Hence a_t is not $\frac{1}{2}$-fpo. Similarly, if b_s for $0.5<s<1$ and $0<r\le\frac{1}{2}$, are not $\frac{1}{2}$-fpo.

(1)For each $a_t\overline{q}b_s$, there exist $\frac{1}{2}$-fpo $a_1,b_1\in I^X$ such that

$$a_t\in a_1, b_s\overline{q}a_1, b_s\in b_1, a_t\overline{q}b_1.$$

For each $a_t\overline{q}a_s$, either $t\le 0.5$ or $s\le 0.5$. Put $t\le 0.5$. there exists $\frac{1}{2}$-fpo $a_t\in I^X$ such that

$$a_t\in a_t, a_t\overline{q}a_s.$$

For each $b_t\overline{q}b_s$, it is similarly proved. Hence (X,τ) is $\frac{1}{2}$-PR_0.

(2) By Theorem 3.4, $a_{0.3}$ is not $\frac{1}{2}$-fpc because, $0<r\le\frac{1}{2}$,

$$a_{0.3}\not\ge\Big(cl_\tau(int_\tau(a_{0.3},r),r)=a_{0.5}\Big).$$

Thus, (X,τ) is not $\frac{1}{2}$-PR_1.

(3) Since $a_{0.7}\ge\Big(cl_\tau(int_\tau(a_{0.7},r),r)=a_{0.5}\Big)$, then $a_{0.3}\overline{q}a_{0.7}$. For all $\frac{1}{2}$-fpo set λ and μ with $a_{0.3}\in\lambda$ and $a_{0.7}\in\mu$, we have $\lambda\ q\mu$. Thus, (X,τ) is not $\frac{1}{2}$-PR_2.

Theorem 3.6. Let (X,τ) be a fts. Then the following statements are equivalent:

(1) (X,τ) is r-PR_3.

(2) If $x_t\in\lambda$ for each r-fpo $\lambda\in I^X$, there exists r-fpo $\mu\in I^X$ such that $x_t\in\mu\le Pcl_\tau(\mu,r)\le\lambda$.

(3) If $x_t\overline{q}\lambda$ for each r-fpc $\lambda\in I^X$, there exist r-fpo sets $\mu_1,\mu_2\in I^X$ such that $x_t\in\mu_1,\lambda\le\mu_2$ and $Pcl_\tau(\mu_1,r)\overline{q}Pcl_\tau(\mu_2,r)$.

Proof. (1)$\Rightarrow$ (2). Let $x_t\in\lambda$ for each r-fpo λ. Then $x_t\overline{q}(\overline{1}-\lambda)$ for r-fpc $\overline{1}-\lambda$. Since (X,τ) is r-PR_3, there exist r-fpo sets $\mu,\rho\in I^X$ such that $x_t\in\mu,\overline{1}-\lambda\le\rho$ and $\mu\overline{q}\rho$. It implies $x_t\in\mu\le$

$\overline{1}-\rho \leq \lambda$. Since $\overline{1}-\rho$ is r-fpc,

$$x_t \in \mu \leq Pcl_\tau(\mu, r) \leq \lambda.$$

(2)$\Rightarrow$ (3). Let $x_t \overline{q} \lambda$ for each r-fpc λ. Then $x_t \in \overline{1}-\lambda$ for r-fpo $\overline{1}-\lambda$. By (2), there exists a r-fpo set $\mu \in I^X$ such that

$$x_t \in \mu \leq Pcl_\tau(\mu, r) \leq \overline{1}-\lambda.$$

Since μ is r-fpo and $x_t \in \mu$, by (2), there exists a r-fpo set $\mu_1 \in I^X$ such that

$$x_t \in \mu_1 \leq Pcl_\tau(\mu_1, r) \leq \mu \leq Pcl_\tau(\mu, r) \leq \overline{1}-\lambda.$$

It implies $\lambda \leq \Big(\overline{1}-Pcl_\tau(\mu, r) = Pint_\tau(\overline{1}-\mu, r)\Big) \leq \overline{1}-\mu$. Put $\mu_2 = Pint_\tau(\overline{1}-\mu, r)$. Then μ_2 is a r-fpo set from the definition of $Pint_\tau$. So, $Pcl_\tau(\mu_2, r) \leq \overline{1}-\mu \leq \overline{1}-Pcl_\tau(\mu_1, r)$, that is, $Pcl_\tau(\mu_1, r)\overline{q}Pcl_\tau(\mu_2, r)$.

(3)$\Rightarrow$ (1). It is trivial.

Theorem 3.7. Let (X, τ) be a fts. Then the following statements are equivalent:

(1) (X, τ) is r-PR_4.

(2) If $\lambda \leq \rho$ for each r-fpc $\lambda \in I^X$ and r-fpo $\rho \in I^X$, there exists a r-fpo set $\mu \in I^X$ such that

$$\lambda \leq \mu \leq Pcl_\tau(\mu, r) \leq \rho.$$

(3)If $\lambda_1 \overline{q} \lambda_2$ for each r-fpc sets λ_i with $i \in \{1, 2\}$, then there exist r-fpo sets $\mu_i \in I^X$ such that $\lambda_i \leq \mu_i$ and $Pcl_\tau(\mu_1, r)\overline{q}Pcl_\tau(\mu_2, r)$.

Proof. It is similarly proved as in Theorem 3.6.

Theorem 3.8. Let (X, τ) be a fts. Then the following implications hold:

$$\begin{aligned}(r-PR_4 \text{ and } r-PR_1) \quad &\Rightarrow (r-PR_3 \text{ and } r-PR_1)\\ \Rightarrow r-PR_{2\frac{1}{2}} \qquad &\Rightarrow r-PR_2 \Rightarrow r-PR_1 \Rightarrow r-PR_0\end{aligned}$$

Proof. We show that $(r-PR_3 \text{ and } r-PR_1) \Rightarrow r-PR_{2\frac{1}{2}}$.

For each $x_t \overline{q} y_s$, by Theorem 3.4 , y_s is r-fpc . Since (X, τ) is r-PR_3, by Theorem 3.6(3), there exist r-fpo sets $\mu_i \in I^X$ such that $x_t \in \mu_1, y_s \in \mu_2$ and $Pcl_\tau(\mu_1, r)\overline{q}Pcl_\tau(\mu_2, r)$. Hence (X, τ) is r-$PR_{2\frac{1}{2}}$.

Other cases are easily proved.

Example 3.9. Let R be a real number set. We define a fuzzy topology $\tau : I^R \to I$ as follows:

$$\tau(\lambda) = \begin{cases} 1, & \text{if } \lambda \in \{\overline{0}, \overline{1}\}, \\ \frac{1}{3}, & \text{if } \lambda = \chi_G, \\ 0, & \text{otherwise.} \end{cases}$$

where each $\emptyset \neq G^c$ is a finite set.

(A) If $supp(\lambda) = \{x \in R \mid \lambda(x) > 0\}$ is denumerable and $0 \le r \le \frac{1}{2}$, then $\lambda \le int_\tau(cl_\tau(\lambda, r), r) = \overline{1}$

$$\lambda \le int_\tau(cl_\tau(\lambda, r), r) = \overline{1}, \ \ \lambda \ge cl_\tau(int_\tau(\lambda, r), r) = \overline{0}.$$

Thus, λ is $\frac{1}{2}$-fpo and $\frac{1}{2}$-fpc.

(1) For $x_t \,\overline{q}\, y_s$ with $x \neq y$ and $0 \le r \le \frac{1}{2}$, there exist $\mu, \rho \in I^R$ with $x_t \in \mu$, $y_s \in \rho$ and $supp(\mu) \cap supp(\rho) = \emptyset$ which $supp(\mu)$ and $supp(\rho)$ are denumerable. So, $\mu \,\overline{q}\, \rho$.

(2) For $x_t \,\overline{q}\, x_s$ and $0 \le r \le \frac{1}{2}$, there exist $\mu, \rho \in I^R$ with $\mu(x) = t$, $\rho(x) = s$ and $supp(\mu) \cap supp(\rho) = \{x\}$ which $supp(\mu)$ and $supp(\rho)$ are denumerable. So, $\mu \,\overline{q}\, \rho$. By (A), μ and ρ are $\frac{1}{2}$-fpo. By (1) and(2), (R, τ) is $r - PR_2$ for $0 < r \le \frac{1}{2}$.

Furthermore, $Pcl(\mu, r) = \mu$ and $Pcl(\rho, r) = \rho$ from (A). Hence (R, τ) is $r - PR_{2\frac{1}{2}}$ for $0 < r \le \frac{1}{2}$.

Example 3.10. Let $X = \{a, b, c\}$ be a set. We define a fuzzy topology $\tau : I^X \to I$ as follows:

$$\tau(\lambda) = \begin{cases} 1, & \text{if } \lambda \in \{\overline{0}, \overline{1}\}, \\ \frac{1}{2}, & \text{if } \lambda = a_1, \\ 0, & \text{otherwise.} \end{cases}$$

(a) For $\lambda \not\le \chi_{\{b,c\}}$ and $0 \le r \le \frac{1}{2}$, then λ is $\frac{1}{2}$-fpo because

$$\lambda \le int_\tau(cl_\tau(\lambda, r), r) = \overline{1}.$$

(b) For $\lambda(a) = t$ with $0 < t < 1$ and $0 \le r \le \frac{1}{2}$, then λ is $\frac{1}{2}$-fpo from (a) and $\frac{1}{2}$-fpc because

$$\lambda \ge cl_\tau(int_\tau(\lambda, r), r) = \overline{0}.$$

(c) For $\lambda \le \chi_{\{a,b\}}$ and $0 \le r \le \frac{1}{2}$, then λ is $\frac{1}{2}$-fpc because

$$\lambda \ge cl_\tau(int_\tau(\lambda, r), r) = \overline{0}.$$

(1) For $0 \leq r \leq \frac{1}{2}$, we have the following cases:

(Case1) For $\lambda_1 \overline{q} \lambda_2$ with λ_1, λ_2 in (b), λ_1 and λ_2 are both $\frac{1}{2}$-fpo and $\frac{1}{2}$-fpc

(Case2) For $\lambda_1 \overline{q} \lambda_2$ with λ_1 in (b) and λ_2 in (c) and $\lambda_1(a) = t$, there exists $\frac{1}{2}$-fpo $\lambda_2 \vee a_{1-t}$ such that $\lambda_1 \overline{q} (\lambda_2 \vee a_{1-t})$.

(Case3) For $\lambda_1 \overline{q} \lambda_2$ with λ_1 and λ_2 in (c), there exist $\frac{1}{2}$-fpo $\lambda_i \vee a_{0.2}$ for $i \in \{1,2\}$ such that $(\lambda_1 \vee a_{0.2}) \overline{q} (\lambda_2 \vee a_{0.2})$.

Thus, (X, τ) is r-PR_3 from the above cases.

(2) Let $a_1 \overline{q} b_1$ and $0 \leq r \leq \frac{1}{2}$. For every r-fpo set λ with $b_1 \in \lambda$, $a_1 \; q \; \lambda$. Thus (X, τ) is not $\frac{1}{2}$-PR_1. Moreover, since b_1 is $\frac{1}{2}$-fpc by (c), (X, τ) is not r-PR_3.

Theorem 3.11. Let (X, τ) and (Y, η) be fts's. Let $f : (X, \tau) \to (Y, \eta)$ be an injective fuzzy irresolute map. If (Y, η) is r-PR_i for $i \in \{0,1,2\}$, then (X, τ) is r-PR_i for $i \in \{0,1,2\}$.

Proof. For each $x_t \overline{q} y_s$, since f is injective, then $f(x)_t \overline{q} f(y)_s$. Since (Y, η) is r-PR_2, there exist r-fpo sets $\mu_1, \mu_2 \in I^Y$ such that $f(x)_t \in \mu_1, f(y)_s \in \mu_2$ and $\mu_1 \overline{q} \mu_2$. Since f is a fuzzy irresolute map, there exist r-fpo sets $f^{-1}(\mu_1), f^{-1}(\mu_2) \in I^X$ such that $x_t \in f^{-1}(\mu_1), y_s \in f^{-1}(\mu_2)$ and $f^{-1}(\mu_1) \overline{q} f^{-1}(\mu_2)$. Hence (X, τ) is r-PR_2.

Other cases are similarly proved.

Theorem 3.12. Let (X, τ) and (Y, η) be fts's. Let $f : (X, \tau) \to (Y, \eta)$ be an injective fuzzy irresolute and fuzzy irresolute closed map. If (Y, η) is r-PR_i for $i \in \{3,4\}$, then (X, τ) is r-PR_i for $i \in \{3,4\}$.

Proof. For each $x_t \overline{q} \lambda$ with r-fpc set λ , since f is an injective fuzzy irresolute closed map, then $f(x)_t \overline{q} f(\lambda)$ with r-fpc set $f(\lambda)$. Since (Y, η) is r-PR_3, there exist r-fpo sets $\mu_1, \mu_2 \in I^Y$ such that $f(x)_t \in \mu_1, f(\lambda) \leq \mu_2$ and $\mu_1 \overline{q} \mu_2$. Since f is a fuzzy irresolute map, there exist r-fpo sets $f^{-1}(\mu_1), f^{-1}(\mu_2) \in I^X$ such that $x_t \in f^{-1}(\mu_1), \lambda \leq f^{-1}(\mu_2)$ and $f^{-1}(\mu_1) \overline{q} f^{-1}(\mu_2)$. Hence (Y, τ) is r-PR_3.

Other case is similarly proved.

Conflict of Interests

The author declares that there is no conflict of interests.

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