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ON GENERALIZATION OF SOME COMMON FIXED POINT THEOREMS FOR MULTIVALUED MAPS IN CONE METRIC SPACES

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Abstract: Let P be a subset of a Banach space E and P is normal and regular cone on $E,$ we prove some common fixed point theorems for multivalued maps in cone metric spaces and these theorems generalize the recent results of various authors.

Key Words: Cone metric spaces, multivalued mappings, fixed point.

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1. Introduction and Preliminaries

Recently, Huang and Zhang [1] have generalized the concept of a metric space, replacing the set of real numbers by an ordered Banach space and obtained some fixed point theorems for mapping satisfying different contractive conditions. Subsequently, many authors have studied the strong convergence to a fixed point with contractive constant in cone metric space, see for instance [2],[3],[7],[8].SeongHoon Cho and Mi Sun Kim [6] have proved certain fixed point theorems by using multivalued mapping in the setting of contractive constant in metric spaces. In sequel, R. Krishnakumar and M. Marudai [3] proved fixed point theorems of multivalued mappings in cone metric spaces.

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The purpose of this paper is to extend and improves the fixed point theorems of Krishnakumar and Marudai [3]. First, we recall some definitions and other results that will be needed in the sequel.

Definition 1.1 [1]. Let E be a real Banach space and P a subset of E. P is called a cone if and only if

i) P is closed, non-empty and $P \neq \{0\}$;

ii) $ax + by \in \mathrm{P}$ for all $x, y \in P$ and non-negative real numbers a, b;

iii) $x \in P$ and $-x \in P \Rightarrow x = 0 \Leftrightarrow \mathrm{P} \cap (-\mathrm{P}) = \{0\}$.

Given a cone $P \subset E$, a partial ordering is defined as $\leq$ on $E$with respect to P by $x \leq y$ if and only if $y - x \in P$. It is denoted as $x \ll y$ will stand for $y - x \in int\ P$, where $int\ P$ denotes the interior of P . The cone P is called normal if there is a number $K > 0$ such that for all $x, y \in \mathrm{E}, 0 \leq x \leq y$ implies $\|x\| \leq K\ \|y\|$.

The least positive number K satisfying above is called normal constant of P. The cone P is called regular if every increasing sequence which is bounded above is convergent and every decreasing sequence which is bounded below is convergent.

Definition 1.2[1]. Let X be a non-empty set of E.Suppose that the map $d: X \times X \to E$ satisfies:

(i) $0 \leq d(x, y)$for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;

(ii) $d(x, y) = d(y, x)$for all $x, y \in X$;

(iii) $d(x, y) \leq d(x, z) + d(z, y)$for all $x, y, z \in X$.

Then d is called cone metric on X and (X, d) is called cone metric space.

Example 1.1. Let $E = R^2, P = \{(x, y) \in E: x, y \geq 0\} \subset R^2, X = R$ and $d: X \times X \to E$ defined by

$$d(x, y) = (|x - y|, \propto |x - y|)$$

where $\propto \geq 0$ is a constant. Then(X, d) is a cone metric space [1].

Definition 1.3. Let (X, d) be a cone metric space ,$x \in X$ and $\{x_n\}$ a sequence in X. Then

(i) $\{x_n\}$ converges to x whenever for every $c \in E$ with $0 \ll c$ there is a natural number N such that $\mathrm{d}(x_n, x) \ll c$ for all $n \geq N$.

(ii) $\{x_n\}$ is Cauchy sequence whenever for every $c \in E$ with $0 \ll c$ there is a natural number N such that $d(x_n, x_m) \ll c$ for all $n, m \geq N$.

(iii) (X, d) is said to be a complete cone metric space, if every Cauchy sequence is convergent in X.

Definition 1.4. Let (X, d) be a metric space. We denote by $CB(X)$ the family of non-empty closed bounded subset of X. Let $H(.,.)$ be the Hausdorff distance on $CB(X)$.That is, for $A, B \in CB(X)$,

$$H(A, B) = \max\{Sup_{a\in A} d(a, B), Sup_{b\in B}\, d(A, b)\}$$

where $d(a, B) = \inf\{d(a, b): b \in B\}$ is the distance from the point a to the subset B. An element $x \in X$ is said to be a fixed point of a multivalued mapping $T: X \to 2^X$ if $x \in T(X)$.

2. Main Results

Theorem 2.1. Let (X, d) be a complete cone metric space and the mappings $T_1, T_2: X \to CB(X)$ be multivalued maps satisfy the condition for each $x, y \in X$,

$$\begin{aligned} H(T_1x, T_2y) \leq\ & a[d(x, T_1x) + d(y, T_2y)] \\ & +b[d(x, T_2y) + d(y, T_1x)] \\ & +c[d(x, y) + d(T_1x, T_2y)] \qquad (2.1) \end{aligned}$$

for all $x, y \in X$ and $a + b + c < \frac{1}{2}, a, b, c, \in \left[0, \frac{1}{2}\right)$. Then T_1 and T_2 have a common fixed point in X.

Proof: For every $x_o \in X$ and $n \geq 1$, $x_1 \in T_1x_0$ and $x_2 \in T_2x_1$ that is

$x_{2n+1} \in T_1x_{2n}$ and $x_{2n+2} \in T_2x_{2n+1}$,

$$\begin{aligned} d(x_{2n+1}, x_{2n}) &\leq H(T_1x_{2n}, T_2x_{2n-1}) \\ &\leq a[d(x_{2n}, T_1x_{2n}) + d(x_{2n-1}, T_2x_{2n-1})] \\ &\quad +b[d(x_{2n}, T_2x_{2n-1}) + d(x_{2n-1}, T_1x_{2n})] \\ &\quad +c[d(x_{2n}, x_{2n-1}) + d(T_1x_{2n}, T_2x_{2n-1})] \\ &\leq a[d(x_{2n}, x_{2n+1}) + d(x_{2n-1}, x_{2n})] \\ &\quad +b[d(x_{2n}, x_{2n}) + d(x_{2n-1}, x_{2n+1})] \\ &\quad +c[d(x_{2n}, x_{2n-1}) + d(x_{2n+1}, x_{2n})] \\ &\leq a[d(x_{2n}, x_{2n+1}) + d(x_{2n-1}, x_{2n})] \\ &\quad +b[d(x_{2n}, x_{2n-1}) + d(x_{2n+1}, x_{2n})] \end{aligned}$$

$$+c[d(x_{2n}\ ,x_{2n-1})+d(x_{2n+1},x_{2n})]$$

$$d(x_{2n+1},x_{2n})\leq(a+b+c)[d(x_{2n},x_{2n+1})+d(x_{2n-1},x_{2n})]$$

$$d(x_{2n+1},x_{2n})\leq Ld(x_{2n},x_{2n-1}),\text{ where } L=\frac{a+b+c}{(1-(a+b+c))}$$

Similarly, $d(x_{2n+3},x_{2n+2})\leq H(T_1x_{2n+2},T_2x_{2n+1})$

$$\leq a[d(x_{2n+2}\ ,T_1x_{2n+2})+d(x_{2n+1},T_2x_{2n+1})]$$

$$+b[d(x_{2n+2}\ ,T_2x_{2n+1})+d(x_{2n+1},T_1x_{2n+2})]$$

$$+c[d(x_{2n+2}\ ,x_{2n+1})+d(T_1x_{2n+2},T_2x_{2n+1})]$$

$$\leq a[d(x_{2n+2}\ ,x_{2n+3})+d(x_{2n+1},x_{2n+2})]$$

$$+b[d(x_{2n+2}\ ,x_{2n+2})+d(x_{2n+1},x_{2n+3})]$$

$$+c[d(x_{2n+2}\ ,x_{2n+1})+d(x_{2n+3},x_{2n+2})]$$

$$\leq a[d(x_{2n+2}\ ,x_{2n+3})+d(x_{2n+1},x_{2n+2})]$$

$$+b[d(x_{2n+1}\ ,x_{2n+2})+d(x_{2n+2},x_{2n+3})]$$

$$+c[d(x_{2n+1}\ ,x_{2n+2})+d(x_{2n+2},x_{2n+3})]$$

$$\leq(a+b+c)[d(x_{2n+1},x_{2n+2})+d(x_{2n+2},x_{2n+3})]$$

$$d(x_{2n+2},x_{2n+3})\leq Ld(x_{2n+1},x_{2n+2}),\text{ where } L=\frac{a+b+c}{(1-(a+b+c))}\ .$$

Thus $d(x_{n+1},x_n)\leq L^nd(x_1,x_0)$.

For $n>m$ we have

$$d(x_n,x_m)\leq d(x_n,x_{n-1})+d(x_{n-1},x_{n-2})+\ ----$$

$$-----\ +d(x_{m+1},x_m)$$

$$\leq[L^{n-1}+L^{n-2}+\ ----+L^m]\ d(x_1,x_0)$$

$$\leq\frac{L^m}{(1-L)}d(x_1,x_0).$$

Let $0\ll c$ be given, choose $\delta>0$ such that $c+N_\delta(0)\subseteq P$where $N_\delta(0)=\{y\in E:\|y\|<\delta\}$.

Also choose a natural number N_1such that $\frac{L^m}{(1-L)}d(x_1,x_0)\in N_\delta(0)$ for all $m\geq N_1$.

Then $\frac{L^m}{(1-L)}d(x_1,x_0)\ll c$

for all $m\geq N_1$. Then we get $d(x_n,x_m)\ll c$ for all $n>m$. Therefore $\{x_n\}$ is a Cauchy sequence in (X,d) is a complete cone metric space, there exists $z\in X$ such that $lim_{n\to\infty}x_n=$

z. Since $\{x_n\}$ has a convergent subsequence $\{x_m\}$ such that $lim_{m\to\infty} x_m = u$. By the uniqueness of the limit, $z = u$. Since T_1 and T_2 are continuous,

$$lim_{m\to\infty} T_1 x_m = T_1 u \quad and \quad lim_{m\to\infty} T_2 x_m = T_2 u.$$

Now, if m is odd, then

$$lim_{n\to\infty} T_2 x_{2n+1} = T_2 u$$

Choose a natural number N_2 such that $d(x_{2n+1}, u) \ll \frac{c(1-L)}{3}$ for all $n \geq N_2$.

Hence for $n \geq N_2$ we have $d(x_{2n+1}, u) \ll \frac{c(1-k)}{3}$ where $k = a + b + c$.

$$\begin{aligned}
d(T_2u, u) &\leq H(T_1x_{2n}, T_2u) + d(T_1x_{2n}, u) \\
&\leq a[d(x_{2n}, T_1x_{2n}) + d(u, T_2u)] \\
&\quad +b[d(x_{2n}, T_2u) + d(u, T_1x_{2n})] \\
&\quad +c[d(x_{2n}, u) + d(T_1x_{2n}, T_2u)] + d(T_1x_{2n}, u) \\
&\leq a[d(x_{2n}, x_{2n+1}) + d(u, T_2u)] \\
&\quad +b[d(x_{2n}, T_2u) + d(u, x_{2n+1})] \\
&\quad +c[d(x_{2n}, u) + d(x_{2n+1}, T_2u)] + d(x_{2n+1}, u) \\
&\leq a[d(x_{2n}, x_{2n+1}) + d(u, T_2u)] \\
&\quad +b[d(x_{2n}, u) + d(u, T_2u) + d(u, x_{2n+1})] \\
&\quad +c[d(x_{2n}, u) + d(u, T_2u) + d(u, x_{2n+1})] + d(x_{2n+1}, u)
\end{aligned}$$

$$\begin{aligned}
(1-k)d(T_2u, u) &\leq a[d(x_{2n}, u) + d(x_{2n+1}, u)] \\
&\quad +b[d(x_{2n}, u) + d(x_{2n+1}, u)] \\
&\quad +c[d(x_{2n}, u) + d(x_{2n+1}, u)] + d(x_{2n+1}, u) \\
&\leq kd(x_{2n}, u) + kd(x_{2n+1}, u) + d(x_{2n+1}, u) \\
&\leq d(x_{2n}, u) + d(x_{2n+1}, u) + d(x_{2n+1}, u)
\end{aligned}$$

$$d(T_2u, u) \leq \frac{[d(x_{2n}, u) + d(x_{2n+1}, u) + d(x_{2n+1}, u)]}{1-k}$$

$$d(T_2u, u) \ll \frac{c}{3} + \frac{c}{3} + \frac{c}{3}$$

$$d(T_2u, u) \ll c$$

for all $n \geq N_2$.Therefore $d(T_2u, u) \ll \frac{c}{i}$ for all $i \geq 1$,we get $\frac{c}{i} - d(T_2u, u) \in P$ for all $i \geq 1$. As $i \to \infty$ we get $\frac{c}{i} \to 0$ and P is closed, $-d(T_2u, u) \in P$, but $d(T_2u, u) \in P$. Hence

$d(T_2u, u) = 0$ and so $u = T_2u$ .Thus u is the fixed point of T_2. And if m is even, then we have

$$lim_{n\to\infty} T_1x_{2n} = T_1u$$

Now by using (2.1), we have

$$d(T_1u, u) \leq \frac{[d(x_{2n+1}, u) + d(x_{2n+2}, u) + d(x_{2n+2}, u)]}{1-k}$$

$$d(T_1u, u) \ll \frac{c}{3} + \frac{c}{3} + \frac{c}{3}$$

$$\ll c$$

for all $n \geq N_2$. Therefore $d(T_1u, u) \ll \frac{c}{i}$ for all $i \geq 1$.Hence $\frac{c}{i} - d(T_1u, u) \in P$ for all $i \geq 1$. As $i \to \infty$ we get $\frac{c}{i} \to 0$ and P is closed, $-d(T_1u, u) \in P$, but $d(T_1u, u) \in P$. Hence $d(T_1u, u) = 0$ and so $u = T_1u$ i.e. u is the fixed point of T_1, too.

Hence $u = T_1u = T_2u$, u is the common fixed point of T_1 and T_2.

Theorem 2.2. Let (X, d) be a complete cone metric space and the mappings $T_1, T_2: X \to CB(X)$ be multivalued maps satisfy the condition

$$H(T_1x, T_2y) \leq a_1d(x, y) + a_2d(x, T_1x) + a_3d(y, T_2y)$$
$$+a_4d(x, T_2y) + a_5d(y, T_1x) \quad (2.2)$$

for all $x, y \in X$ and $a_i, i = 1, 2, 3, 4, 5$ are all non negative constants such that $a_1 + a_2 + a_3 + a_4 + a_5 < 1$.Then T_1 and T_2 have a common fixed point in X.

Proof: For every $x_o \in X$ and $n \geq 1$, $x_1 \in T_1x_0$ and $x_2 \in T_2x_1$ that is

$x_{2n+1} \in T_1x_{2n}$ and $x_{2n+2} \in T_2x_{2n+1}$

$$d(x_{2n+1}, x_{2n}) \leq H(T_1x_{2n}, T_2x_{2n-1})$$
$$\leq a_1d(x_{2n} , x_{2n-1}) + a_2d(x_{2n}, T_1x_{2n})$$
$$+a_3d(x_{2n-1} , T_2x_{2n-1}) + a_4d(x_{2n}, T_2x_{2n-1})$$
$$+a_5d(x_{2n-1}, T_1x_{2n})$$
$$\leq a_1d(x_{2n}, x_{2n-1}) + a_2d(x_{2n} , x_{2n+1})$$
$$+a_3d(x_{2n-1}, x_{2n}) + a_5d(x_{2n-1}, x_{2n})$$
$$+a_5d(x_{2n}, x_{2n+1})$$
$$\leq (a_1 + a_3 + a_5)d(x_{2n}, x_{2n-1}) + (a_2 + a_5)\, d(x_{2n}, x_{2n+1})$$

$d(x_{2n+1}, x_{2n}) \leq Ld(x_{2n}, x_{2n-1})$, where $L = \frac{a_1+a_3+a_5}{(1-(a_2+a_5))}$

Similarly,

$$\begin{aligned} d(x_{2n+3}, x_{2n+2}) &\leq H(T_1x_{2n+2}, T_2x_{2n+1}) \\ &\leq a_1d(x_{2n+2}\ , x_{2n+1}) + a_2d(x_{2n+2}, T_1x_{2n+2}) \\ &\quad +a_3d(x_{2n+1}\ , T_2x_{2n+1}) + a_4d(x_{2n+2}, T_2x_{2n+1}) \\ &\quad +a_5d(x_{2n+1}, T_1x_{2n+2}) \\ &\leq a_1d(x_{2n+2}\ , x_{2n+1}) + a_2d(x_{2n+2}, x_{2n+3}) \\ &\quad +a_3d(x_{2n+1}\ , x_{2n+2}) + a_5d(x_{2n+1}, x_{2n+2}) \\ &\quad +a_5d(x_{2n+2}, x_{2n+3}) \\ &\leq (a_1 + a_3 + a_5)d(x_{2n+1}, x_{2n+2}) \\ &\quad +(a_2 + a_5)d(x_{2n+2}, x_{2n+3}) \end{aligned}$$

$d(x_{2n+3}, x_{2n+2}) \leq Ld(x_{2n+2}, x_{2n+1})$,where $L = \frac{a_1+a_3+a_5}{(1-(a_2+a_5))}$.

Thus $d(x_{n+1}, x_n) \leq L^n d(x_1, x_0)$.

For $n > m$ we have

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n-1}) + d(x_{n-1}, x_{n-2}) + \cdots + d(x_{m+1}, x_m) \\ &\leq [L^{n-1} + L^{n-2} + \cdots + L^m]d(x_1, x_0) \\ &\leq \frac{L^m}{(1-L)} d(x_1, x_0). \end{aligned}$$

Let $0 \ll c$ be given, choose $\delta > 0$ such that $c + N_\delta(0) \subseteq P$ where $N_\delta(0) = \{y \in E: \|y\| < \delta\}$. Also choose a natural number N_1such that $\frac{L^m}{(1-L)} d(x_1, x_0) \in N_\delta(0)$ for all $m \geq N_1$.Then $\frac{L^m}{(1-L)} d(x_1, x_0) \ll c$ for all $m \geq N_1$.

Then we get $d(x_n, x_m) \ll c$ for all $n > m$. Therefore $\{x_n\}$ is a Cauchy sequence in (X, d) is a complete cone metric space, there exists $z \in X$ such that $lim_{n\to\infty} x_n = z$. Since $\{x_n\}$ has a convergent subsequence $\{x_m\}$ such that $lim_{m\to\infty} x_m = u$. By the uniqueness of the limit, $z = u$.Since T_1 and T_2 are continuous,

$$lim_{m\to\infty} T_1x_m = T_1u$$

$$\text{and } lim_{m\to\infty} T_2x_m = T_2u.$$

Now, if m is odd, then

$$lim_{n\to\infty}T_2x_{2n+1} = T_2u.$$

Choose a natural number N_2 such that $d(x_{2n+1}, u) \ll \frac{c(1-L)}{2}$ for all $n \geq N_2$.

$$d(T_2u, u) \leq H(T_1x_{2n}, T_2u) + d(T_1x_{2n}, u)$$
$$\leq a_1d(x_{2n}\ , u) + a_2d(x_{2n}, T_1x_{2n}) + a_3d(u, T_2u)$$
$$+a_4d(x_{2n}\ , T_2u) + a_5d(u, T_1x_{2n}) + d(x_{2n+1}\, , u)$$
$$\leq a_1d(x_{2n}\ , u) + a_2d(x_{2n}, x_{2n+1}) + a_3d(u, T_2u)$$
$$+a_4d(x_{2n}\ , T_2u) + a_5d(u, x_{2n+1}) + d(x_{2n\ +1}\, , u)$$
$$\leq a_1d(x_{2n}\ , u) + a_2[d(x_{2n}, u) + d(u, x_{2n+1})]$$
$$+a_3d(u, T_2u) + a_4[d(x_{2n}, u) + d(u, T_2u)]$$
$$+a_5d(x_{2n+1}, u) + d(x_{2n\ +1}\, , u)$$
$$(1 - a_3 - a_4)d(T_2u, u) \leq (a_1 + a_2 + a_4)d(x_{2n}\ , u)$$
$$+(a_2 + a_5 + 1)d(\ x_{2n+1}, u)$$
$$d(T_2u, u) \leq \frac{(a_1 + a_2 + a_4)}{(1 - a_3 - a_4)}d(x_{2n}, u)$$
$$+\frac{(1 + a_2 + a_5)}{(1 - a_3 - a_4)}d(x_{2n+1}, u)$$
$$d(T_2u, u) \ll \frac{c}{2} + \frac{c}{2}$$
$$d(T_2u, u) \ll c$$

for all $n \geq N_2$. Therefore $d(T_2u, u) \ll \frac{c}{i}$ for all $i \geq 1$, we get $\frac{c}{i} - d(T_2u, u) \in P$ for all $i \geq 1$. As $i \to \infty$ we get $\frac{c}{i} \to 0$ and P is closed, $-d(T_2u, u) \in P$, but $d(T_2u, u) \in P$. Hence $d(T_2u, u) = 0$ and so $u = T_2u$. Thus u is the fixed point of T_2. And if m is even, then we have

$$lim_{n\to\infty}T_1x_{2n} = T_1u$$

Now by using (2.2), we have

$$d(T_1u, u) \leq \frac{(a_1 + a_2 + a_4)}{(1 - a_3 - a_4)}d(x_{2n+1}, u)$$
$$+\frac{(1 + a_2 + a_5)}{(1 - a_3 - a_4)}d(x_{2n+2}, u)$$
$$d(T_1u, u) \ll \frac{c}{2} + \frac{c}{2}$$
$$d(T_1u, u) \ll c$$

for all $n \geq N_2$.Therefore $d(T_1 u, u) \ll \frac{c}{i}$ for all $i \geq 1$, Hence$\frac{c}{i} - d(T_1 u, u) \in P$ for all $i \geq 1$. As $i \to \infty$ we get $\frac{c}{i} \to 0$ and P is closed, $-d(T_1 u, u) \in P$, but $d(T_1 u, u) \in P$. Hence $d(T_1 u, u) = 0$ and so $u = T_1 u$ i.e. u is the fixed point of T_1, too.

Hence $u = T_1 u = T_2 u$, u is the common fixed pointof $T_1$and T_2.

Conflict of Interests

The author declares that there is no conflict of interests.

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