Available online at http://scik.org
J. Math. Comput. Sci. 4 (2014), No. 2, 267-277
ISSN: 1927-5307

ON CERTAIN CLASS OF SEQUENCE SPACES OF INVARIANT MEAN DEFINED BY ORLICZ FUNCTION

KHALID EBADULLAH

Department of Applied Mathematics, Faculty of Engg. Tech. Aligarh Muslim University, Aligarh-202002, India

Copyright © 2014 Khalid Ebadullah. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Abstract. In this article, we introduce the sequence space $BV_\sigma(M,p,r,\triangle_v^u)$, where $p=(p_k)$ sequence of positive reals, $v=(v_k)$ is any fixed sequence of non zero complex numbers, $u \in N$ is a fixed number and study some of the properties and inclusion relations on this space.

Keywords: invariant mean; paranorm; orlicz function and difference sequence.

2010 Mathematics Subject Classification: 40C05.

1. Introduction

Let N, R and C be the sets of all natural, real and complex numbers respectively. We write

$$\omega = \{x=(x_k) : x_k \in R \text{ or } C\},$$

the space of all real or complex sequences. Let l_∞, c and c_0 denote the Banach spaces of bounded, convergent and null sequences respectively. The following subspaces of ω were first introduced and discussed by Maddox [10-11]. $l(p)=\{x\in\omega : \sum_k |x_k|^{p_k} < \infty\}$,
$l_\infty(p)=\{x\in\omega : \sup_k |x_k|^{p_k} < \infty\}$,

Received January 2, 2014

$c(p)=\{x\in\omega:\lim_k|x_k-l|^{p_k}=0,$ for some $l\in\mathbb{C}$ $\}$, $c_0(p)=\{x\in\omega:\lim_k|x_k|^{p_k}=0\}$,

where $p=(p_k)$ is a sequence of striclty positive real numbers. The idea of Difference sequence sets

$$X_\triangle=\{x=(x_k)\in\omega:\triangle x=(x_k-x_{k+1})\in X\},$$

where $X=l_\infty,\ c$ or c_0 was introduced by Kizmaz [7]. Kizmaz [7] defined the sequence spaces,

$$l_\infty(\triangle)=\{x=(x_k)\in\omega:(\triangle x_k)\in l_\infty\},$$

$$c(\triangle)=\{x=(x_k)\in\omega:(\triangle x_k)\in c\},$$

$$c_0(\triangle)=\{x=(x_k)\in\omega:(\triangle x_k)\in c_0\},$$

where $\triangle x=(x_k-x_{k+1})$. These are Banach spaces with the norm

$$||x||_\triangle=|x_1|+||\triangle x||_\infty.$$

After then Et [4] defined the sequence spaces

$$l_\infty(\triangle^2)=\{x=(x_k)\in\omega:(\triangle^2 x_k)\in l_\infty\},$$

$$c(\triangle^2)=\{x=(x_k)\in\omega:(\triangle^2 x_k)\in c\},$$

$$c_0(\triangle^2)=\{x=(x_k)\in\omega:(\triangle^2 x_k)\in c_0\},$$

where $(\triangle^2 x)=(\triangle^2 x_k)=(\triangle x_k-\triangle x_{k+1})$. The sequence spaces $l_\infty(\triangle^2),c(\triangle^2)$ and $c_0(\triangle^2)$ are Banach spaces with the norm

$$||x||_\triangle=|x_1|+|x_2|+||\triangle^2 x||_\infty.$$

After then R. Colak and M. Et [5] defined the sequence spaces

$$l_\infty(\triangle^m)=\{x=(x_k)\in\omega:(\triangle^m x_k)\in l_\infty\},$$

$$c(\triangle^m)=\{x=(x_k)\in\omega:(\triangle^m x_k)\in c\},$$

$$c_0(\triangle^m)=\{x=(x_k)\in\omega:(\triangle^m x_k)\in c_0\},$$

where $m\in N$,

$$\triangle^0 x=(x_k),$$

$$\triangle x=(x_k-x_{k+1}),$$

$$\triangle^m x = (\triangle^{m-1} x_k - \triangle^{m-1} x_{k+1}),$$

and so that

$$\triangle^m x_k = \sum_{i=0}^{m} (-1)^i \begin{bmatrix} m \\ i \end{bmatrix} x_{k+i}$$

and showed that these are Banach spaces with the norm

$$||x||_\triangle = \sum_{i=1}^{m} |x_i| + ||\triangle^m x||_\infty.$$

Esi and Isik [3] defined the sequence spaces

$$l_\infty(\triangle_v^m, s, p) = \{x = (x_k) \in \omega : \sup\lim_k k^{-s} |\triangle_v^m x_k|^{p_k} < \infty, s \geq 0\},$$

$$c(\triangle_v^m, s, p) = \{x = (x_k) \in \omega : k^{-s} |\triangle_v^m x_k - L|^{p_k} \to 0 (k \to \infty), s \geq 0, \text{forsome L}\},$$

$$c_0(\triangle_v^m, s, p) = \{x = (x_k) \in \omega : k^{-s} |\triangle_v^m x_k|^{p_k} \to 0 (k \to \infty), s \geq 0\},$$

where $p = (p_k)$ is a sequence of striclty positive real numbers, $v = (v_k)$ is any fixed sequence of non zero complex numbers, $m \in \mathbb{N}$ is a fixed number,

$$\triangle_v^0 x_k = (v_k x_k),\ \triangle_v x_k = (v_k x_k - v_{k+1} x_{k+1})$$

and

$$\triangle_v^m x_k = (\triangle_v^{m-1} x_k - \triangle_v^{m-1} x_{k+1})$$

and so that

$$\triangle_v^m x_k = \sum_{i=0}^{m} (-1)^i \begin{bmatrix} m \\ i \end{bmatrix} v_{k+i} x_{k+i}.$$

When s=0, m=1, v=(1,1,1,.......) and $p_k = 1$ for all $k \in \mathbb{N}$, they are just $l_\infty(\triangle), c(\triangle)$ and $c_0(\triangle)$ defined by Kizmaz[7]. When s=0 and $p_k = 1$ for all $k \in \mathbb{N}$, they are the following sequence spaces defined by Et and Esi[6]

$$l_\infty(\triangle_v^m) = \{x = (x_k) \in \omega : (\triangle_v^m x_k) \in l_\infty\},$$

$$c(\triangle_v^m) = \{x = (x_k) \in \omega : (\triangle_v^m x_k) \in c\},$$

$$c_0(\triangle_v^m) = \{x = (x_k) \in \omega : (\triangle_v^m x_k) \in c_0\}.$$

The concept of paranorm is closely related to linear metric spaces.It is a generalization of that of absolute value.(see[11]) Let X be a linear space. A function $g : X \longrightarrow R$ is called paranorm, if for all $x, y, z \in X$,

(PI) $g(x) = 0 \; if \; x = \theta$,

(P2) $g(-x) = g(x)$,

(P3) $g(x+y) \leq g(x) + g(y)$,

(P4) If (λ_n) is a sequence of scalars with $\lambda_n \to \lambda \; (n \to \infty)$ and $x_n, a \in X$ with $x_n \to a \; (n \to \infty)$, in the sense that $g(x_n - a) \to 0 \; (n \to \infty)$, in the sense that $g(\lambda_n x_n - \lambda a) \to 0 \; (n \to \infty)$.

An Orlicz function is a function $M : [0, \infty) \to [0, \infty)$, which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$; see [2],[14] and the references therein.

Lindenstrauss and Tzafriri[8] used the idea of Orlicz functions to construct the sequence space

$$\ell_M = \{x \in \omega : \sum_{k=1}^{\infty} M(\frac{|x_k|}{\rho}) < \infty, \text{for some } \rho > 0\}.$$

The space ℓ_M is a Banach space with the norm

$$||x|| = \inf\{\rho > 0 : \sum_{k=1}^{\infty} M(\frac{|x_k|}{\rho}) \leq 1\}.$$

The space ℓ_M is closely related to the space l_p which is an Orlicz sequence space with $M(x) = x^p$ for $1 \leq p < \infty$.

An Orlicz function M is said to satisfy $\triangle_2$ condition for all values of x if there exists a constant $K > 0$ such that $M(Lx) \leq KLM(x)$ for all values of $L > 1$.

A sequence space E is said to be solid or normal if $(x_k) \in E$ implies $(\alpha_k x_k) \in E$ for all sequence of scalars (α_k) with $|\alpha_k| < 1$ for all $k \in \mathbb{N}$.

A sequence space E is said to be symmetric if $(x_{\pi(k)}) \in E$ whenever $(x_k) \in E$ where $\pi(k)$ is a permutation on $\mathbb{N}$.

Let σ be an injection on the set of positive integers $\mathbb{N}$ into itself having no finite orbits and T be the operator defined on l_∞ by $T(x_k) = (x_{\sigma(k)})$. A positive linear functional functional Φ, with $||\Phi|| = 1$, is called a σ-mean or an invariant mean if $\Phi(x) = \Phi(Tx)$ for all $x \in l_\infty$.

A sequence x is said to be σ-convergent, denoted by $x \in V_\sigma$, if $\Phi(x)$ takes the same value, called $\sigma - \lim x$, for all σ-means Φ. We have

$$V_\sigma = \{x = (x_k) : \sum_{m=1}^{\infty} t_{m,n}(x) = L \text{ uniformly in n, L} = \sigma - \lim x\},$$

where for $m \geq 0, n > 0$.

$$t_{m,n}(x) = \frac{x_k + x_{\sigma(k)} + ... + x_{\sigma^m(k)}}{m+1}, \text{and } t_{-1,n} = 0,$$

where $\sigma^m(k)$ denotes the mth iterate of σ at n. In particular, if σ is the translation, a σ-mean is often called a Banach limit and V_σ reduces to f, the set of almost convergent sequences; see [9],[15],[16] and the references therein. Mursaleen [12] defined the sequence space

$$BV_\sigma = \{x \in l_\infty : \sum_m |\phi_{m,n}(x)| < \infty, \text{uniformly in n}\},$$

where

$$\phi_{m,n}(x) = t_{m,n}(x) - t_{m-1,n}(x)$$

assuming that

$$t_{m,n}(x) = 0, \text{ for m = -1}.$$

A straight forward calculation shows that

$$\phi_{m,n}(x) = \begin{cases} \frac{1}{m(m+1)} \sum_{j=1}^{m} J(x_{\sigma^j(k)} - x_{\sigma^{j-1}(k)}) \text{ (m}\geq 1\text{)}, \\ x_k, \text{ (m = 0)}. \end{cases}$$

Note that for any sequence x, y and scalar λ we have

$$\phi_{m,n}(x+y) = \phi_{m,n}(x) + \phi_{m,n}(y) \text{ and } \phi_{m,n}(\lambda x) = \lambda \phi_{m,n}(x).$$

After then Khan[17] introduced and studied the space

$$BV_\sigma(M,p,r) = \{x = (x_k) : \sum_{m=1}^{\infty} \frac{1}{m^r} [M(\frac{|\phi_{m,n}(x)|}{\rho})]^{p_k} < \infty \text{ uniformly in n, } \rho > 0\},$$

where M is an Orlicz function, $p = (p_k)$ is any sequence of strictly positive real numbers and $r \geq 0$. Recently Khan and Ebadullah[18] introduced and studied the sequence space

$$BV_\sigma(M,p,r,\triangle) = \{x = (x_k) : \sum_{m=1}^{\infty} \frac{1}{m^r} [M(\frac{|\phi_{m,n}(\triangle x)|}{\rho})]^{p_k} < \infty \text{ uniformly in n, } \rho > 0\}.$$

Subsequently the spaces of invariant mean and Orlicz function have been studied by various authors; see [1],[2],[9],[12],[13],[14],[15],[16],[17] and the references therein.

2. Main Results

In this article, we introduce the sequence space

$$BV_{\sigma}(M,p,r,\triangle_{v}^{u})=\{x=(x_k):\sum_{m=1}^{\infty}\frac{1}{m^r}[M(\frac{|\phi_{m,n}(\triangle_{v}^{u}x)|}{\rho})]^{p_k}<\infty \text{ uniformly in n}, \rho>0\},$$

where $u\in\mathbb{N}$ is a fixed number, $v=(v_k)$ is any fixed sequence of non zero complex numbers and study some of the properties and inclusion relations on this space.

Let M be an Orlicz function, $p=(p_k)$ be any sequence of strictly positive real numbers, $u\in\mathbb{N}$ be a fixed number and $r\geq 0$. Now we define the sequence spaces as follows:

We have

$$BV_{\sigma}(M,p,r,\triangle)=\{x=(x_k):\sum_{m=1}^{\infty}\frac{1}{m^r}[M(\frac{|\phi_{m,n}(\triangle x)|}{\rho})]^{p_k}<\infty \text{ uniformly in n}, \rho>0\}.$$

For $M(x)=x$ we get

$$BV_{\sigma}(p,r,\triangle_{v}^{u})=\{x=(x_k):\sum_{m=1}^{\infty}\frac{1}{m^r}|\phi_{m,n}(\triangle_{v}^{u}x)|^{p_k}<\infty \text{ uniformly in n}\}.$$

For $p_k=1$, for all m, we get

$$BV_{\sigma}(M,r,\triangle_{v}^{u})=\{x=(x_k):\sum_{m=1}^{\infty}\frac{1}{m^r}[M(\frac{|\phi_{m,n}(\triangle_{v}^{u}x)|}{\rho})]<\infty \text{ uniformly in n}, \rho>0\}.$$

For r = 0 we get

$$BV_{\sigma}(M,p,\triangle_{v}^{u})=\{x=(x_k):\sum_{m=1}^{\infty}[M(\frac{|\phi_{m,n}(\triangle_{v}^{u}x)|}{\rho})]^{p_k}<\infty \text{ uniformly in n}, \rho>0\}.$$

For $M(x)=x$ and r=0 we get

$$BV_{\sigma}(p,\triangle_{v}^{u})=\{x=(x_k):\sum_{m=1}^{\infty}|\phi_{m,n}(\triangle_{v}^{u}x)|^{p_k}<\infty \text{ uniformly in n}, \rho>0\}.$$

For $p_k=1$, for all m and r=0, we get

$$BV_{\sigma}(M,\triangle_{v}^{u})=\{x=(x_k):\sum_{m=1}^{\infty}[M(\frac{|\phi_{m,n}(\triangle_{v}^{u}x)|}{\rho})]<\infty \text{ uniformly in n}, \rho>0\}.$$

For $M(x) = x$, $p_k = 1$, for all m and r=0, we get

$$BV_\sigma(\triangle_v^u) = \{x = (x_k) : \sum_{m=1}^{\infty} |\phi_{m,n}(\triangle_v^u x)| < \infty \text{ uniformly in n}\}.$$

Theorem 2.1. *The sequence space $BV_\sigma(M,p,r,\triangle_v^u)$ is a linear space over the field $\mathbb{C}$ of complex numbers.*

Proof. Let $x,y \in BV_\sigma(M,p,r,\triangle_v^u)$ and $\alpha,\beta \in \mathbb{C}$ then there exists positive numbers ρ_1 and ρ_2 such that

$$\sum_{m=1}^{\infty} \frac{1}{m^r}[M(\frac{|\phi_{m,n}(\triangle_v^u x)|}{\rho_1})]^{p_k} < \infty,$$

and

$$\sum_{m=1}^{\infty} \frac{1}{m^r}[M(\frac{|\phi_{m,n}(\triangle_v^u y)|}{\rho_2})]^{p_k} < \infty$$

uniformly in n. Define ρ_3 = max($2|\alpha|\rho_1$, $2|\beta|\rho_2$). Since M is non decreasing and convex we have

$$\sum_{m=1}^{\infty} \frac{1}{m^r}[M(\frac{|\alpha\phi_{m,n}(\triangle_v^u x) + \beta\phi_{m,n}(\triangle_v^u y)|}{\rho_3})]^{p_k}$$

$$\leq \sum_{m=1}^{\infty} \frac{1}{m^r}[M(\frac{|\alpha\phi_{m,n}(\triangle_v^u x)|}{\rho_3} + \frac{|\beta\phi_{m,n}(\triangle_v^u y)|}{\rho_3})]^{p_k}$$
$$\leq \sum_{m=1}^{\infty} \frac{1}{m^r}\frac{1}{2}[M(\frac{\phi_{m,n}(\triangle_v^u x)}{\rho_1}) + M(\frac{\phi_{m,n}(\triangle_v^u y)}{\rho_2})] < \infty$$

uniformly in n. This proves that $BV_\sigma(M,p,r,\triangle_v^u)$ is a linear space over the field $\mathbb{C}$ of complex numbers.

Theorem 2.2. *For any Orlicz function M and a bounded sequence $p = (p_k)$ of strictly positive real numbers, $BV_\sigma(M,p,r,\triangle_v^u)$ is a paranormed space with*

$$g(\triangle_v^u x) = \inf_{n\geq 1}\{\rho^{\frac{p_k}{K}} : (\sum_{m=1}^{\infty} \frac{1}{m^r}[M(\frac{|\phi_{m,n}(\triangle_v^u x)|}{\rho})]^{p_k})^{\frac{1}{K}} \leq 1, \textit{ uniformly in n}\}$$

where K = max(1, supp_k).

Proof. It is clear that $g(\triangle_v^u x) = -g(\triangle_v^u x)$. Since $M(0) = 0$, we get $\inf\{\rho^{\frac{p_k}{K}}\} = 0$, for $\triangle_v^u x = 0$. Now for α=β=1, we get $g(\triangle_v^u x + \triangle_v^u y) \leq g(\triangle_v^u x) + g(\triangle_v^u y)$. For the continuity of scalar multiplication let $l \neq 0$ be any complex number. Then by the definition we have

$$g(l\triangle_v^u x) = \inf_{n\geq 1}\{\rho^{\frac{p_k}{K}} : (\sum_{m=1}^{\infty} \frac{1}{m^r}[M(\frac{|\phi_{m,n}(l\triangle_v^u x)|}{\rho})]^{p_k})^{\frac{1}{K}} \leq 1, \text{ uniformly in n}\}$$

$$g(l\triangle_v^u x) = \inf_{n\geq 1}\{(|l|s)^{\frac{p_k}{K}} : (\sum_{m=1}^{\infty} \frac{1}{m^r}[M(\frac{|\phi_{m,n}(l\triangle_v^u x)|}{(|l|s)})]^{p_k})^{\frac{1}{K}} \leq 1, \text{ uniformly in n}\},$$

where $s = \frac{\rho}{|l|}$. Since $|l|^{p_k} \leq \max(1,|l|^H)$, we have

$$g(l\triangle_v^u x) \leq max(1,|l|^H) \inf_{n\geq 1}\{s^{\frac{p_k}{K}} : (\sum_{m=1}^{\infty} \frac{1}{m^r}[M(\frac{|\phi_{m,n}(l\triangle_v^u x)|}{(|l|s)})]^{p_k})^{\frac{1}{K}} \leq 1, \text{ uniformly in n}\}$$

$g(l\triangle_v^u x) \leq max(1,|l|^H) g(\triangle_v^u x)$. Therefore $g(l\triangle_v^u x)$ converges to zero when $g(\triangle_v^u x)$ converges to zero in $BV_\sigma(M,p,r,\triangle_v^u)$. Now let x be fixed element in $BV_\sigma(M,p,r,\triangle_v^u)$. There exists $\rho > 0$ such that

$$g(\triangle_v^u x) = \inf_{n\geq 1}\{\rho^{\frac{p_k}{K}} : (\sum_{m=1}^{\infty} \frac{1}{m^r}[M(\frac{|\phi_{m,n}(\triangle_v^u x)|}{\rho})]^{p_k})^{\frac{1}{K}} \leq 1, \text{ uniformly in n}\}.$$

Now

$$g(l\triangle_v^u x) = \inf_{n\geq 1}\{\rho^{\frac{p_k}{K}} : (\sum_{m=1}^{\infty} \frac{1}{m^r}[M(\frac{|\phi_{m,n}(l\triangle_v^u x)|}{\rho})]^{p_k})^{\frac{1}{K}} \leq 1, \text{ uniformly in n}\} \to 0 \text{ as } l \to 0.$$

This completes the proof.

Theorem 2.3. *Suppose that* $0 < p_m < t_m < \infty$ *for each* $m \in \mathbb{N}$ *and* $r > 0$. *Then*

(a) $BV_\sigma(M,p,\triangle_v^u) \subseteq BV_\sigma(M,t,\triangle_v^u)$.

(b) $BV_\sigma(M,\triangle_v^u) \subseteq BV_\sigma(M,r,\triangle_v^u)$.

Proof. (a) Suppose that $x \in BV_\sigma(M,p,\triangle_v^u)$. This implies that $[M(\frac{|\phi_{i,n}(\triangle_v^u x)|}{\rho})]^{p_k}) \leq 1$ for sufficiently large value of i, say $i \geq m_0$ for some fixed $m_0 \in \mathbb{N}$. Since M is non decreasing, we have

$$\sum_{m=m_0}^{\infty} [M(\frac{|\phi_{i,n}(\triangle_v^u x)|}{\rho})]^{t_m} \leq \sum_{m=m_0}^{\infty} [M(\frac{|\phi_{i,n}(\triangle_v^u x)|}{\rho})]^{p_m} < \infty.$$

Hence $x \in BV_\sigma(M,t,\triangle_v^u)$.

(b) The proof is trivial.

Corollary 2.4. $0 < P_m \leq 1$ *for each m, then* $BV_\sigma(M,p,\triangle_v^u) \subseteq BV_\sigma(M,\triangle_v^u)$

If $P_m \geq 1$ *for all m , then* $BV_\sigma(M,\triangle_v^u) \subseteq BV_\sigma(M,p,\triangle_v^u)$.

Theorem 2.5. *The sequence space* $BV_\sigma(M,p,r,\triangle_v^u)$ *is solid.*

Proof. Let $x \in BV_\sigma(M,p,r,\triangle_v^u)$. This implies that

$$\sum_{m=1}^{\infty} \frac{1}{m^r}[M(\frac{|\phi_{m,n}(\triangle_v^u x)|}{\rho})]^{p_k} < \infty.$$

Let α_k be a sequence of scalars such that $|\alpha_k| \leq 1$ for all $m \in \mathbb{N}$. Then the result follows from the following inequality.

$$\sum_{m=1}^{\infty} \frac{1}{m^r}[M(\frac{|\alpha_k \phi_{m,n}(\triangle_v^u x)|}{\rho})]^{p_k} \leq \sum_{m=1}^{\infty} \frac{1}{m^r}[M(\frac{|\phi_{m,n}(\triangle_v^u x)|}{\rho})]^{p_k} < \infty.$$

Hence $\alpha x \in BV_\sigma(M,p,r,\triangle_v^u)$ for all sequence of scalars (α_k) with $|\alpha_k| \leq 1$ for all $m \in \mathbb{N}$ whenever $x \in BV_\sigma(M,p,r,\triangle_v^u)$.

Corollary 2.6. *The sequence space* $BV_\sigma(M,p,r,\triangle_v^u)$ *is monotone.*

Theorem 2.7. *Let* M_1, M_2 *be Orlicz function satisfying* $\triangle_2$ *condition and*
$r, r_1, r_2 \geq 0$. *Then we have*

(a) If $r > 1$ *then* $BV_\sigma(M_1,p,r,\triangle_v^u) \subseteq BV_\sigma(M0M_1,p,r,\triangle_v^u)$,

(b) $BV_\sigma(M_1,p,r,\triangle_v^u) \cap BV_\sigma(M_2,p,r,\triangle_v^u) \subseteq BV_\sigma(M_1+M_2,p,r,\triangle_v^u)$,

(c) If $r_1 \leq r_2$ *then* $BV_\sigma(M,p,r_1,\triangle_v^u) \subseteq BV_\sigma(M,p,r_2,\triangle_v^u)$.

Proof. (a) Since M is continuous at 0 from right, for $\varepsilon > 0$ there exists $0 < \delta < 1$ such that $0 \leq c \leq \delta$ implies $M(c) < \varepsilon$.

If we define

$$I_1 = \{m \in \mathbb{N} : M_1(\frac{|\phi_{m,n}(\triangle_v^u x)|}{\rho}) \leq \delta \text{ for some } \rho > 0,$$

$$I_2 = \{m \in \mathbb{N} : M_1(\frac{|\phi_{m,n}(\triangle_v^u x)|}{\rho}) > \delta \text{ for some } \rho > 0,$$

when

$$M_1(\frac{|\phi_{m,n}(\triangle_v^u x)|}{\rho}) > \delta$$

we get

$$M(M_1(\frac{|\phi_{m,n}(\triangle_v^u x)|}{\rho})) \leq \{\frac{2M_1}{\delta}\}M_1(\frac{|\phi_{m,n}(\triangle_v^u x)|}{\rho}).$$

Hence for $x \in BV_\sigma(M_1, p, r, \triangle_v^u)$ and $r > 1$

$$\sum_{m=1}^{\infty} \frac{1}{m^r}[M0M_1(\frac{|\phi_{m,n}(\triangle_v^u x)|}{\rho})]^{p_k} = \sum_{m\in I_1} \frac{1}{m^r}[M0M_1(\frac{|\phi_{m,n}(\triangle_v^u x)|}{\rho})]^{p_k} + \sum_{m\in I_2} \frac{1}{m^r}[M0M_1(\frac{|\phi_{m,n}(\triangle_v^u x)|}{\rho})]^{p_k}.$$

$$\sum_{m=1}^{\infty} \frac{1}{m^r}[M0M_1(\frac{|\phi_{m,n}(\triangle_v^u x)|}{\rho})]^{p_k} \leq max(\varepsilon^h, \varepsilon^H) \sum_{m=1}^{\infty} \frac{1}{m^r} + max(\{\frac{2M_1}{\delta}\}^h, \{\frac{2M_1}{\delta}\}^H)$$

where $0 < h = \inf p_k \leq p_k \leq H = \sup_k p_k < \infty$.

(b) The proof follows from the following inequality

$$\frac{1}{m^r}[(M_1+M_2)(\frac{|\phi_{m,n}(\triangle_v^u x)|}{\rho})]^{p_k} \leq C\frac{1}{m^r}[M_1(\frac{|\phi_{m,n}(\triangle_v^u x)|}{\rho})]^{p_k} + C\frac{1}{m^r}[M_2(\frac{|\phi_{m,n}(\triangle_v^u x)|}{\rho})]^{p_k}.$$

(c) The proof is straightforward.

Corollary 2.8. *Let M be an Orlicz function satisfying $\triangle_2$ condition. Then we have*

(a) If $r > 1$ then $BV_\sigma(p, r, \triangle_v^u) \subseteq BV_\sigma(M, p, r, \triangle_v^u)$,

(b) $BV_\sigma(M, p, \triangle_v^u) \subseteq BV_\sigma(M, p, r, \triangle_v^u)$,

(c) $BV_\sigma(p, \triangle_v^u) \subseteq BV_\sigma(p, r, \triangle_v^u)$,

(d) $BV_\sigma(M, \triangle_v^u) \subseteq BV_\sigma(M, r, \triangle_v^u)$.

Proof. The proof is straightforward.

Conflict of Interests

The author declares that there is no conflict of interests.

REFERENCES

[1] Z.U. Ahmad, M.Mursaleen,An application of Banach limits, Proc. Amer. Math. soc. 103 (1983), 244-246.

[2] S.T. Chen, Geometry of Orlicz spaces, Dissertation Math. The Institute of Mathematics, Polish Academy of Sciences 1996.

[3] A. Esi, M. Isik, Some generalized difference sequence spaces, Thai. J. Math. 3 (2005), 241-247.

[4] M. Et, On some difference sequence spaces, Dogra-Tr. J. Math. 17 (1993), 18-24.

[5] M. Et, R. Colak, On some generalized difference sequence spaces, Soochow J. Math 21 (1995), 377-386.

[6] M. Et, A. Esi, On Köthe-Toeplitz duals of generalised difference sequence spaces, Bull. Malaysian Math. Sci. Soc. 23 (2000), 25-32.

[7] H. Kizmaz, On Certain sequence spaces, Canad. Math. Bull. 24 (1981), 169-176.

[8] J. Lindenstrauss, L. Tzafriri, On Orlicz sequence spaces, Israel J. Math. 101 (1971), 379-390.

[9] C.G. Lorentz, A contribution to the theoty of divergent series, Acta Math. 80 (1948), 167-190.

[10] I.J. Maddox, Elements of Functional Analysis, Cambridge University Press 1970.

[11] I.J. Maddox, Some properties of paranormed sequence spaces, J. London. Math. Soc. 1 (1969), 316-322.

[12] M. Mursaleen, Matrix transformation between some new sequence spaces, Houston J. Math. 9 (1983), 505-509.

[13] M. Mursaleen, On some new invariant matrix methods of summability, Quart. J. Math. Oxford, 34 (1983), 77-86.

[14] J. Musielak, Orlicz spaces and Modular spaces, Lecture notees in Math. Springer-Verlag , 1983.

[15] R.A. Raimi, Invariant means and invariant matrix methods of summability, Duke J. Math. 30 (1963), 81-94.

[16] P. Schafer, Infinite matrices and Invariant means, Proc. Amer. Math. Soc. 36 (1972), 104-110.

[17] V. A.Khan. On a new sequence space defined by Orlicz Functions, Commun. Fac. Sci. Univ. Ank. Series A1. 57 (2008), 25-33.

[18] V.A. Khan, K. Ebadullah, On a new difference sequence space of invariant means defined by Orlicz functions, Bull. Allahabad Math. Soc. 26 (2011),259-272.