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CONVERGENCE OF THE VARIATIONAL ITERATION METHOD FOR SOLVING FRACTIONAL KLEIN-GORDON EQUATION

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Abstract. In this study, the variational iteration method is implemented to solve linear and nonlinear fractional Klein-Gordon equations. To illustrate the reliability of the method some examples are presented. The convergence of the VIM solutions to the exact solutions is shown.

Keywords: fractional differential equation; Klein-Gordon equation; variational iteration method.

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1. Introduction

Nonlinear phenomena that appear in many areas of scientific fields such as solid state physics, plasma physics, fluid dynamics, mathematical biology and chemical kinetics are modeled in terms of nonlinear partial differential equations and in many scientific and engineering applications one of the corner stones of modeling are partial differential equations. For example, the klein- Gordeon equations which are of the form

$$u_{tt}(x,t)+bu(x,t)+g(u(x,t))=f(x,t), \tag{0.1}$$

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with initial conditions

$$u(x,0) = h(x),\ u_t(x,o) = k(x), \tag{0.2}$$

appears in modeling of problems in quantum field theory, relavistic physics, dispersive wave-phenomena, plasma physic, nonlinear optics and applied physical sciences. The complexity of the equations though requires the use of numerical and analytical methods in most cases. A broad class of analytical solution and numerical solution methods were used to handle these problems. Perturbation techniques are currently the main stream. Perturbation techniques are based on the existence of small/large parameters, so-called perturbations quantities, but many nonlinear problems in science and engineering do not contain such kind of perturbation quantities at all. Some non-perturbatiion techniques, such as the Lyapunov's artificial small parameter method [1], the δ-expansion method [3], the Adomian's decomposition method [4, 5] and the homotopy perturbation method [2] have been developed. In 1978 Inokuti et al [6] proposed a general lagrange multiplier method to solve nonlinear problems, which was intended to solve problems in quantum mechanics. Subsequntly, in 1997 He [7] has modified this method to an iterative method and named it variational iteration method (VIM). VIM has been presented by many authors to be a powerful mathematical tool for solving various types of nonlinear problems which present a plenty of modern science branches [8]-[13]. This method has been applied to solve many ordinary differential equations, partial differential equations and integral differential equations. VIM is very convenient, efficient and accurate. On the other hand recently there is a increasing interest to study of the fractional differential equations because of their frequent appearance in various applications in physics, biology, engineering, signal processing, system identification, control theory, finance and fractional dynamics [14]-[16] and so a large number of research papers devoted to the study of the solutions of fractional differential equations. In this paper we consider the fractional Klein- Gordon equation

$$\frac{\partial^\alpha}{\partial t^\alpha}u(x,t) + bu(x,t) + g(u(x,t)) = f(x,t), \tag{0.3}$$

and try to show the convergence of variational iteration method in solving this equation.

2. Fractional calculus

In this section we describe some necessary definitions and mathematical preliminaries of the fractional calculus theory required for our subsequent development.

Definition 2.1. A real valued function $f(x), x > 0$, is said to be in the space $C_\mu, \mu \in R$ if there exists a real number $p > \mu$, such that $f(x) = x^p f_1(x)$ where $f_1(x) \in C[0,\infty)$ and it is said to be in the space C_μ^m if $f^{(m)} \in C_\mu, m \in N$.

Definition 2.2. The Riemann-Liouville fractional integral of order $\alpha \geq 0$, of a function $f \in C_\mu (\mu \geq -1)$ is defined as

$$J^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)dt, \quad \alpha > 0, x > 0, \tag{0.4}$$
$$J^0 f(x) = f(x),$$

where $\Gamma(\alpha)$ is the well-known gamma function. We mention only the following properties of the operator J^α:

For $f \in C_\mu, \mu \geq -1, \alpha, \beta \geq 0$ and $\gamma > -1$

$$\begin{aligned} J^\alpha J^\beta f(x) &= J^{\alpha+\beta}, \\ J^\alpha J^\beta f(x) &= J^\beta J^\alpha f(x), \\ J^\alpha x^\gamma &= \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)}. \end{aligned} \tag{0.5}$$

Definition 2.3. Caputo fractional derivative operator $D_*^\alpha$of order α is defined in the following form

$$D_*^\alpha f(x) = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-t)^{m-\alpha-1} f^m(t)dt, & m-1 < \alpha \leq m, \ m \in N, \\ \frac{d^n}{dx^n} f(x), & \alpha = n. \end{cases} \tag{0.6}$$

We need here two basic properties of the Caputo fractional derivative:

If $m-1 < \alpha \leq m$, $m \in N$ and $f \in C_\mu^m$, $\mu \geq -1$, then

$$D_*^\alpha J^\alpha f(x) = f(x), \tag{0.7}$$

$$J^\alpha D_*^\alpha f(x) = f(x) - \sum_{k=0}^{m-1} f^{(k)}(0^+) \frac{x^k}{k!}, \ x > 0. \tag{0.8}$$

3. Variational iteration method

The principles of the variational iteration method and its applicability for various kinds of differential equations can be found in [7]-[13]. To illustrate (VIM), we consider the general nonlinear equation

$$Lu(x,t)+Nu(x,t)=g(x,t),$$

where L is a linear operator, N a nonlinear operator and g(x,t) a known analytical function. According to variational iteration method, we can construct a correction functional as follows

$$u_{n+1}(x,t)=u_n(x,t)+\int_0^t \lambda(\xi)(Lu_n(x,\tau)+N\tilde{u}_n(x,\xi)-g(x,\xi))d\xi, \tag{0.9}$$

where λ is a general lagrange multiplier, which can be identified optimally via variational theory and $\tilde{u}_n$ is considered as a restricted variation, i.e. $\delta\tilde{u}_n=0$. Therefore, we first determine the Lagrange multiplier that will be identified optimally via integration by parts. The initial guess u_0 may be selected by any function that satisfies the two prescribed initial conditions (in this case). The other components of the solution can easily be determined iteratively and consequently we may obtain the exact solution by using

$$u(x,t)=\lim_{k\to\infty} u_k(x,t). \tag{0.10}$$

We consider the following time-fractional partial differential equation

$$D_{*t}^{\alpha}u(x,t)=f(u,u_x,u_{xx})+g(x,t), \tag{0.11}$$

where $D_{*t}^{\alpha}=\frac{\partial^{\alpha}}{\partial t^{\alpha}}$ is the Caputo derivative of order α, $m\in N$, f is a nonlinear function and g is the source function. The initial and boundary conditions associated with (0.11) are of the form

$$0<\alpha\leq 1:\ u(x,0)=h(x),\ u(x,t)\to 0\ as\ |x|\to\infty,\ t>0, \tag{0.12}$$

and

$$1<\alpha\leq 2:\ u(x,0)=h(x),\ \frac{\partial u(x,0)}{\partial t}=k(x),\ u(x,t)\to 0\ as\ |x|\to\infty,\ t>0. \tag{0.13}$$

The correction functional for Eq.(0.11) can be approximately expressed as follows

$$u_{k+1}(x,t)=u_k(x,t)+\int_0^t \lambda(\xi)(\frac{\partial^m}{\partial\xi^m}u(x,\xi)-f(\tilde{u}_k,(\tilde{u}_k)_x,(\tilde{u}_k)_{xx})-g(x,\xi))d\xi, \tag{0.14}$$

where λ is the general Lagrange multiplier which can be identified optimally via variational theory. Here $\tilde{u}_k, (\tilde{u}_k)_x, (\tilde{u}_k)_{xx}$ are considered variations, *i.e.*, $\delta \tilde{u}_n = 0$. Making the above functional stationary,

$$\delta u_{k+1}(x,t) = \delta u_k(x,t) + \delta \int_0^t \lambda(\xi)(\frac{\partial^m}{\partial \xi^m} u(x,\xi) - g(x,\xi))d\xi, \tag{0.15}$$

yields the following Lagrange multipliers

$\lambda = -1, \quad for \quad m = 1,$

$\lambda = \xi - t, \quad for \quad m = 2.$

Therefore, for $m = 2$ we obtain the following iteration formula

$$u_{k+1}(x,t) = u_k(x,t) + \int_0^t (\xi - t)(\frac{\partial^m}{\partial \xi^m} u(x,\xi) - f(u_k,(u_k)_x,(u_k)_{xx}) - g(x,\xi))d\xi. \tag{0.16}$$

For $m = 1$, we begin with the initial approximation

$$u_0(x,t) = h(x). \tag{0.17}$$

Also, for $m = 2$ we begin with the initial approximation

$$u_0(x,t) = h(x) + tk(x). \tag{0.18}$$

The correction functional (0.14) will give several approximations and the exact solution is obtained as

$$u(x,t) = \lim_{k \to \infty} u_k(x,t). \tag{0.19}$$

4. Numerical examples

Example 4.1. Consider the time-fractional partial differential Klein-Gordon equation

$$D^{\alpha}_{*t} u(x,t) - u_{xx} + u = 0, \quad 1 < \alpha \leq 2, \tag{0.20}$$

subject to the initial conditions

$$u(x,0) = 0, \; u_t(x,0) = x. \tag{0.21}$$

To solve (0.20), by means of He's variational iteration method, a correction functional for Eq.(0.20) can be approximately expressed as follows

$$u_{k+1}(x,t) = u_k(x,t) + \int_0^t \lambda(t,\xi)\{\frac{\partial^2}{\partial \xi^2}u(x,\xi) - \frac{\partial^2}{\partial x^2}\tilde{u}_k(x,\xi) + \tilde{u}_k(x,\xi)\}d\xi, \tag{0.22}$$

where λ is the Lagrange multiplier and $\tilde{u}_n$ is considered as a restricted variation. Its stationary conditions can be obtain by the followings

$$\frac{\partial^2}{\partial \xi^2}\lambda(t,\xi) = 0, \tag{0.23}$$

$$1 - \frac{\partial}{\partial \xi}\lambda(t,\xi)|_{t=\xi} = 0, \tag{0.24}$$

$$\lambda(t,\xi)|_{t=\xi} = 0. \tag{0.25}$$

The Lagrange multiplier, therefore is of the form

$$\lambda(t,\xi) = \xi - t. \tag{0.26}$$

So we have the following variational iteration formula

$$u_{k+1}(x,t) = u_k(x,t) + \int_0^t (\xi - t)(\frac{\partial^\alpha}{\partial \xi^\alpha}u_k(x,\xi) - \frac{\partial^2}{\partial x^2}u_k(x,\xi) + u_k(x,\xi))d\xi. \tag{0.27}$$

We start with an initial approximation

$$u_0(x,t) = xt,$$

which satisfies the initial conditions (0.21) and obtain the following successive approximations

$$u_1(x,t) = xt - \frac{xt^3}{3!},$$

$$u_2(x,t) = xt - \frac{xt^3}{3!} + \frac{xt^5}{5!} - \frac{xt^{5-\alpha}}{\alpha^2 - 9\alpha + 20},$$

$$\vdots$$

If we set $\alpha = 2$ we get the followings

$u_1(x,t) = xt - \frac{xt^3}{3!}$,
$u_2(x,t) = xt - \frac{xt^3}{3!} + \frac{xt^5}{5!}$,
$u_3(x,t) = xt - \frac{xt^3}{3!} + \frac{xt^5}{5!} - \frac{xt^7}{7!}$,

$\vdots$

The approximate solution

$$u = \lim_{n\to\infty} u_n = x \sin t. \tag{0.28}$$

is obtained, upon using the Taylor expansion of $\sin t$, which is the exact solution of the Eq. (0.20) when $\alpha = 2$.

Example 4.2. Consider the inhomogeneous linear time-fractional partial differential Klein-Gordon equation

$$D^{\alpha}_{*t}u(x,t) - u_{xx} + u = 2sinx, \quad 1 < \alpha \le 2, \tag{0.29}$$

with the initial conditions

$$u(x,0) = sinx,\ u_t(x,0) = 1. \tag{0.30}$$

To solve (0.29), by means of He's variational iteration method, a correction functional for Eq.(0.29) can be approximately expressed as follows

$$u_{k+1}(x,t) = u_k(x,t) + \int_0^t \lambda(t,\xi)\{\frac{\partial^2}{\partial \xi^2}u(x,\xi) - \frac{\partial^2}{\partial x^2}\tilde{u}_k(x,\xi) + \tilde{u}_k(x,\xi) - 2sinx)d\xi, \tag{0.31}$$

where λ is the Lagrange multiplier and $\tilde{u}_n$ is considered as a restricted variation. The Lagrange multiplier, can be identified as $\xi - t$. Hence, we have the following correction functional

$$u_{k+1}(x,t) = u_k(x,t) + \int_0^t (\xi - t)(\frac{\partial^\alpha}{\partial \xi^\alpha}u_k(x,\xi) - \frac{\partial^2}{\partial x^2}u_k(x,\xi) + u_k(x,\xi) - 2sinx)d\xi. \tag{0.32}$$

We start with an initial approximation

$$u_0(x,t) = sinx + t,$$

which satisfies the initial conditions (0.30) and obtain the following successive approximations

$$u_1(x,t) = sinx + t - \frac{t^3}{3!},$$

$$u_2(x,t) = sinx + t - \frac{t^3}{3} + \frac{t^5}{5!} + \frac{t^{5-\alpha}}{\Gamma(6-\alpha)},$$

$$u_3(x,t) = sinx + t - \frac{t^3}{2} + \frac{t^5}{40} + \frac{t^7}{5040} + t^{5-\alpha}(\frac{1}{\Gamma(6-\alpha)} + \frac{84}{\Gamma(8-\alpha)} - \frac{26\alpha}{\Gamma(8-\alpha)} + \frac{2\alpha^2}{\Gamma(8-\alpha)})$$

$$-\frac{2t^{(7-\alpha)}}{\Gamma(8-\alpha)}+\frac{t^{(7-2\alpha)}}{\Gamma(8-2\alpha)},$$

$$\vdots$$

If we set $\alpha = 2$ we get the followings

$u_1(x,t) = sinx + t - \frac{t^3}{3!}$,

$u_2(x,t) = sinx + t - \frac{t^3}{3} + \frac{t^5}{5!}$,

$u_3(x,t) = sinx + t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!}$.

$\vdots$

The approximate solution

$$u = \lim_{n\to\infty} u_n = \sin x + \sin t, \tag{0.33}$$

is obtained, upon using the Taylor expansion of $\sin t$, which is the exact solution of the Eq. (0.29) when $\alpha = 2$.

Example 4.3. Consider the non-linear time-fractional partial differential Klein-Gordon equation

$$D_{*t}^{\alpha} u(x,t) - u_{xx} + u^2 = 2x^2 - 2t^2 + x^4 t^4, \quad 1 < \alpha \leq 2, \tag{0.34}$$

with the initial conditions

$$u(x,0) = 0, u_t(x,0) = 0. \tag{0.35}$$

To solve (0.34) by means of He's variational iteration method, a correction functional for Eq.(0.34) can be approximately expressed as follows

(0.36)
$$u_{k+1}(x,t) = u_k(x,t) + \int_0^t \lambda(t,\xi)\{\frac{\partial^2}{\partial \xi^2} u(x,\xi) - \frac{\partial^2}{\partial x^2} \tilde{u}_k(x,\xi) + \tilde{u}_k^2(x,\xi) - 2x^2 + 2\xi^2 - x^4\xi^4) d\xi,$$

where λ is the Lagrange multiplier and $\tilde{u}_n$ is considered as a restricted variation. The Lagrange multiplier, can be identified as $\xi - t$. Hence, we have the following correction functional

(0.37)
$$u_{k+1}(x,t) = u_k(x,t) + \int_0^t (\xi - t)(\frac{\partial^\alpha}{\partial \xi^\alpha} u_k(x,\xi) - \frac{\partial^2}{\partial x^2} u_k(x,\xi) + u_k^2(x,\xi) - 2x^2 + 2\xi^2 - x^4\xi^4) d\xi.$$

We start with an initial approximation

$$u_0(x,t) = x^2 t^2,$$

which satisfies the initial conditions (0.35), the following approximation is obtained in the first iteration

$$u_1(x,t) = 2t^2 x^2 - \frac{2t^{4-\alpha} x^2}{\Gamma(5-\alpha)},$$

and the rest of the components of the iteration formula can be obtained using the Mathematica program. If we set $\alpha = 2$ we get the followings

$u_1(x,t) = x^2 t^2,$

$u_2(x,t) = x^2 t^2,$

$u_3(x,t) = x^2 t^2,$

$\vdots$

So the approximate solution

$$u = \lim_{n\to\infty} u_n = x^2 t^2, \tag{0.38}$$

is obtained which is the exact solution of the Eq. (0.34) when $\alpha = 2$.

Conflict of Interests

The authors declare that there is no conflict of interests.

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