Available online at http://scik.org
J. Math. Comput. Sci. 2 (2012), No. 4, 982-998
ISSN: 1927-5307

SEMI-INVARIANT SUBMANIFOLDS OF KENMOTSU MANIFOLD IMMERSED IN AN ALMOST r-CONTACT STRUCTURE ADMITTING A QUARTER-SYMMETRIC NON-METRIC CONNECTION

MOBIN AHMAD, MOHD DANISH SIDDIQI* AND JANARDAN PARSAD OJHA

Integral University Department of Mathematics Kursi Road, Lucknow-226026, India

Abstract. We consider a Kenmotsu manifold immersed in almost r-contact manifold admitting a quarter- symmetric non-metric connection and study semi-invariant submanifolds of an almost r-contact Kenmotsu manifold immersed in almost r-contact Riemannian manifold endowed with a quarter- symmetric non- metric connection. We also discuss the integrability conditions of distribution on Kenmotsu manifold.

Keywords: Semi-invariant submanifolds, almost r-contact manifold, quarter- symmetric non- metric connection, integrability conditions.

2000 AMS Subject Classification: 53D12; 53C05

1. Introduction

Aurel Bejancu [5] introduced the notion of semi-invariant or contact CR-submanifolds [6], as a generalization of invariant and anti-invariant submanifolds of an almost contact-metric manifold and was followed by Several authors ([1], [9], [11], [13], [14], [15]). Kenmotsu manifold immersed a generalized almost r-contact structure was first defined by

*Corresponding author

E-mail addresses: mobinahmad@rediffmail.com (M. Ahmad), anallali@yahoo.com (M. Siddiqi), janardanojha@gmail.com (J. Ojha)

Received January 24, 2012

R.Nivas [12]. In [1], M.Ahmad and Jun, J.B. studied semi-invariant submanifolds of n-early Kenmotsu manifolds with semi-symmetric non-metric connection.In this paper, we study semi-invariant subamnifolds of Kenmotsu manifold immersed in a generalized almost r-contact structure admitting a quarter symmetric non-metric connection.

Let ∇ be a linear connection in an n-dimensional differentiable manifold M. The torsion tensor T and curvature tensor R of ∇ are given respectively by

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y],$$
$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

The connection ∇ is symmetric if its torsion tensor T vanishes, otherwise it is non-symmetric. The connection ∇ is metric connection if there is a Riemannian metric g in M such that $\nabla g = 0$, otherwise it is non-metric. It is well known that a linear connection is symmetric and metric if and only if it is the Levi-Civita connection. In [8], S. Golab introduced the idea of a quarter symmetric connection. A linear connection is said to be a quarter symmetric connection if its torsion tensor T is of the form

$$T(X,Y) = \eta(Y)\phi X - \eta(X)\phi Y.$$

Some properties of quarter symmetric non-metric connection was studied by several authors in ([3], [4], [7], [8], [11]).

This paper is organized as, we study quarter symmetric non-metric connection in a semi-invariant submanifold of Kenmotsu manifold with generalized r-contact structure .We consider semi-invariant submanifold of Kenmotsu manifold of generalised r-contact struture endowed with a quarter- symmetric non-metric connection. We obtain Gauss and Wiengarten equation for semi-invariant submanifolds of Kenmotsu manifild with generalized r-contact structure with quarter-symmertic symmetric non metric connection. Certain intresting results have been obtained.

2. Preliminaries.

Let $\bar{M}$ be an $(2n+r)$-dimensional Kenmotsu manifold with generalized almost r-contact structure (ϕ, ξ_p, η_p, g) where ϕ is a tensor field of type (1, 1), ξ_p are r-vector fields, η_p are r 1-forms and g the associated Riemannian metric on $\bar{M}$, satisfying.

(a) $\phi^2 = a^2 I + \sum_{p=1}^{r} \eta_p \otimes \xi_p$,

(b) $\eta_p(\xi_p) = \delta_{pq}$, $p, q, \in (r) := 1, 2, 3....., r$, (2.1)

(c) $\phi(\xi_p) = 0$, $p \in (r)$,

(d) $\eta_p(\phi X) = 0$

and

$$g(\phi X, \phi Y) + a^2 g(X, Y) + \sum_{p=1}^{r} \eta_p(X)\eta_p(Y) = 0, \tag{2.2}$$

$$\eta_p(X) = g(X, \xi_p), \tag{2.3}$$

$$(\bar{\bar{\nabla}}_X \phi)Y = -\sum_{p=1}^{r} \eta_p(Y)\phi X - g(X, \phi Y)\sum_{p=1}^{r} \xi_p, \tag{2.4}$$

$$\bar{\bar{\nabla}}_X \xi_p = X - \sum_{p=1}^{r} \eta_p(X)\xi_p, \tag{2.5}$$

where I is the identity tensor field and X, Y are vector fields on $\bar{M}$ and $\bar{\bar{\nabla}}$ denotes the Riemannian connection.

An n-dimensional Riemannian submanifold M of a Kenmotsu manifold with almost r-contact structure $\bar{M}$ is called a semi-invariant submanifold if ξ_p is tangent to M and there exists on M a pair of orthogonal distributions $(D, D^{\perp})$ such that [1]

(i) $TM = D \oplus D^{\perp} \oplus \xi$,

(ii) distribution D is invariant under ϕ, that is $\phi D_x = D_x$ for all $x \in M$,

(iii) distribution $D^{\perp}$ is anti-invariant uunder ϕ, that is $\phi D_x^{\perp} \subset T_x^{\perp} M$ for all $x \in M$, where $T_x M$ and $T_x^{\perp}$ are respectively the tangent and normal space of M at x.

The distribution D(resp. $D^{\perp}$) can be defined by projection P(resp. Q) which satisfies the conditions

$$P^2 = P,\ Q^2 = Q,\ PQ = QP = 0. \tag{2.6}$$

The distribution D(resp. $D^{\perp}$) is called the *horizontal*(resp. *vertical*) distribution.
A semi-invariant submanifold M is said to be an invariant (resp. *anti-invariant*) submanifold if we have $D^{\perp}_x = \{0\}$(resp.$D_x = \{0\}$) for each $x \in M$, we also called M is proper if neither D nor $D^{\perp}$ is null. It is easy to check that each hypersurface of M which is tangent to ξ_p inherits a structure of the semi-invariant submanifold of $\bar{M}$.

Now we define a quarter-symmetric non-metric connection $\bar{\nabla}$ in a kenmotsu manifold with generalized almost r-contact structure

$$\bar{\nabla}_X Y = \bar{\bar{\nabla}}_X Y + \eta_p(Y)\phi X \tag{2.7}$$

$$\text{such that } (\bar{\nabla}_X g) = -\eta_p(Y)g(X,Z) - \eta_p(Z)g(X,Y)$$

for any X and $Y \in TM$, where $\bar{\bar{\nabla}}$is the induced connection on M.
From (2.4) and (2.7), we have

$$(\bar{\nabla}_X \phi)Y = -\sum_{p=1}^{r}\eta_p(Y)\phi X - g(X,\phi X)\sum_{p=1}^{r}\xi_p$$

$$-a^2\sum_{p=1}^{r}\eta_p(Y)X - \sum_{p=1}^{r}\eta_p(Y)\eta_p(X)\xi_p. \tag{2.8}$$

We denote by g the metric tensor of $\bar{M}$ as well as that is induced on M. Let $\bar{\nabla}$ be the quarter-symmetric non-metric connection on $\bar{M}$ and ∇ be the induced connection M with respect to the unit normal N.

Theorem 3.2. The connection induced on the semi-invariant submanifolds of a generalized Kenmotsu manifold with quarter-symmetric non-metric connection is also a quarter-symmetric non-metric connection.

Proof. Let ∇ be the induced connection with respect to unit normal N on semi-invariant submanifolds of a generalized Kenmotsu manifold with semi-symmetric non-metric connection $\bar{\nabla}$. Then

$$\bar{\nabla}_X Y = \nabla_X Y + m(X, Y),$$

where m is a tensor field of type (0,2) on semi-invariant submanifold M . If ∇^* be the induced connection on semi-invariant submanifolds from Riemannian connection $\bar{\bar{\nabla}}$, then

$$\bar{\bar{\nabla}}_X Y = \nabla^*_X Y + h(X, Y),$$

where h is second fundamental tensor. Now using (3.2), we have

$$\nabla_X Y + m(X, Y) = \nabla^*_X Y + h(X, Y) + \eta_p(Y)\phi X.$$

Equating the tangential and normal components from the both sides in the above equation, we get

$$h(X, Y) = m(X, Y)$$

and

$$\nabla_X Y = \nabla^*_X Y + \eta_p(Y)\phi X.$$

Thus ∇ is also a semi-symmetric non-metric connection.

Now, the Gauss formula for semi-invariant submanifold of a generalized Kenmotsu manifold with quarter-symmetric non-metric connection is

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y) \tag{2.9}$$

and the Weingarten formulas for M is given by

$$\bar{\nabla}_X N = -A_N X + \eta_p(N)\phi X + \nabla^\perp_X N \tag{2.10}$$

for $X,Y \in TM$, $N \in T^\perp M$, where h and A are called the second fundamental tensors of M in and $\nabla^\perp$ denotes the operator of the normal connection. Moreover, we have

(2.11). $$g(h(X, Y), N) = g(A_N X, Y)$$

Any vector X tangent to M is given as

$$X = PX + QX + \eta_p(X)\xi_P, \tag{2.12}$$

where PX and QX belong to the distribution D and $D^{\perp}$ respectively.
For any vector field N normal to M, we get

$$\phi N = BN + CN, \tag{2.13}$$

where BN(resp. CN) denotes the tangential (resp. normal) component of ϕN.

3. Semi-invariant submanifolds

Lemma 3.1. Let M be a semi-invariant submanifold of a generalized Kenmotsu manifold with a quarter-symmetric non-metric connection. Then we have

$$2(\bar{\nabla}_X\phi)Y = \nabla_X\phi Y - \nabla_Y\phi X + h(X,\phi Y) - h(Y,\phi X) - \phi[X,Y]$$

Proof. By the Gauss formula, we have

$$\bar{\nabla}_X\phi Y - \bar{\nabla}_Y\phi X = \nabla_X\phi Y - \nabla_Y\phi X + h(X,\phi Y) - h(Y,\phi X) \tag{3.1}$$

Also by use of (2.9), the covariant differentiation yields

$$\bar{\nabla}_X\phi Y - \bar{\nabla}_Y\phi X = (\bar{\nabla}_X\phi)Y - (\bar{\nabla}_Y\phi)X + \phi[X,Y]. \tag{3.2}$$

From (3.1) and (3.2) we get

$$(\bar{\nabla}_X\phi)Y - (\bar{\nabla}_Y\phi)X = \nabla_X\phi Y - \nabla_Y\phi X + h(X,\phi Y) - h(Y,\phi X) - \phi[X,Y]. \tag{3.3}$$

Using $\eta_p(X) = 0$ for each $X \in D$ in (2.8), we get

$$(\bar{\nabla}_X\phi)Y + (\bar{\nabla}_Y\phi)X = 0. \tag{3.4}$$

Adding (3.3) and (3.4) we get the result.
Similar computations also yields the following.

Lemma 3.2. Let M be a semi-invariant submanifold of a generalized Kenmotsu manifold $\bar{M}$ with a quarter-symmetric non-metric connection. Then we have

$$2(\bar{\nabla}_X\phi)Y = -A_{\phi Y}X + \nabla^{\perp}_X\phi Y - \nabla_Y\phi X - h(Y,\phi X) - \phi[X,Y]$$

for any $X \in D$ and $Y \in D^{\perp}$.

Lemma 3.3. Let M be a semi-invariant submanifold of generalized Kenmotsu manifold $\bar{M}$ with a quarter-symmetric non-metric connection. Then we have

$$P\nabla_X \phi PY - PA_{\phi QY}X = \phi P\nabla_X Y + g(\phi X, Y)P\sum_{p=1}^{r}\xi_p - \sum_{p=1}^{r}\eta_p(Y)\phi PX$$

$$-a^2\left(\sum_{p=1}^{r}\eta_p(Y)PX + P\sum_{p=1}^{r}\eta_p(Y)\eta_p(X)\xi_p\right) - P\sum_{p=1}^{r}\eta_p(Y)\eta_p(X)\xi_p, \tag{3.5}$$

$$Q\nabla_X \phi PY - QA_{\phi QY}X = Bh(X,Y) + g(\phi X, Y)Q\sum_{p=1}^{r}\xi_p - \sum_{p=1}^{r}\eta_p(Y)\phi QX$$

$$-a^2\left(\sum_{p=1}^{r}\eta_p(Y)QX + Q\sum_{p=1}^{r}\eta_p(Y)\eta_p(X)\xi_p\right) - Q\sum_{p=1}^{r}\eta_p(Y)\eta_p(X)\xi_p, \tag{3.6}$$

$$h(X, \phi PY) + \nabla^{\perp}_X \phi QY = \phi Q\nabla_X Y + Ch(X,Y) \tag{3.7}$$

$$\sum_{p=1}^{r}\eta_p\left(\nabla_X \phi PY - A_{\phi QY}X\right) + a^2\sum_{p=1}^{r}\eta_p(Y)\eta_p(X) = 0 \tag{3.8}$$

for all X and $Y \in TM$.

Proof. We know that

$$(\bar{\nabla}_X\phi)Y = -\sum_{p=1}^{r}\eta_p(Y)\phi X - g(X,\phi X)\sum_{p=1}^{r}\xi_p - a^2\sum_{p=1}^{r}\eta_p(Y)X$$

$$-\sum_{p=1}^{r}\eta_p(Y)\eta_p(X)\xi_p.$$

Using (2.12), we get

$$(\bar{\nabla}_X\phi)Y = g(\phi X, Y)(P\sum_{p=1}^{r}\xi_p + Q\sum_{p=1}^{r}\xi_p) - \sum_{p=1}^{r}\eta_p(Y)(\phi PX + \phi QX)$$

$$-a^2\sum_{p=1}^{r}\eta_p(Y)(PX + QX + \eta(X)\xi_p) - \sum_{p=1}^{r}\eta_p(Y)\eta_p(X)(P\sum_{p=1}^{r}\xi_p + Q\sum_{p=1}^{r}\xi_p). \tag{3.9}$$

We know that

$$(\bar{\nabla}_X\phi)Y = \bar{\nabla}_X\phi Y - \phi(\bar{\nabla}_X Y).$$

Using (2.9) and (2.12), the above equation takes the form

$$(\bar{\nabla}_X\phi)Y = \bar{\nabla}_X\phi PY + \bar{\nabla}_X\phi QY - \phi\nabla_X Y - \phi h(X,Y).$$

By using the Gauss and Weingarten formulae and (2.13), we get

$$(\bar{\nabla}_X\phi)Y = \nabla_X\phi PY + h(X,\phi PY) - A_{\phi QY}X + \nabla^{\perp}_X\phi QY$$

$$-\phi P\nabla_X Y - \phi Q\nabla_X Y - Bh(X,Y) - Ch(X,Y)$$

$$= P\nabla_X\phi PY + Q\nabla_X\phi PY + \eta_p(\nabla_X\phi PY)\xi_p + h(X,\phi PY)$$

$$-PA_{\phi QY}X - QA_{\phi QY}X - \eta_p(A_{\phi QY}X)\xi_p + \nabla^{\perp}_X\phi QY$$

$$-\phi P\nabla_X Y - \phi Q\nabla_X Y - Bh(X,Y) - Ch(X,Y). \tag{3.10}$$

Comparing (3.9) and (3.10) and equating the horizontal, vertical and normal components, we get (3.5), (3.6), (3.7) and (3.8) respectively.

Definition 3.4. The horizontall distribution D is said to be parallel with respect to the connection ∇ on M, if $\nabla_X Y \in D$ for all vector fields X and $Y \in D$.

Theorem 3.5. Let M be a semi-invariant submanifold of generalized Kenmotsu manifold $\bar{M}$ with a quarter-symmetric non-metric connection. If M is ξ_p-horizontal, then the distribution D is integrable if and only if

$$h(X,\phi Y) = h(\phi X, Y) \tag{3.11}$$

for all X and $Y \in D$.

Proof. Let M be ξ_p-horizontal, then (4.7) reduces to

$$h(X,\phi Y) = \phi Q\nabla_X Y + Ch(X,Y)$$

and hence, we have

$$h(X,\phi Y) - h(\phi X, Y) = \phi Q[X,Y].$$

Thus If M is ξ_p horizontal then $[X,Y] \in D$ i,e $Q[X,Y] = 0$ if and only if

$$h(X,\phi Y) = h(\phi X, Y)$$

for all X, $Y \in D$.

Theorem 3.6. Let M be a semi-invariant submanifold of a generalized Kenmotsu manifold $\bar{M}$ with quarter-symmetric non-metric connection.If M is ξ_p-vertical then the distribution $D^{\perp}$ is integrable if and only if.

$$A_{\phi X}Y - A_{\phi Y}X = \sum_{p=1}^{r}\eta_p(X)\phi Y - \sum_{p=1}^{r}\eta_p(Y)\phi X + 2g(\phi X, Y)\sum_{p=1}^{r}\xi_p$$

$$+a^2\left(\sum_{p=1}^{r}\eta_p(X)Y - \sum_{p=1}^{r}\eta_p(Y)X\right). \tag{3.12}$$

Proof. Let M be ξ_p-vertical, then (3.7) reduces to

$$\nabla^{\perp}_X \phi Y = \phi Q \nabla_X Y = Ch(X, Y) \tag{3.13}$$

for all X, $Y \in D^{\perp}$.

Using (2.8) and (2.12), we get

$$\bar{\nabla}_X \phi Y = g(\phi X, Y)\sum_{p=1}^{r}\xi_p - \sum_{p=1}^{r}\eta_p(Y)\phi X - a^2\sum_{p=1}^{r}\eta_p(Y)X - \sum_{p=1}^{r}\eta_p(Y)\eta_p(X)\xi_p$$

$$+\phi P\nabla_X Y + \phi Q\nabla_X Y + Bh(X, Y) + Ch(X, Y). \tag{3.14}$$

Since M is ξ_p-vertical then by Wiengarten formula, we have

$$\bar{\nabla}_X \phi Y = -A_{\phi Y}X + \nabla^{\perp}_X \phi Y$$

$$\nabla^{\perp}_X \phi Y = \bar{\nabla}_X \phi Y + A_{\phi Y}X.$$

Using (3.14), we have

$$\nabla^{\perp}_X \phi Y = g(\phi X, Y)\sum_{p=1}^{r}\xi_p - \sum_{p=1}^{r}\eta_p(Y)\phi X - a^2\sum_{p=1}^{r}\eta_p(Y)X - \sum_{p=1}^{r}\eta_p(Y)\eta_p(X)\xi_p$$

$$+\phi P\nabla_X Y + \phi Q\nabla_X Y + Bh(X, Y) + Ch(X, Y) + A_{\phi Y}X. \tag{3.15}$$

From (3.12) and (3.14), we have

$$\phi P\nabla_X Y = \sum_{p=1}^{r}\eta_p(Y)\phi X + a^2\sum_{p=1}^{r}\eta_p(Y)X + \sum_{p=1}^{r}\eta_p(Y)\eta_p(X)\xi_p$$

$$-Bh(X, Y) - Ch(X, Y) - A_{\phi Y}X.$$

Similarly,

$$\phi P\nabla_Y X = \sum_{p=1}^{r}\eta_p(X)\phi Y + a^2\sum_{p=1}^{r}\eta_p(X)Y + \sum_{p=1}^{r}\eta_p(X)\eta_p(Y)\xi_p$$
$$-Bh(X,Y) - Ch(Y,X) - A_{\phi X}Y.$$

Therefore, we have

$$\phi P\nabla_X Y - \phi P\nabla_Y X = \sum_{p=1}^{r}\eta_p(Y)\phi X + a^2\sum_{p=1}^{r}\eta_p(Y)X - g(\phi X, Y)\sum_{p=1}^{r}\xi_p$$
$$+\sum_{p=1}^{r}\eta_p(Y)\eta_p(X)\xi_p - Bh(X,Y) - A_{\phi Y}X$$
$$-\sum_{p=1}^{r}\eta_p(X)\phi Y + a^2\sum_{p=1}^{r}\eta_p(X)Y + g(\phi Y, X)\sum_{p=1}^{r}\xi_p$$
$$-\sum_{p=1}^{r}\eta_p(Y)\eta_p(X)\xi_p + Bh(Y,X) + A_{\phi X}Y,$$
$$\phi P[X,Y] = \sum_{p=1}^{r}\eta_p(Y)\phi X - \sum_{p=1}^{r}\eta_p(X)\phi Y - 2g(\phi X, Y)\sum_{p=1}^{r}\xi_p$$
$$+a^2\sum_{p=1}^{r}\eta_p(Y)X - a^2\sum_{p=1}^{r}\eta_p(X)Y + A_{\phi X}Y - A_{\phi Y}X.$$

Thus if M is ξ_p-vertical, we seen that $[X,Y] \in D^{\perp}$ i,e $P[X,Y] = 0$ if and only if the equation (3.12) holds.

4. Parallel Horizontal Distributios

Definition 4.1. A non-zero normal vector field N is said to be D-parallel normal section if

$$\nabla^{\perp}_X N = 0 \text{ for all } X \in D. \tag{4.1}$$

Definiion 4.2. M is said to be totally r-contact Umbilical if there exist a normal vector H on M such that

$$h(X,Y) = g(\phi X, \phi Y)H + \sum_{p=1}^{r}\eta_p(X)h(Y,\xi_p) + \sum_{p=1}^{r}\eta_p(Y)h(X,\xi_p) \tag{4.2}$$

for all vector fields X, Y tangent to M [4].

If $H=0$, that is the fundamental form is given by

$$h(X,Y)=\sum_{p=1}^{r}\eta_p(X)h(Y,\xi_p)+\sum_{p=1}^{r}\eta_p(Y)h(X,\xi_p). \tag{4.3}$$

Then M is totally r-contact geodesic.

Theorem 4.1. If M is totally r-contact umbilical semi-invariant submanifold of a generalized Kenmotsu manifold $\bar{M}$ with a quarter-symmetric non-metric connection with parallel horizontal distribution, then M is totally r-contact geodesic.

Proof. Since M is semi-invariant submanifold of a generalized Kenmotsu manifold $\bar{M}$ with quarter-symmetric non-metric connection. From (3.5) and (3.6), we have

$$P\nabla_X\phi PY-PA_{\phi QY}X=\phi P\nabla_XY+g(\phi X,Y)P\sum_{p=1}^{r}\xi_p-\sum_{p=1}^{r}\eta_p(Y)\phi PX$$

$$-a^2\sum_{p=1}^{r}\eta_p(Y)PX-P\sum_{p=1}^{r}\eta_p(Y)\eta_p(X)\xi_p,$$

$$Q\nabla_X\phi PY-QA_{\phi QY}X=Bh(X,Y)+g(\phi X,Y)Q\sum_{p=1}^{r}\xi_p-2\sum_{p=1}^{r}\eta_p(Y)\phi QX$$

$$-a^2\sum_{p=1}^{r}\eta_p(Y)QX-Q\sum_{p=1}^{r}\eta_p(Y)\eta_p(X)\xi_p.$$

Adding the above equations, we have

$$P\nabla_X\phi PY+Q\nabla_X\phi PY-(PA_{\phi QY}X+QA_{\phi QY}X)$$

$$=\phi P\nabla_XY+Bh(X,Y)+g(\phi X,Y)\sum_{p=1}^{r}\xi_p-\sum_{p=1}^{r}\eta_p(Y)\phi X$$

$$-a^2\sum_{p=1}^{r}\eta_p(Y)X-\sum_{p=1}^{r}\eta_p(Y)\eta_p(X)\xi_p. \tag{4.4}$$

Interchanging X and Y in (4.4), we have

$$\nabla_Y\phi PX-A_{\phi QX}Y=\phi P\nabla_YX+Bh(Y,X)+g(\phi Y,X)\sum_{p=1}^{r}\xi_p-\sum_{p=1}^{r}\eta_p(X)\phi Y$$

$$-a^2\sum_{p=1}^{r}\eta_p(X)Y-\sum_{p=1}^{r}\eta_p(X)\eta_p(Y)\xi_p. \tag{4.5}$$

Adding (4.4) and (4.5), we get

$$\begin{aligned}\nabla_X\phi PY+\nabla_Y\phi PX-A_{\phi QY}X-A_{\phi QX}Y&=\phi P\nabla_XY+\phi P\nabla_YX+2Bh(X,Y)\\ &-\sum_{p=1}^{r}\eta_p(Y)\phi X-\sum_{p=1}^{r}\eta_p(X)\phi Y\\ &-a^2\sum_{p=1}^{r}\eta_p(X)Y-a^2\sum_{p=1}^{r}\eta_p(Y)X-2\sum_{p=1}^{r}\eta_p(X)\eta_p(Y)\xi_p.\end{aligned}$$

Taking inner product with Z, we get

$$\begin{aligned}g\left(\nabla_X\phi PY+\nabla_Y\phi PX-A_{\phi QY}X-A_{\phi QX}Y,Z\right)&=g\left(\phi P\nabla_XY+\phi P\nabla_YX+2Bh(X,Y),Z\right)\\ &-\sum_{p=1}^{r}\eta_p(Y)g(\phi X,Z)-\sum_{p=1}^{r}\eta_p(X)g(\phi Y,Z)\\ &-a^2\sum_{p=1}^{r}\eta_p(Y)g(X,Z)-a^2\sum_{p=1}^{r}\eta_p(X)g(Y,Z)\\ &-2\sum_{p=1}^{r}\eta_p(Y)g(X,Z)\eta_p(X)g(Y,Z)\xi_p.\end{aligned}$$

Using (4.2), we get

$$g\left(\nabla_X\phi PY,Z\right)+g\left(\nabla_Y\phi PX,Z\right)-g(A_{\phi QY}X,Z)-g\left(A_{\phi QX}Y,Z\right)$$

$$\begin{aligned}&=g\left(\phi P\nabla_XY,Z\right)+g\left(\phi P\nabla_YX,Z\right)+g[(2B\{g(\phi X,\phi Y)H+\sum_{p=1}^{r}\eta_p(X)h(Y,\xi_p)\\ &+\sum_{p=1}^{r}\eta_p(Y)h(X,\xi_p)\},Z)]-\sum_{p=1}^{r}\eta_p(Y)g(\phi X,Z)-\sum_{p=1}^{r}\eta_p(X)g(\phi Y,Z)\\ &-a^2\sum_{p=1}^{r}\eta_p(Y)g(X,Z)-a^2\sum_{p=1}^{r}\eta_p(X)g(Y,Z)\\ &-2\sum_{p=1}^{r}\eta_p(Y)g(X,Z)\eta_p(X)g(Y,Z)\xi_p.\end{aligned}$$

$$g\left(\nabla_X\phi PY,Z\right)+g\left(\nabla_Y\phi PX,Z\right)-g(h(X,Z),\phi QY)-g(h(Y,Z),\phi QX)$$

$$\begin{aligned}&=g\left(\phi P\nabla_XY,Z\right)+g\left(\phi P\nabla_YX,Z\right)+2g(\phi X,\phi Y)g(BH,Z)\\ &+2\sum_{p=1}^{r}\eta_p(X)g(Bh(Y,\xi_p),Z)+2\sum_{p=1}^{r}\eta_p(Y)g(Bh(X,\xi_p),Z)\end{aligned}$$

$$-\sum_{p=1}^{r}\eta_p(Y)g(\phi X,Z)-\sum_{p=1}^{r}\eta_p(X)g(\phi Y,Z)$$

$$-a^2\sum_{p=1}^{r}\eta_p(Y)g(X,Z)-a^2\sum_{p=1}^{r}\eta_p(X)g(Y,Z)$$

$$-2\sum_{p=1}^{r}\eta_p(Y)g(X,Z)\eta_p(X)g(Y,Z)\xi_p$$

$$=g\left(\phi P\nabla_X Y,Z\right)+g\left(\phi P\nabla_Y X,Z\right)-2a^2g(X,Y)g(BH,Z)-2\sum_{p=1}^{r}\eta_p(X)\eta_p(Y)g(BH,Z)$$

$$+2\sum_{p=1}^{r}\eta_p(X)g(h(Y,\xi_p),\phi Z)+2\sum_{p=1}^{r}\eta_p(Y)g(h(X,\xi_p),\phi Z)$$

$$-\sum_{p=1}^{r}\eta_p(Y)g(\phi X,Z)-\sum_{p=1}^{r}\eta_p(X)g(\phi Y,Z)$$

$$-a^2\sum_{p=1}^{r}\eta_p(Y)g(X,Z)-a^2\sum_{p=1}^{r}\eta_p(X)g(Y,Z)$$

$$-2\sum_{p=1}^{r}\eta_p(Y)g(X,Z)\eta_p(X)g(Y,Z)\xi_p.$$

Replacing Y by BH and Z by X and using (4.2), we get (4.6) given below

$$g\left(\nabla_X\phi PBH,X\right)+g\left(\nabla_{BH}\phi PX,X\right)-g(X,X)g(H,\phi QBH)-g(BH,X)g(H,\phi QX)$$

$$=g\left(\phi P\nabla_X BH,X\right)+g\left(\phi P\nabla_{BH}X,X\right)-2a^2g(X,BH)g(BH,X)$$

$$-2\sum_{p=1}^{r}\eta_p(X)\eta_p(BH)g(BH,X)$$

$$+2\sum_{p=1}^{r}\eta_p(X)g(h(BH,\xi_p),\phi X)+2\sum_{p=1}^{r}\eta_p(BH)g(h(X,\xi_p),\phi X)$$

$$-\sum_{p=1}^{r}\eta_p(BH)g(\phi X,X)-\sum_{p=1}^{r}\eta_p(X)g(\phi BH,X)$$

$$-a^2\sum_{p=1}^{r}\eta_p(BH)g(X,X)-a^2\sum_{p=1}^{r}\eta_p(X)g(BH,X)$$

$$-2\sum_{p=1}^{r}\eta_p(Bh)g(X,X)\eta_p(X)g(BH,X)\xi_p. \tag{4.6}$$

For $X \in D$, we have.

$$g(X, BH) = g(\phi X, BH) = 0.$$

Differentiating above covariantly along X, we find

$$g(\nabla_X \phi X, BH) + g(\phi X, \nabla_X BH) = 0.$$

Since, the horizontal distribution D is parallel, we have

$$g(\phi X, \nabla_X BH) = 0. \tag{4.7}$$

Using (4.7) in (4.6) and taking X in D as a unit vector, we get

$$g\left(\nabla_{BH}\phi PX, X\right) - g(H, \phi QBH) = g(\phi P \nabla_{BH} X, X) - a^2 \sum_{p=1}^{r} \eta_p(BH),$$

$$g((\nabla_{BH}\phi P)X, X) + \phi P \nabla_{BH} X, X) - g(H, \phi QBH) = g(\phi P \nabla_{BH} X, X) - a^2 \sum_{p=1}^{r} \eta_p(BH),$$

$$g((\nabla_{BH}\phi P)X, X)) + g(\phi P \nabla_{BH} X, X) - g(H, \phi QBH) = g(\phi P \nabla_{BH} X, X) - a^2 \sum_{p=1}^{r} \eta_p(BH),$$

(4.8)

$$g((\nabla_{BH}\phi P)X, X)) = g(H, \phi QBH) = -g(\phi H, QBH) = -g(BH, QBH) - a^2 \sum_{p=1}^{r} \eta_p(BH).$$

Now

$$g((\nabla_{BH}\phi P)X, X)) = -a^2 \sum_{p=1}^{r} \eta_p(BH).$$

From (4.8), we have

$$g(BH, QBH) + a^2 \sum_{p=1}^{r} \eta_p(BH) = 0.$$

Now from (4.3) and (4.4), we have

$$BH = 0.$$

Since $\phi H \in D^{\perp}$, we have $CH = 0$, hence $\phi H = 0$, thus $H = 0$.

Hence M is totally r-contact geodesic.

Remark. For a generalized Kenmotsu manifold with quarter-symmetric non-metric connection, we have

$$\bar{\nabla}_X \xi_p = \nabla_X \xi_p + h(X, \xi_p)$$

$$= PX + QX - 2\sum_{p=1}^{r} \eta_p(X) - \sum_{p=1}^{r} \eta_p(X)\xi_p. \tag{4.9}$$

Equating the tangential and normal components, we have

$$\nabla_X \xi_p = PX - \sum_{p=1}^{r} \eta_p(X) P\xi_p, \tag{4.10}$$

$$h(X, \xi_p) = QX, \tag{4.11}$$

$$\eta_p(X)\xi_p = 0. \tag{4.12}$$

From (5.10) and (5.11), we can easily obtain

$$\nabla_X \xi_p = 0 \text{ for X} \in \text{D}^{\perp},$$

$$h(X, \xi_p) = 0 \text{ for X} \in \text{D}.$$

Also for $X \in D$, we have

$$g(A_N \xi_p, X) = g(h(X, \xi_p), N) = 0$$

and so we have $A_N \xi_p \in D^{\perp}$.

Theorem 4.2. Let M be D-umbilic(resp.$D^{\perp}$-umbilic) semi-invariant submanifold of a generalized Kenmotsu manifold$\bar{M}$ with quarter-symmetric non-metric connection. If ξ_p-horizontal(resp. ξ_p-vertical) then M is totally geodesic(resp. $D^{\perp} -$ totally geodesic).

Proof. If M is D-umbilic semi-invariant submanifold of a generalized Kenmotsu manifold with quarter-symmetric non-metric connection with ξ_p-horizontal then, we have

$$h(X, \xi_p) = g(X, \xi_p)L,$$

which means that $L = 0$ from which we get $h(X, \xi_p) = 0$.

Hence M is D-totally geodesic.

Similarly, we can prove that if M is a $D^{\perp}$-umbilic semi-invariant submanifold with ξ_p-vertical, then M is $D^{\perp}$-totally geodesic.

References

[1] Ahmad. M., On semi-invariant submanifolds of a nearly Kenmotsu manifold with a canonical semi symmetric semi-metric connection, Mathematicki Vesnik. 62 (2010), 189-198.

[2] Ahmad. M, Jun. J. B., On semi-invariant submanifolds of a nearly Kenmotsu manifold with semi symmetric non-metric connection, Journal of Chungcheong Math.Soc. 23(2010), no.2, 257-266.

[3] Ahmad. M., Ozgur. C and Haseeb. A., Hypersurfaces of almost r-paracontact Riemannian manifold endowed with quarter symmetric non- metric connection, Kyungpook Math. J. 49 (2009), 533-543.

[4] Ahmad. M., Jun. J. B., and Siddiqi. M. D., On Some properties of semi-invariant submanifolds of a nearly trans- Sasakian manifolds admitting a quarter symmetric non- metric connection", Journal of Chungchuong Mathematical society Vol. 25, (2012), no. 1, 73-90.

[5] Ali. S. and Nivas. R., On submanifolds of a manifold with quarter symmetric connection, Riv. Mat. Uni. Parma. (6) 3 (2000), 11-23.

[6] Bejancu. A., On semi-invariant submanifolds of an almost contact metric manifold, An. Stiint. Univ."AI.I.Cuza" Iasi Sect Ia Mat. 27(supplement)(1981),17-21.

[7] Bejancu. A., Geometry of CR-submanifolds,D.Reidel Publishing Company, Holland, 1986.

[8] Das. L.S., Ahmad. M and Haseeb, A. On semi-invariant submanifolds of a nearly Kenmotsu manifold with semi symmetric non-metric connection, Journal of Applied Analysis, vol. 17, (2011), no. 1, 119-130.

[9] Golab. S., On semi-symmetric and quarter-symmetric linear connections. Tensor 29 (1975), 249-254.

[10] Kobayashi. M., Semi-invariant submanifolds of a certain class of almost contact manifolds, Tensor 43(1986)28-36.

[11] Matsumoto. K., Shahid. M.H, and Mihai. I., Semi-invariant submanifolds of certain almost contact manifolds, Bull.Yamagata Univ.Natur.Sci. 13(1994), no.3,183-192.

[12] Nivas. R and Verma. G., On quarter-symmetric non-metric connection in a Riemannian manifold, J. Rajasthan Acad. Sci. 4 (2005), no. 1, 57-68.

[13] Nivas. R and Yadav. S., Semi-invariant submanifolds of a Kenmotsu manifold with generalized almost r-contact strucrure, J. T. S. 3(2009), 125-135.

[14] Papaghiuc. N., Semi-invariant subamnifolds in Kenmotsu manifolds, Rend.Mat.3(1983), 607-622.

[15] Shahid. M.H., On semi-invariant submanifolds of a nearly Sasakian manifold. Indian Journal of Pure and Applied Math. 94(10)(1993), 571-580.

[16] Tripathi. M. M. and Shukla. S.S., semi-invariant submanifolds of nearly Kenmotsu manifolds, Bull.Calcutta Math.Soc. 95(2003), no. 1, 17-30.