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ANALYTICITY OF THE GENERALIZED TWO-DIMENSIONAL FRACTIONAL COSINE TRANSFORM

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Abstract. Fourier transform can be generalized into the fractional Fourier transform (FRFT), Linear Canonical Transform (LCT) and simplified fractional Fourier transform (SFRFT). They extend the utilities of original Fourier transform, and can solve many problems that can't solved well by original Fourier transform.

In this paper, we study distributional generalized two-dimensional fractional Cosine transform. Testing function space and distributional two-dimensional fractional Cosine transform is defined. Analyticity of the generalized two-dimensional fractional Cosine transform is proved.

Keywords: Fractional Cosine Transform, Fractional Fourier Transform, Generalized Function, Optics, Signal Processing.

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1. Introduction

In recent years, the concept of fractional operator and measure have been investigated extensively in many engineering applications and science. Four typical examples are

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described as follows.

The first is fractional derivative and integral are defined by many mathematicians and applied to solve some physical problems [1]. The second is the fractional Fourier transform has been studied in the optic community and signal processing area [2]. The third is the fractional dimension is used to measure some real world data such as coastline, cloud, dust in the air, and network of neurons in the body [3]. The fractional dimension has being applied widely to pattern recognition and classification. The last is fractional lower-order moment has been used to analyse non-Gaussian signal which is more realistic than the Gaussian model in signal processing application [4].

Fractional transform are used to compute the mined time and frequency components of signals. Fractional operators particularly, Fractional Fourier Transform (FrFT) have been investigated in some depth in recent years. The FrFT is an extension of the ordinary Fourier Transform (FT) and successfully applied in the areas of optics, quantum mechanics and signal processing. It gives more complete representation of the signal in phase space and enlarge the number of applications of the ordinary FT [5]. In addition to the FT, the Cosine Transform (CT), which are based on half range expression of a function over Cosine basis function are also important tools in signal processing. Despite of some lack of elegance in there properties compared to the FT, CT has their own areas of applications.

The idea of fractionalization of the CT was proposed in [6]. The real part of the FrFT kernel was chosen as the kernels for a fractional Cosine transform (FrCT) as in the case of CT where real part of FT is chosen as a CT kernel. Thus FrFT and FrCT with parameter are finding its place in many application where FT and CT are found to be useful like beam forming, image compression, noise removal and signal restoration.

Distributional Generalized one-dimensional Fractional Fourier transform is defined as

$$FrFT\{f(x)\} = F_\alpha(u) =< f(x), k_\alpha(x,u) >,$$

Where the kernel

$$k_\alpha(x,u) = \sqrt{\frac{1-i\cot\phi}{2\pi}}\, e^{\frac{i}{2\sin\alpha}[(x^2+u^2)\cos\alpha - 2xu]}, where \quad 0<\alpha<\frac{\pi}{2} \ldots (1.1)$$

In this paper, Two-dimensional fractional Cosine transform is extended in the distributional generalized sense. Section 2 presents definition of two-dimensional fractional Cosine transform and testing function space. Section 3 explains distributional two-dimensional FrCT. Section 4 proved analyticity theorem. Section 5 concludes the paper.

2. Two-dimensional Generalized fractional Cosine transform

2.1.Two-dimensional fractional Cosine transform with parameter α of $f(x,y)$ denoted by $F_c^{\alpha}\{f(x,y)\}$ performs a linear operation, given by the integral transform.

$$F_c^{\theta}\{f(x,y)\} = F^{\theta}\{f(x,y)\}(u,v) = F^{\theta}(u,v) \quad = \int_0^{\infty}\int_0^{\infty} f(x,y)K_{\theta}(x,y,u,v)dxdy, \tag{2.1}$$

Where the kernel

$$K_{\theta}(x,y,u,v) = \sqrt{\frac{2}{\pi}}\frac{e^{\frac{i\theta}{2}}}{\sqrt{i\sin\theta}}.e^{\frac{i}{2}(x^2+u^2+y^2+v^2)\cot\theta}\cos(cosec\theta.ux).\cos(cosec\theta.vy) \tag{2.2}$$

2.2.The Test Function Space E.

An infinitely differentiable complex valued function ϕ on R^n belongs to $E(R^n)$ if for each compact set $I \subset S_{a,b}$ where

$$S_{a,b} = \{x,\, y : x,\, y \in R^n,\, |x| \leq a,\, |y| \leq b,\, a>0,\, b>0\}, I \in R^n.$$

$$\gamma_{E,p,q}(\phi) = \sup_{x,y\in I} \mid D_{x,y}^{p,q}\phi(x,y) \mid$$

$$< \infty, \qquad where\, p,q = 1,2,3.....$$

Thus $E(R^n)$ will denote the space of all $\phi \in E(R^n)$ with support contained in $S_{a,b}$.

Note that the space E is complete and therefore a Frechet space. Moreover, we say that f is a fractional Cosine transformable if it is a member of E^*, the dual space of E.

3. Distributional two-dimensional fractional Cosine transform

The two-dimensional distributional fractional Cosine transform of $f(x,y) \in E^*(R^n)$ defined by

$$F_\theta^c\{f(x,y)\} = F^\theta(u,v) = \langle f(x,y),\ K_\theta(x,y,u,v)\rangle,$$

where

$$K_\theta(x,y,u,v) = \sqrt{\frac{2}{\pi}} \frac{e^{\frac{i\theta}{2}}}{\sqrt{i\sin\theta}} . e^{\frac{i}{2}(x^2+u^2+y^2+v^2)\cot\theta} \cos(cosec\theta.ux).\cos(cosec\theta.vy) \tag{3.1}$$

R.H.S. of eq. (3.1) has a meaning as the application of $f \in E^*$ to $K_\theta(x,y,u,v) \in E$

It can be extended to the complex space as an entire function given by

$$F_c^\theta\{f(x,y)\} = F_\theta(g,h) = \langle f(x,y),\ K_\theta(x,y,g,h)\rangle. \tag{3.2}$$

The right hand side is meaningful because for each $g,h \in c^n,\ K_\theta(x,y,g,h) \in E$, as a function of x,y.

4. Analyticity of the Generalized two-dimensional fractional Cosine transform

Theorem 4.1. *Let $f(x,y) \in E^*(R^n)$ and let its fractional Cosine transform be defined by (3.2). The $F_c^\theta(g,h)$ is an analytic on c^n if the $supf \subset S_{a,b}$ where*
$S_{a,b} = \{x,\ y : x,\ y \in R^n,\ |x| \leq a,\ |y| \leq b,\ a > 0,\ b > 0\}$
Moreover $F_\theta^c(g,h)$ is differentiable and

$$D_{g,h}^{p,q} F_c^\theta(g,h) = \langle f(x,y),\ D_{g,h}^{p,q} K_\theta(x,y,g,h)\rangle \tag{4.1}$$

Proof.

Let

$$g : (g_1, g_2, ...g_j, ...g_n) \in c^n$$

and

$$h : (h_1, h_2, ..., h_j, ...h_n) \in c^n.$$

We first prove that

$$\frac{\partial}{\partial q_j} F_c^{\theta}(g,h) = \langle f(x,y),\ \frac{\partial}{\partial q_j} K_{\theta}(x,y,g,h)\rangle$$

For fined $g_j \neq 0$, chose two concentric circle c and c' with centre at g_j and radii r and r_1 respectively such that $0 < r < r_1 < |g_j|$.

Let $\triangle g_j$ be a complex increment satisfying $0 < |\triangle g_j| < x$.

Consider,

$$\frac{F_c^{\theta}(g_j + \triangle g_j) - F_c^{\theta}(g_j)}{\triangle g_j} - \langle f(x,y), \frac{\partial}{\partial q_j} K_{\theta}(x,y,g,h)\rangle = \langle f(x,y), \psi_{\triangle q_j}(x,y)\rangle \tag{4.2}$$

$$\Rightarrow \langle f(x,y), \frac{1}{\triangle q_j} K_{\theta}(x,y,g_j+\triangle g_j,h) - K_{\theta}(x,y,g,h) - \frac{\partial}{\partial q_j} k_{\theta}(x,y,g,h)\rangle = \langle f(x,y), \psi_{\triangle q_j}(x,y)\rangle$$

For any fixed $x, y \in R_n$ and fixed integer $p = (p_1, p_2, ..., p_n)$

$$D_x^p K_{\theta}(x,y,g,h) = D_x^p [A e^{\frac{i}{2}(x^2+g^2)\cot\theta} \cos(cosec\theta.gx).B(y)],$$

where $B(y) = e^{\frac{i}{2}(y^2+h^2)\cot\theta} \cos(cosec\theta.hy)$ and $A = \sqrt{\frac{2}{\pi}} \frac{e^{\frac{i\theta}{2}}}{\sqrt{i\sin\theta}}$

$$\begin{aligned} D_x^p K_{\theta}(x,y,g,h) = {} & \sqrt{\frac{2}{\pi}} \frac{e^{\frac{i\theta}{2}}}{\sqrt{i\sin\theta}} e^{\frac{i}{2}(x^2+g^2+y^2+h^2)\cot\theta} \cos(cosec\theta.hy) \\ & \sum_{n=0}^{p} \sum_{r=0}^{k} \binom{p}{n} \frac{n!}{(k-2r)!r!} (i\cot\theta)^{k-r} 2^{-r} x^{k-2r} (cosec\theta.gx)^{p-n} \\ & \cos[cosec\theta.gx + \frac{(p-n)\pi}{2}]. \end{aligned} \tag{4.3}$$

Since for any fixed $x, y \in R^n$ and fixed integer p and θ is ranging from 0 to $\frac{\pi}{2}$.

$D_x^p K_{\theta}(x,y,g,h)$ is analytic inside and on c', we have by Cauchy's integral formula

$$D_x^p \psi_{\triangle g_j}(x,y) = \frac{1}{2\pi i} D_x^p \int_{c'} k_{\theta}(x,y,\tilde{g},h) \left[\frac{1}{\triangle g_j}\left(\frac{1}{z-g_j-\triangle g_j} - \frac{1}{z-g_j}\right) - \frac{1}{(z-g_j)^2}\right] dz$$

Where $\tilde{g} = g_1, g_2 \ldots g_{j-1}, z, g_{j+1} \ldots g_n$

$$D_x^p \psi_{\triangle q_j}(x,y) = \frac{\triangle g_j}{2\pi i} \int_{c'} \frac{A(x,y,\tilde{g},h)}{(z-g_j-\triangle g_j)(z-g_j)^2} dz$$

But for all $z \in c'$ and x restricted to a compact subset of $R^n, 0 < \theta < \frac{\pi}{2}$,
$A(x,y,\tilde{g},h) = D_x^p k_\theta(x,y,\tilde{g},h)$ is bounded by a constant Q.
Moreover, $|z-g_j-\triangle g_j| > r_1 - r > 0$ and $|z-g_j| = r_1$.
Therefore we have,

$$\begin{aligned} |D_x^p \psi_{\triangle g_j}(x,y)| &= \left|\frac{\triangle g_j}{2\pi i} \int_{c'} \frac{A(x,y,\tilde{g},h)}{(z-g_j-\triangle g_j)(z-g_j)^2} dz\right| \\ &\leq \frac{|\triangle g_j| Q}{(r_1-r)r_1} \end{aligned}$$

Similarly, $|D_y^q \psi_{\triangle h_j}(x,y)| \leq \dfrac{|\triangle h_j| P}{(r_1-r)r_1}$,
where $B(x,y,g,\tilde{h}) = D_y^q K_\theta(x,y,g,\tilde{h})$ is bounded by a constant P.
Thus, as $|\triangle g_j| \longrightarrow 0$, $D_x^p \psi_{\triangle g_j}(x,y)$ tends to zero uniformly on the compact subset of R^n, therefore it follows that $\psi_{\triangle g_j}(x,y)$ converges in $E(R^n)$ to zero.

Since $f(x,y) \in E^*$ we conclude that equation (4.2) also tends to zero.
Therefore, $F_\theta(g,h)$ is differentiable with respect to g_j and h_j. But this is true for all $j = 1, 2, 3 \ldots, n$. Hence $F_\theta(g,h)$ is analytic on c^n and

$$D_{x,y}^{p,q} F_\theta(g,h) = \langle f(x,y),\ D_{g,h}^{p,q} K_\theta(x,y,g,h) \rangle$$

5. Conclusion

We have extended the two-dimensional fractional Cosine transform in the distributional generalized sense. The testing function space and Distributional generalized two-dimensional fractional Cosine transform is defined. Analyticity theorem is also proved.

Fractional Cosine transform is closely related to fractional Fourier transform which is most essential tool in the theory of optics and signal processing. In particular, when the function denoting the signal is impulse type, the generalized two-dimensional fraction Cosine transform is useful.

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