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SOME NEW FAMILIES OF FIFTH -ORDER METHODS FOR FINDING SIMPLE ZEROS OF NON-LINEAR EQUATIONS

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Abstract. In this paper, we developed some new families of fifth-order methods for solving simple zeros of non-linear equations. This new family of methods is constructed such that the convergence is of order five. Each member of the families requires two evaluations of the given function and two of its derivative per iteration. These methods have more advantages than Newtons method and other methods with the same convergence order, as shown in the illustration examples.

Keywords: Iterative methods; fifth order; Simple zeros of nonlinear equations; Newton method.

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1. Introduction

Iterative methods for finding the simple zeros of a nonlinear equation $f(x) = 0$, i.e., $f(\alpha) = 0$ and $f(\alpha) \neq 0$, have been studied by a number of authors [1-9]. In these methods new approximations to a zero of $f(x)$ is obtained by calculating $f(x)$, and possibly its derivatives at a number of values of the independent variable, at each step.
The classical Newtons method for a single non-linear equation is written as

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{1}$$

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The aim of this paper is to develop two new families of fifth-order methods for finding simple roots of non-linear equations by using the method of undetermined coefficients. In Section 2, we consider a general iterative scheme, then analyze it to present a family of fifth-order methods. in Section3, The formulae are tested and their performance is compared with some known methods. Lastly, conclusions are stated in the last section.

2. The methods and analysis of convergence

Consider the following fifth-order iterative scheme, defined in [6], as

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{5f'^2(x_n) + 3f'^2(y_n)}{f'^2(x_n) + 7f'^2(y_n)} . \frac{f(y_n)}{f'(x_n)}, \tag{2}$$

where

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)} \tag{3}$$

Throughout this paper, we assume that y_n is defined by (3).

We begin with the multipoint iteration scheme

$$x_{n+1} = x_n - H(u_n, v_n, w_n) \tag{4}$$

where $u_n = \frac{f(x_n)}{f'(x_n)}$,$v_n = \frac{f'(y_n)}{f'(x_n)}$ and $w_n = \frac{f(y_n)}{f'(x_n)}$.

It can be verified that, the method (2) is a special case of the iterative scheme (4), for $H(u, v, w) = u + \frac{5+3w^2}{1+7w^2} v$.

Our aim is to find the conditions for weigh function $H(u_n, v_n, w_n)$ such that the proposed scheme (4), may yield a fifth-order method.

Let $e_n = x_n - \alpha$ and $C_k = (1/k!) f^{(k)}(\alpha)/f'(\alpha)$ for $k \geqslant 2$. Considering the iteration function of (4):

$$F(x) = x - H(u(x), v(x), w(x))$$

Furthermore $u(x) = \frac{f(x)}{f'(x)}$,$v(x) = \frac{f'(x-u(x))}{f'(x)}$ and $w(x) = \frac{f(x-u(x))}{f'(x)}$.

For iterative scheme(4), in fact we have

$$x_{n+1} = F(x_n)$$

Now if we employ the symbolic computation of the Maple package to compute the Taylor expansion of $F(x)$ around α and taking into account that $f(\alpha) = 0$ and $f'(\alpha) \neq 0$, after the algebraic manipulation, we have

$$x_{n+1} = F(x_n) = \alpha + K_1\, e_n + K_2\, e_n^2 + K_3\, e_n^3 + K_4\, e_n^4 + O(e_n^5).$$

where

$$K_1 = 1 - H_u(0, 1, 0) + [2H_v(0, 1, 0)]\, C_2.$$

So, under the conditions

$$H_u(0,1,0) = 1, \tag{5.1}$$

$$H_v(0,1,0) = 0. \tag{5.2}$$

By using the conditions (5) in K_2, we have

$$K_2 = -\frac{1}{2}H_{uu}(0,1,0) + [1 - H_w(0,1,0) + 2H_{uv}(0,1,0)]\,C_2 - [2H_{vv}(0,1,0)]\,C_2^2$$

It can be verified that $K_2 = 0$, if we have

$$H_{uu}(0,1,0) = 0, \tag{6.1}$$

$$H_w(0,1,0) = 1 + 2H_{uv}(0,1,0), \tag{6.2}$$

$$H_{vv}(0,1,0) = 0, \tag{6.3}$$

Substituting of (5) and (6) into K_3, leads to

$$K_3 = \frac{4}{3}\,[H_{vvv}(0,1,0)]\,C_2^3 + 2\,[1 - H_{vw}(0,1,0) + H_{uvv}(0,1,0)]\,C_2^2 - [H_{uuv}(0,1,0) - H_{vw}(0,1,0)]\,C_2$$
$$- [H_{uv}(0,1,0)]\,C_3 - \frac{1}{6}H_{uuu}(0,1,0).$$

K_3 can be vanished, for

$$H_{vvv}(0,1,0) = 0, \tag{7.1}$$

$$H_{uv}(0,1,0) = 0, \tag{7.2}$$

$$H_{uvv}(0,1,0) = 1 + H_{vw}(0,1,0), \tag{7.3}$$

$$H_{uw}(0,1,0) = H_{uuv}(0,1,0), \tag{7.4}$$

$$H_{uuu}(0,1,0) = 0, \tag{7.5}$$

In a similar manner we obttain the conditions for $K_4 = 0$, as follows

$$H_{vvvv}(0,1,0) = 0, \tag{8.1}$$

$$H_{uw}(0,1,0) = -1, \tag{8.2}$$

$$H_{uuw}(0,1,0) = \frac{2}{3}H_{uuuv}(0,1,0), \tag{8.3}$$

$$H_{uuv}(0,1,0) = 0, \tag{8.4}$$

$$H_{ww}(0,1,0) = 4H_{uvw}(0,1,0) - 2H_{uuvv}(0,1,0), \tag{8.5}$$

$$H_{vvw}(0,1,0) = \frac{5}{2} + \frac{2}{3} + H_{uvvv}(0,1,0), \tag{8.6}$$

$$H_{uuuu}(0,1,0) = 0. \tag{8.7}$$

Indeed, we have proved the following convergence theorem for iterative scheme (4).

Theorem 2.1. *Let $f : I \to R$ denote a sufficiently differentiable function defined on I, where I is a neighbourhood of a simple zero α of $f(x)$. Then the iteration scheme (4) defines a family of fifth-order convergence if $H(u, v, w)$ satisfies the conditions (5)-(8).*

Many new methods can be obtained by considering the different function H.

Furthermore, the family (4), includes following known fifth-order methods as its particular cases. For example, it is not difficult to find the following functions satisfying conditions (5)-(8):

$$H_1(u,v,w) = u + \frac{3+v}{-1+5v}w$$

$$H_2(u,v,w) = u + \frac{5+3v^2}{1+7v^2}w$$

$$H_3(u,v,w) = u + \frac{6+v+v^2}{-1+7v+2v^2}w$$

For $H_1(u,v,w)$, scheme (4) turns into the method by Ham et al. [5].

For $H_2(u,v,w)$, scheme (4) turns into the method by Fang et al. (2)[6].

For $H_3(u,v,w)$, scheme (4) turns into the method by Fang et al. in [7].

In terms of computational cost, the developed methods (4), require evaluations of only two functions and two first-order derivatives per iteration. Consider the definition of efficiency index [10] as $\sqrt[r]{p}$, where p is the order of the method and r the number of function evaluations per iteration required by the method. The new methods have the efficiency index equal to $\sqrt[4]{5} \approx 1.5874$, which is better than the one of Newtons method $\sqrt{2} \approx 1.4142$.

3. Numerical examples

In this section, the results of the numerical tests are presented to compare the efficiency of the developed methods with that of the other fifth-order methods. The tested methods are the classical Newtons method (NM) in [1], the FLM method in [6] (2), the Chun et al. method (CM) [8] defined by

$$x_{n+1} = z_n - \frac{3f'(x_n) - f'(y_n)}{f'(x_n) + f'(y_n)} \frac{f(z_n)}{f'(x_n)}$$

where

$$z_n = x_n - \frac{f(x_n)}{2}\left(\frac{1}{f'(x_n)} + \frac{1}{f'(y_n)}\right)$$

the method of Kou et al. (KM)[9] defined by

$$x_{n+1} = u_{n+1} - \frac{f(u_{n+1})}{f'(y_n)}$$

where

$$u_{n+1} = x_n - \frac{2f(x_n)}{f'(x_n) + f'(y_n)}$$

with the new presented method in this contribution, Namely BGM, with $H(u,v,w) = u + \frac{3+v}{-1+5v}\sin(w)$ in(4)).

Numerical computations reported here have been carried out in the MAPLE with 128 digit floating point arithmetics (Digits =128). The stopping criterion is $|f(x_n)| < 10^{-15}$. The test functions are listed as follows

$$f_1(x) = x^3 + 4x^2 - 10,$$

$$f_2(x) = x^2 - e^x - 3x + 2,$$

$$f_3(x) = x\,e^{x^2} - \sin^2(x) + 3\cos(x) + 5,$$

$$f_4(x) = \sin(x)\,e^x + \ln(x^2+1),$$

$$f_5(x) = (x-1)^3 - 2,$$

$$f_6(x) = (x+2)\,e^x - 1,$$

$$f_7(x) = \sin^2(x) - x^2 + 1.$$

The computing results are displayed in Table 1, where $f(x)$ is for the test function, x_0 is the initial guess and NC means that the method does not converge to the zero of function. Obtained results in Table 1 confirm that the new methods have advantages over the Newtons method and the other compared fifth-order methods.

4. Conclusions

In this paper we have presented a family of fifth-order methods for solving equations. It is obvious that the proposed iterative family contains several well-known fifth-order methods as its special cases. Application of each method requires two evaluations of function f and two evaluations of its derivative f' per step. The numerical results show that proposed methods converge more quickly than the other compared methods.

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Table 1. Comparison of Number of iterations of various fifth-order convergent iterative methods

$f(x)$	x_0	NM	FLM	CM	KM	BGM
f_1	$x_0 = -0.3$	55	11	27	24	8
	$x_0 = 1$	6	3	4	4	3
f_2	$x_0 = 0$	5	3	3	3	2
	$x_0 = 1$	5	3	4	4	3
f_3	$x_0 = -1$	6	4	4	4	3
	$x_0 = -2$	9	5	5	NC	6
f_4	$x_0 = 2$	6	4	4	4	4
	$x_0 = -5$	8	4	6	4	5
f_5	$x_0 = 3$	7	4	4	4	3
	$x_0 = 4$	8	4	5	4	4
f_6	$x_0 = 2$	9	5	5	NC	5
	$x_0 = 3.5$	11	5	6	6	6
f_7	$x_0 = 1$	7	4	4	5	6
	$x_0 = 2$	6	4	2	4	3