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A NOTE ON THE CARTESIAN PRODUCT OF ISOTONIC SPACES

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Abstract. A closure operator may be defined on the Cartesian product of two sets $\boldsymbol{X}$ and $\boldsymbol{Y}$. It is convenient when the closure operators defined separately on $\boldsymbol{X}$ and on $\boldsymbol{Y}$ have the same Kuratowski properties. This is a note on the extension of topological product spaces onto isotonic product spaces.

Keywords: closure spaces; isotonic spaces; product spaces; Cartesian product.

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1. Introduction

In an attempt to extend the boundaries of topology, it has been shown that topological spaces do not constitute a natural boundary for the validity of theorems and results in topology (Mashhour and Ghanim, 1982). Many results therefore, can be extended to closure spaces in which some of the basic axioms in such a space have been dropped. On the other hand, almost all approaches to extend the framework of topology, including Hammer (1964), at least use the closure function with the assumption that it is isotonic. Consequently, many properties which hold in basic topological spaces hold in spaces possessing the isotonic property.

Some meaningful topological concepts that can be defined on a set X endowed with an arbitrary set-valued set-function interpreted as a generalized closure operator were explored by Stadler and Stadler (2002). Here, the importance of the isotony property in extending the framework of topology is emphasized.

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2. Literature review

2.1 Closure operator and generalized closure space

A closure operator is an arbitrary set-valued, set-function $cl: \mathcal{P}(X) \to \mathcal{P}(X)$, where $\mathcal{P}(X)$ is the power set of a non-void set X that satisfies some closure axioms (Thron, 1981). Consequently, various combinations of the following axioms have been used in the past in an attempt to define closure operators (Sunitha, 1994). Let $A, B \subset \mathcal{P}(X)$.

i. Grounded: $cl(\emptyset) = \emptyset$
ii. Expansive: $A \subset cl(A)$
iii. Sub-additive: $cl(A \cup B) \subset cl(A) \cup cl(B)$. This axiom implies the Isotony axiom: $A \subset B$ implies $cl(A) \subset cl(B)$
iv. Idempotent: $cl(cl(A)) = cl(A)$

The structure (X, cl), where cl satisfies the first three axioms is called a closure space. If in addition the idempotent axiom is satisfied, then the structure is a topological space.

The grounded, expansive and isotonic axioms are hereditary in closure spaces (Habil, 2009). This means that if a closure space X has these axioms defined on it, and A is a subset of X, then A with the subspace topology also has these properties.

2.2 Interpretation of the closure operator

There are at least two possible interpretations of the closure operator (Mashhour and Ghanim, 1982). The first is to think of the closure operator with respect to a given family of sets, such as the class $\mathcal{F}$ of closed sets. This interpretation leads to $cl(A) = \cap \{A \subset F: F \in \mathcal{F}\}$. An operator so defined must satisfy the expansive, isotony and idempotent axioms. It need not satisfy the grounded and sub-additive axioms.

The second approach involves defining $cl(A)$ in terms of the cluster points of the set A, such that $cl(A) = A \cup \{x: x \text{ is a cluster point of } A\}$. Clearly, this operator must satisfy the expansive axiom and any other axioms that depend on the properties assigned to the cluster points (Mashhour and Ghanim, 1982).

2.3 Isotonic space

A closure space (X, cl) satisfying only the grounded and the isotony closure axioms is called an isotonic space (Elzenati and Habil, 2006). This is the space of interest in this study and clearly, it is more general than a closure space.

In a dual formulation, a space (X, cl) is isotonic if and only if the interior function $int: \mathcal{P}(X) \to \mathcal{P}(X)$ satisfies;

i. $int(X) = X$.

ii. $A \subseteq B \subseteq X$ implies $int(A) \subseteq int(B)$.

Clearly, if $A \subset X$ where X is an isotonic space, then from section 2.2.1, A is an isotonic space in the subspace topology. This result will be very integral especially in instances where the heredity of a property needs to be investigated.

2.4 Product spaces

A topological structure can be defined on the Cartesian product of topological spaces in some natural and useful way (Willard, 1970).

Let X_α be a set for each $\alpha \in I$. Then the Cartesian product of the sets X_α is the set $\prod_{\alpha \in I} X_\alpha = \{x: I \to \bigcup_{\alpha \in I} X_\alpha \mid x(\alpha) \in X_\alpha \ \ \forall \alpha \in I\}$. Thus, $\prod X_\alpha$is a set of functions defined on the indexing set I (Willard, 1970).

The map $\pi_\beta: \prod X_\alpha \to X_\beta$ defined by $\pi_\beta(x) = x_\beta$ is called the projection map of $\prod X_\alpha$ on X_β or simply the βth projection map.

2.5 Product topology

The product topology is precisely that topology which has for a subbase, the collection $\{\pi_\alpha^{-1}(U_\alpha): \alpha \in I, U_\alpha$is open in $X_\alpha\}$. The sets U_α can be restricted to come from some fixed base (in this case, subbase) in X_α (Joshi, 1983).

3. Results

3.1 Cartesian product of isotonic spaces

Let X and Y be two isotonic spaces and let $cl_X: \mathcal{P}(X) \to \mathcal{P}(X)$ and $cl_Y: \mathcal{P}(Y) \to \mathcal{P}(Y)$ be the closure operators defined on X and Y respectively. If $X \times Y$ is the Cartesian product of the underlying sets X and Y, then it is possible to define a closure operator on $X \times Y$ by;

$cl: \mathcal{P}(X \times Y) \to \mathcal{P}(X \times Y)$, where $cl = cl_X \times cl_Y$ such that $cl(\emptyset) = \emptyset$ and $(\mu) \subseteq cl(\varphi) \ \forall \ \mu \subseteq \varphi$.

It follows that this closure operator will inherit the same axioms as those possessed by the operators in both X and Y. Therefore, since X and Y are isotonic spaces, then the closure operator defined on $X \times Y$ also has the grounded and isotonic closure axioms.

3.2 Definition

As in topological spaces, a projection may be defined from the isotonic space $X \times Y$ to the coordinate space X or Y as follows;

Let $p: (X \times Y, cl) \rightarrow X$ be a continuous function from $(X \times Y, cl)$ onto (X, cl). Then for every $A \in X \times Y$, $p(A) \in X$. That is p maps the sets in $(X \times Y, cl)$ to sets in (X, cl). p is called the projection of $(X \times Y, cl)$ onto (X, cl).

3.3 Theorem

A function $f: X \rightarrow X_1 \times X_2$ is continuous if and only if $p_1 of$ is continuous where p_1 is the projection $p_1: X_1 \times X_2 \rightarrow X_1$.

Proof:

Let f be continuous, then $p_1 of$ is always continuous since every projection p is continuous and the composition of two continuous functions is continuous as well.

Conversely, let $p_1 of$ be continuous and $U = int(U) \in \mathcal{N}(X)$. Then, $(p_1 of\ )^{-1}(U) = f^{-1}\left(p_1^{-1}(U)\right)$. But $p_1^{-1}(U) = p_1^{-1}(int(U))$ is open since p_1^{-1} is continuous. Therefore $f^{-1}\left(p_1^{-1}(U)\right)$ is open and hence f is continuous.

3.4 Perfect mappings and Cartesian product

A perfect mapping in general topological spaces may be defined as a projection of the product of a compact space and a Hausdorff space onto the Hausdorff space. This characterization can also be extended to isotonic spaces in the theorem below without requiring a lot of modifications.

3.4.1 Theorem

If X is a c-compact isotonic space and Y is a T_2-isotonic space, then the projection $p: X \times Y \rightarrow Y$ is perfect.

From this theorem, it follows that if a topological property $\wp$ is an inverse invariant of perfect mappings, then the Cartesian product $X \times Y$ of a c-compact isotone space X and a T_2-isotonic space Y which has property $\wp$, also has the property $\wp$. According to Engelking (1989), theorems on inverse invariants under perfect mappings are generalizations of corresponding theorems on Cartesian products.

Conclusion

This paper dealt with the product of a pair of sets endowed with a closure operator that is both grounded and isotonic. It turns out that the properties of the closure operator defined on the coordinate spaces are induced onto their product. Further, the idea of a projection is intuitively induced from the Cartesian product to the coordinate space.

Conflict of Interests

The authors declare that there is no conflict of interests.

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