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A GENERALIZATION OF ROUGH SETS IN TOPOLOGICAL ORDERED SPACES

A. KANDIL[1], O. TANTAWY[2], S. A. EL-SHEIKH[3], M. HOSNY[3,*]

[1]Department of Mathematics, Helwan University, Cairo, Egypt

[2]Department of Mathematics, Zagazig University, Cairo, Egypt

[3]Department of Mathematics, Ain Shams University, Cairo, Egypt

Abstract. This paper concerns with construct a new rough set structure for an ideal ordered topological spaces. Properties of lower and upper approximation are extended to an ideal order topological approximation spaces. The main aim of the rough set is reducing the boundary region by increasing the lower approximation and decreasing the upper approximation. So, in this paper different new methods are proposed to reduce the boundary region. The properties of these methods are obtained. Comparisons between the current approximations and the previous approximations are introduced.

Keywords: Rough sets; $\mathscr{I}$-increasing and $\mathscr{I}$-decreasing sets; ideal; filter; ordered topological spaces.

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1. Introduction

Rough set theory had been proposed by Pawlak [19] in the early of 1982. Rough set theory has achieved a large amount of applications in various real-life fields, like economics, medical diagnosis, biochemistry, environmental science, biology, chemistry, psychology, conflict analysis, medicine, pharmacology, banking, market research, engineering, speech recognition,

*Corresponding author

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material science, information analysis, data analysis, data mining, linguistics, networking and other fields can be found in [12, 13, 18].

The standard rough set theory starts from an equivalence relation. The theory is a new mathematical tool to deal with vagueness and imperfect knowledge. It is dealing with vagueness(ambiguous) of the set by using the concept of the lower and upper approximations [19]. The set has the same lower and upper approximations, called crisp (exact) set, otherwise known as rough (inexact) set. Therefore, the boundary region is defined as the difference between the upper and lower approximations, and then the accuracy of the set or ambiguous depending on the boundary region is empty or not respectively. Nonempty boundary region of a set means that our knowledge about the set is not sufficient to define the set precisely. The main aim of rough set is reducing the boundary region by increasing the lower approximation and decreasing the upper approximation.

The original rough set theory introduced by Pawlak was based on an equivalence relation on a finite universe X. For practice use, there have been some extensions on Pawlak's original concept. One extension is to replace the equivalence relation by an arbitrary binary relation [2, 11, 21, 22]. If R is a binary general relation on X, then the pair (X,R) is called a generalized approximation space in briefly "GAS" [1]. For example of this extension Abo-Table [2] and Yao [21, 22].

The other direction is to study rough set via topological method [1, 9, 10, 12].

Topological spaces have been generalized by many ways. A topological ordered space (X,τ,ρ) is a set X endowed with both a topology τ and a partially order relation ρ on X. The study of order relations in topological spaces was initiated by Nachbin [16] in 1965. Approximation operators draw close links between rough set theory and topology.

El-Shafei et al. [3] introduced and investigated a new rough set in ordered topological spaces depended on a general binary relation and partially order relation. They used a general binary relation to generate a topology τ_R by using the subbase $\xi = \{xR : x \in X\}$ of the topology τ_R also they used partially order relation to construct the increasing and decreasing sets and hence defined the lower and upper approximation by using the increasing and decreasing sets [16].

In this paper, after the preliminaries, the aim of Section 3 is to define the lower and upper approximations of any set with respect to any relation by using the notion of $\mathscr{I}$-increasing ($\mathscr{I}$-decreasing) sets [8]. The object of Sections 4 is to use the filter properties to present a new method to define the lower and upper approximations by using the notion of increasing and decreasing sets. We consider the filter which is generated by the after sets that has a nonempty finite intersection. The main purpose of Section 5 is to use filter subbase to define the lower and upper approximations of any set by using the notion of ($\mathscr{I}$-increasing and $\mathscr{I}$-decreasing) sets. Some examples are given to illustrate the concepts. Moreover, the properties of all present concepts are obtained. Furthermore, the current approximations are compared with the previous approximations [3]. It's therefore shown that the current approximations are more generally. It's shown that the present method decreases the boundary region.

2. **Preliminaries**

In this section, the needed definitions and results are given.

Definition 2.1. [16] *Let (X,R) be a poset. A set $A \subseteq X$ is said to be:*

(1) *Decreasing if for every $a \in A$ and $x \in X$ such that xRa, then $x \in A$.*

(2) *Increasing if for every $a \in A$ and $x \in X$ such that aRx, then $x \in A$.*

Definition 2.2. [1] *If R is a binary relation on X and $A \subseteq X$, then A is called "the after composed"(respectively after-c composed) set if A contains all the after (respectively fore) sets for all its elements, i.e., $\forall a \in A, aR \subseteq A$ (respectively $Ra \subseteq A$), where $aR = \{b : (a,b) \in R\}$ and $Ra = \{b : (b,a) \in R\}$.*

Definition 2.3. [15] *A subfamily $\mathfrak{F}$ of $P(X)$ is called a filter on X if:*

(1) $\phi \notin \mathfrak{F}$.

(2) *If $A_1, A_2 \in \mathfrak{F}$, then $A_1 \cap A_2 \in \mathfrak{F}$.*

(3) *If $A \in \mathfrak{F}$ and $A \subseteq B \subseteq X$, then $B \in \mathfrak{F}$.*

Definition 2.4. [15] *A subset $\mathfrak{B}$ of $P(X)$ is called a filter base if:*

(1) $\phi \notin \mathfrak{B}$.

(2) *If* $B_1, B_2 \in \mathfrak{B}$, *then* $\exists B_3 \in \mathfrak{B} : B_3 \subseteq B_1 \cap B_2$.

A filter base $\mathfrak{B}$ *can be turned into a filter by including all sets of* $P(X)$ *which contains a set of* $\mathfrak{B}$, *i.e.,* $\mathfrak{F}_{\mathfrak{B}} = \{A \in P(X) : A \supseteq B, B \in \mathfrak{B}\}$.

Definition 2.5. [15] *Let* $\xi \subseteq P(X)$. *Then,* ξ *is called a filter-subbases on* X *if it satisfies the finite intersection property, i.e., any finite subcollection of* ξ *has a non empty intersection.*

Definition 2.6. [6] *A non-empty collection* $\mathscr{I}$ *of subsets of a set* X *is called an ideal on* X*, if it satisfies the following conditions:*

(1) $A \in \mathscr{I}$ *and* $B \in \mathscr{I} \Rightarrow A \cup B \in \mathscr{I}$.

(2) $A \in \mathscr{I}$ *and* $B \subseteq A \Rightarrow B \in \mathscr{I}$.

Definition 2.7. [8] *Let* (X,R) *be a poset and* $\mathscr{I} \subseteq P(X)$ *be an ideal on* X. *Then, a set* $A \subseteq X$ *is called:*

(1) $\mathscr{I}$*-decreasing set iff* $Ra \cap A' \in \mathscr{I}$ $\forall a \in A$, *where* $Ra = \{b : (b,a) \in R\}$.

(2) $\mathscr{I}$*-increasing set iff* $aR \cap A' \in \mathscr{I}$ $\forall a \in A$, *where* $aR = \{b : (a,b) \in R\}$.

Proposition 2.1. [8] *For every ideal* $\mathscr{I}$ *on* X*, any increasing set is* $\mathscr{I}$*-increasing set.*

Theorem 2.1. [8] *Let* (X,R) *be a poset,* $\mathscr{I} \subseteq P(X)$ *be an ideal on* X *and* $A \subseteq X$. *Then,* $\tau_{\mathscr{I} inc} = \{A \subseteq X : A \text{ is } \mathscr{I}\text{-inc set}\}$ *is a topology on* X*, which is finer than the topology that is generated by the increasing sets. In other words,* $\tau_{inc} \subseteq \tau_{\mathscr{I}-inc}$.

The rough set theory was introduced by Pawlak based on an equivalence relation R on a finite universe X. In the approximation space (X,R), he considered two operators, the lower and upper approximations of subsets. Let $A \subseteq X$

$\underline{R}(A) = \{x \in X : [x]_R \subseteq A\}$.

$\overline{R}(A) = \{x \in X : [x]_R \cap A \neq \phi\}$.

Boundary, positive and negative regions are also defined:

$BN_R(A) = \overline{R}(A) - \underline{R}(A)$.

$POS_R(A) = \underline{R}(A)$.

$NEG_R(A) = X - \overline{R}(A)$.

Definition 2.8. [3] *A triple (X,τ_R,ρ), where τ_R is the topology generated by any relation R and ρ is a partially order relation, is called an order topological approximation space "OTAS".*

Definition 2.9. [3] *Let (X,τ_R,ρ) be an OTAS ,$A \subseteq X$. Then, the lower, upper approximations, boundary region and accuracy respectively are given by:*

$\underline{R}_{inc}(A) = \cup\{G \in \tau_R : G$ *is an increasing,* $G \subseteq A\}$.

$\underline{R}_{dec}(A) = \cup\{G \in \tau_R : G$ *is a decreasing,* $G \subseteq A\}$.

$\overline{R}^{inc}(A) = \cap\{F \in \tau_R' : F$ *is an increasing,* $A \subseteq F\}$.

$\overline{R}^{dec}(A) = \cap\{F \in \tau_R' : F$ *is a decreasing,* $A \subseteq F\}$.

$BN_{inc}(A) = \overline{R}^{inc}(A) \setminus \underline{R}_{inc}(A)$.

$BN_{dec}(A) = \overline{R}^{dec}(A) \setminus \underline{R}_{dec}(A)$.

$\alpha^{inc}(A) = \frac{|\underline{R}_{inc}(A)|}{|\overline{R}^{inc}(A)|}$.

$\alpha^{dec}(A) = \frac{|\underline{R}_{dec}(A)|}{|\overline{R}^{dec}(A)|}$, α^{inc} *is an increasing accuracy and* α^{dec} *is a decreasing accuracy.*

3. Rough set in ideal topological ordered spaces

In this section, we introduce a new rough set in ideal topological ordered spaces depends on a general binary relation, partially order relation and ideal. We use a general binary relation to generate a topology τ_R by using the subbase $\xi = \{xR : x \in X\}$ of the topology τ_R also define the lower and upper approximations by using $\mathscr{I}$-increasing and $\mathscr{I}$-decreasing sets. These new approximations are compared with El-Shafei et al.'s approximations [3]. It's therefore shown that the current approximations are more generally and reduce the boundary region by increasing the lower approximation and decreasing the upper approximation. The lower and upper approximations satisfy some properties in analogue of Pawalk's spaces [19]. Moreover, we give several examples and counter examples for comparison between the current approach and the approach in [3].

Definition 3.1. *A quadrable $(X,\tau_R,\rho,\mathscr{I})$, is said to be ideal order topological approximation space (IOTAS, for short), where τ_R is a topology generated by any relation R and ρ is a partially order relation and $\mathscr{I}$ an ideal on X.*

Definition 3.2. *Let $(X,\tau_R,\rho,\mathscr{I})$ be an IOTAS and $A \subseteq X$. Then, the lower, upper approximations, boundary region and accuracy respectively are given by:*

$\underline{R}_{\mathscr{I}-inc}(A) = \cup\{G \in \tau_R : G \text{ is } \mathscr{I}\text{-increasing, } G \subseteq A\}$.

$\underline{R}_{\mathscr{I}-dec}(A) = \cup\{G \in \tau_R : G \text{ is } \mathscr{I}\text{-decreasing, } G \subseteq A\}$.

$\overline{R}^{\mathscr{I}-inc}(A) = \cap\{F \in \tau_R^{'} : F \text{ is } \mathscr{I}\text{-increasing, } A \subseteq F\}$.

$\overline{R}^{\mathscr{I}-dec}(A) = \cap\{F \in \tau_R^{'} : F \text{ is } \mathscr{I}\text{-decreasing, } A \subseteq F\}$.

$BN_{\mathscr{I}-inc}(A) = \overline{R}^{\mathscr{I}-inc}(A) \setminus \underline{R}_{\mathscr{I}-inc}(A)$.

$BN_{\mathscr{I}-dec}(A) = \overline{R}^{\mathscr{I}-dec}(A) \setminus \underline{R}_{\mathscr{I}-dec}(A)$.

$\alpha^{\mathscr{I}-inc}(A) = \frac{|\underline{R}_{\mathscr{I}-inc}(A)|}{|\overline{R}^{\mathscr{I}-inc}(A)|}$.

$\alpha^{\mathscr{I}-dec}(A) = \frac{|\underline{R}_{\mathscr{I}-dec}(A)|}{|\overline{R}^{\mathscr{I}-dec}(A)|}$, *$\alpha^{\mathscr{I}-inc}$ is an $\mathscr{I}$-increasing accuracy and $\alpha^{\mathscr{I}-dec}$ is an $\mathscr{I}$-decreasing accuracy.*

The following proposition presents the relationship between the current approximations and El-Shafei et al.'s approximations [3].

Proposition 3.1. *Let $(X,\tau_R,\rho,\mathscr{I})$ be an IOTAS and $A \subseteq X$. Then*

(1) $\underline{R}_{inc}(A) \subseteq \underline{R}_{\mathscr{I}-inc}(A)$ $(\underline{R}_{dec}(A) \subseteq \underline{R}_{\mathscr{I}-dec}(A))$.

(2) $\overline{R}_{\mathscr{I}-inc}(A) \subseteq \overline{R}_{inc}(A)$ $(\overline{R}_{\mathscr{I}-dec}(A) \subseteq \overline{R}_{dec}(A))$.

(3) $BN_{\mathscr{I}-inc}(A) \subseteq BN_{inc}(A)$ $(BN_{\mathscr{I}-dec}(A) \subseteq BN_{dec}(A))$.

(4) $\alpha^{\mathscr{I}-inc}(A) \geq \alpha^{inc}(A)$ $(\alpha^{\mathscr{I}-dec}(A) \geq \alpha^{dec}(A))$.

Proof. The proof is straightforward from Definitions 2.9, 3.2 and Proposition 2.1.

It is noted from Proposition 3.1 that, Definition 3.2 reduces the boundary region by increasing the lower approximation and decreasing the upper approximation with the comparison of [3]. Moreover, it shows that the current accuracy is greater than the previous one in [3].

The following example is computed the lower, upper approximations, boundary region and accuracy for all subset of X by using El-Shafei et al.'s Definition 2.9[3] and the present method in Definition 3.2.

Example 3.1. *Let $X = \{a,b,c,d\}$ and $R = \{(a,a),(a,d),(b,d),(c,c),(c,a),(d,b)\}$. Then,*

$\xi = \{\{b\},\{d\},\{a,d\},\{a,c\}\}, \beta = \{\{a\},\{b\},\{d\},\{a,d\},\{a,c\}\}$,

$\tau_R = \{X,\phi,\{a\},\{b\},\{d\},\{a,b\},\{a,c\},\{a,d\},\{b,d\},\{a,b,c\},\{a,b,d\},\{a,c,d\}\}$,

$\tau_R^{'} = \{X, \phi, \{b\}, \{c\}, \{d\}, \{a,c\}, \{b,c\}, \{b,d\}, \{c,d\}, \{a,b,c\}, \{a,c,d\}, \{b,c,d\}\}$,
$\rho = \Delta \cup \{(a,c), (a,d), (b,c), (d,c)\}$, *and* $\mathscr{I} = \{\phi, \{c\}, \{d\}, \{c,d\}\}$.

TABLE 1. Comparison between the boundary and accuracy by using El-Shafei et al.'s Definition 2.9[3] and the current method in Definition 3.2 in the case of increasing ($\mathscr{I}$-increasing) sets.

A	El-Shafei et al.'s method 2.9[3]				The current method in Definition 3.2			
	$\underline{R}_{inc}(A)$	$\overline{R}^{inc}(A)$	$BN_{inc}(A)$	$\alpha^{inc}(A)$	$\underline{R}_{\mathscr{I}-inc}(A)$	$\overline{R}^{\mathscr{I}-inc}(A)$	$BN_{\mathscr{I}-inc}(A)$	$\alpha^{\mathscr{I}-inc}(A)$
ϕ	ϕ	ϕ	ϕ	0	ϕ	ϕ	ϕ	0
$\{a\}$	ϕ	$\{a,c,d\}$	$\{a,c,d\}$	0	$\{a\}$	$\{a,c\}$	$\{c\}$	0.5
$\{b\}$	ϕ	$\{b,c\}$	$\{b,c\}$	0	$\{b\}$	$\{b\}$	ϕ	1
$\{c\}$	ϕ	$\{c\}$	$\{c\}$	0	ϕ	$\{c\}$	$\{c\}$	0
$\{d\}$	ϕ	$\{c,d\}$	$\{c,d\}$	0	$\{d\}$	$\{d\}$	ϕ	1
$\{a,b\}$	ϕ	X	X	0	$\{a,b\}$	$\{a,b,c\}$	$\{c\}$	$2/3$
$\{a,c\}$	ϕ	$\{a,c,d\}$	$\{a,c,d\}$	0	$\{a,c\}$	$\{a,c\}$	ϕ	1
$\{a,d\}$	ϕ	$\{a,c,d\}$	$\{a,c,d\}$	0	$\{a,d\}$	$\{a,c,d\}$	$\{c\}$	$1/3$
$\{b,c\}$	ϕ	$\{b,c\}$	$\{b,c\}$	0	$\{b\}$	$\{b,c\}$	$\{c\}$	0.5
$\{b,d\}$	ϕ	$\{b,c,d\}$	$\{b,c,d\}$	0	$\{b,d\}$	$\{b,d\}$	ϕ	1
$\{c,d\}$	ϕ	$\{c,d\}$	$\{c,d\}$	0	$\{d\}$	$\{c,d\}$	$\{c\}$	0.5
$\{a,b,c\}$	ϕ	X	X	0	$\{a,b,c\}$	$\{a,b,c\}$	ϕ	1
$\{a,b,d\}$	ϕ	X	X	0	$\{a,b,d\}$	X	$\{c\}$	0.75
$\{a,c,d\}$	$\{a,c,d\}$	$\{a,c,d\}$	ϕ	1	$\{a,c,d\}$	$\{a,c,d\}$	ϕ	1
$\{b,c,d\}$	ϕ	$\{b,c,d\}$	$\{b,c,d\}$	0	$\{b,d\}$	$\{b,c,d\}$	$\{c\}$	$2/3$
X	X	X	ϕ	1	X	X	ϕ	1

TABLE 2. Comparison between the boundary and accuracy by using El-Shafei et al.'s method in Definition 2.9[3] and the current method in Definition 3.2 in the case of decreasing ($\mathscr{I}$-decreasing) sets.

A	El-Shafei et al.'s method in Definition 2.9[3]				The current method in Definition 3.2			
	$\underline{R}_{dec}(A)$	$\overline{R}^{dec}(A)$	$BN_{dec}(A)$	$\alpha^{dec}(A)$	$\underline{R}_{\mathscr{I}-dec}(A)$	$\overline{R}^{\mathscr{I}-dec}(A)$	$BN_{\mathscr{I}-dec}(A)$	$\alpha^{\mathscr{I}-dec}(A)$
ϕ	ϕ	ϕ	ϕ	0	ϕ	ϕ	ϕ	0
$\{a\}$	$\{a\}$	X	$\{b,c,d\}$	0.25	$\{a\}$	$\{a,b,c\}$	$\{b,c\}$	1/3
$\{b\}$	$\{b\}$	$\{b\}$	ϕ	1	$\{b\}$	$\{b\}$	ϕ	1
$\{c\}$	ϕ	X	X	0	ϕ	$\{a,b,c\}$	$\{a,b,c\}$	0
$\{d\}$	ϕ	X	X	0	ϕ	X	X	0
$\{a,b\}$	$\{a,b\}$	X	$\{c,d\}$	0.5	$\{a,b\}$	$\{a,b,c\}$	$\{c\}$	2/3
$\{a,c\}$	$\{a\}$	X	$\{b,c,d\}$	0.25	$\{a\}$	$\{a,b,c\}$	$\{b,c\}$	1/3
$\{a,d\}$	$\{a,d\}$	X	$\{b,c\}$	0.5	$\{a,d\}$	X	$\{b,c\}$	0.5
$\{b,c\}$	$\{b\}$	X	$\{a,c,d\}$	0.25	$\{b\}$	$\{a,b,c\}$	$\{a,c\}$	1/3
$\{b,d\}$	$\{b\}$	X	$\{a,c,d\}$	0.25	$\{b\}$	X	$\{a,c,d\}$	0.25
$\{c,d\}$	ϕ	X	X	0	ϕ	X	X	0
$\{a,b,c\}$	$\{a,b\}$	X	$\{c,d\}$	0.5	$\{a,b,c\}$	$\{a,b,c\}$	ϕ	1
$\{a,b,d\}$	$\{a,b,d\}$	X	$\{c\}$	0.75	$\{a,b,d\}$	$\{a,b,d\}$	ϕ	1
$\{a,c,d\}$	$\{a,d\}$	X	$\{b,c\}$	0.5	$\{a,d\}$	X	$\{b,c\}$	0.5
$\{b,c,d\}$	$\{b\}$	X	$\{a,c,d\}$	0.25	$\{b\}$	X	$\{a,c,d\}$	0.25
X	X	X	ϕ	1	X	X	ϕ	1

Proposition 3.2. *Let $(X,\tau_R,\rho,\mathscr{I})$ be IOTAS and $A,B\subseteq X$. Then,*

(1) $\underline{R}_{\mathscr{I}-inc}(A)\subseteq A\subseteq\overline{R}^{\mathscr{I}-inc}(A)$ $(\underline{R}_{\mathscr{I}-dec}(A)\subseteq A\subseteq\overline{R}^{\mathscr{I}-dec}(A))$ *equality hold if* $A=\phi$ *or* X.

(2) $A\subseteq B\Rightarrow\overline{R}^{\mathscr{I}-inc}(A)\subseteq\overline{R}^{\mathscr{I}-inc}(B)(\overline{R}^{\mathscr{I}-dec}(A)\subseteq\overline{R}^{\mathscr{I}-dec}(B))$.

(3) $A\subseteq B\Rightarrow\underline{R}_{\mathscr{I}-inc}(A)\subseteq\underline{R}_{\mathscr{I}-inc}(B)$ $(\underline{R}_{\mathscr{I}-dec}(A)\subseteq\underline{R}_{\mathscr{I}-dec}(B))$.

(4) $\overline{R}^{\mathscr{I}-inc}(A\cap B)\subseteq\overline{R}^{\mathscr{I}-inc}(A)\cap\overline{R}^{\mathscr{I}-inc}(B)$ $(\overline{R}^{\mathscr{I}-dec}(A\cap B)\subseteq\overline{R}^{\mathscr{I}-dec}(A)\cap\overline{R}^{\mathscr{I}-dec}(B))$.

(5) $\underline{R}_{\mathscr{I}-inc}(A\cup B)\supseteq\underline{R}_{\mathscr{I}-inc}(A)\cup\underline{R}_{\mathscr{I}-inc}(B)$ $(\underline{R}_{\mathscr{I}-dec}(A\cup B)\supseteq\underline{R}_{\mathscr{I}-dec}(A)\cup\underline{R}_{\mathscr{I}-dec}(B))$.

(6) $\overline{R}^{\mathscr{I}-inc}(A\cup B)=\overline{R}^{\mathscr{I}-inc}(A)\cup\overline{R}^{\mathscr{I}-inc}(B)$ $(\overline{R}^{\mathscr{I}-dec}(A\cup B)=\overline{R}^{\mathscr{I}-dec}(A)\cup\overline{R}^{\mathscr{I}-dec}(B))$.

(7) $\underline{R}_{\mathscr{I}-inc}(A\cap B)=\underline{R}_{\mathscr{I}-inc}(A)\cap\underline{R}_{\mathscr{I}-inc}(B)$ $(\underline{R}_{\mathscr{I}-dec}(A\cap B)=\underline{R}_{\mathscr{I}-dec}(A)\cap\underline{R}_{\mathscr{I}-dec}(B))$.

(8) $\overline{R}^{\mathscr{I}-inc}(\overline{R}^{\mathscr{I}-inc}(A))\supseteq\overline{R}^{\mathscr{I}-inc}(A)$ $(\overline{R}^{\mathscr{I}-dec}(\overline{R}^{\mathscr{I}-dec}(A))\supseteq\overline{R}^{\mathscr{I}-dec}(A))$.

(9) $\underline{R}_{\mathscr{I}-inc}(\underline{R}_{\mathscr{I}-inc}(A))\subseteq\underline{R}_{\mathscr{I}-inc}(A)$ $(\underline{R}_{\mathscr{I}-dec}(\underline{R}_{\mathscr{I}-dec}(A))\subseteq\underline{R}_{\mathscr{I}-dec}(A))$.

Proof.

1.: Straightforward.

2.: Let $x\notin\overline{R}^{\mathscr{I}-inc}(B)$. Then, $\exists\ F\in\tau_R^{'},F$ is $\mathscr{I}$-increasing, $F\supseteq B\supseteq A, x\notin F\Rightarrow x\notin\overline{R}^{\mathscr{I}-inc}(A)$.

3.: Similar to part 2.

4.: It is directly from part 2.

5.: It is directly from part 3.

6.: $\overline{R}^{\mathscr{I}-inc}(A\cup B)\supseteq\overline{R}^{\mathscr{I}-inc}(A)\cup\overline{R}^{\mathscr{I}-inc}(B)$ (by part 4) and to prove $\overline{R}^{\mathscr{I}-inc}(A\cup B)\subseteq\overline{R}^{\mathscr{I}-inc}(A)\cup\overline{R}^{\mathscr{I}-inc}(B)$, let $x\notin\overline{R}^{\mathscr{I}-inc}(A)\cup\overline{R}^{\mathscr{I}-inc}(B)$. Then, $x\notin\overline{R}^{\mathscr{I}-inc}(A)$ and $x\notin\overline{R}^{\mathscr{I}-inc}(B)\Rightarrow\exists F_1,F_2\in\tau_R^{'},F_1,F_2$ are $\mathscr{I}$-increasing, such that $x\notin F_1,F_1\supseteq A,x\notin F_2,F_2\supseteq B\Rightarrow x\notin F_1\cup F_2$, (Which is $\mathscr{I}-inc$ by Theorem 2.1), $F_1\cup F_2\supseteq A\cup B\Rightarrow x\notin\overline{R}^{\mathscr{I}-inc}(A\cup B)$. Then, $\overline{R}^{\mathscr{I}-inc}(A\cup B)\subseteq\overline{R}^{\mathscr{I}-inc}(A)\cup\overline{R}^{\mathscr{I}-inc}(B)$. Hence, $\overline{R}^{\mathscr{I}-inc}(A\cup B)=\overline{R}^{\mathscr{I}-inc}(A)\cup\overline{R}^{\mathscr{I}-inc}(B)$.

7.: Similar to No. 6.

8.: It is directly from part 1.

9.: It is directly from part 1.

Example 3.1 shows that the inclusion in Proposition 3.2 parts 1,4 and 5 can not be replaced by equality relation (for part 1, if $A=\{a,b\},\overline{R}^{\mathscr{I}-inc}(A)=\{a,b,c\}$. Then, $\overline{R}^{\mathscr{I}-inc}(A)\nsubseteq A$, take $A=\{b,c\},\underline{R}_{\mathscr{I}-inc}(A)=\{b\}$. Then, $A\nsubseteq\underline{R}_{\mathscr{I}-inc}(A)$. Also, if $A=\{a,b\},\overline{R}^{\mathscr{I}-dec}(A)=\{a,b,c\}$. Then, $\overline{R}^{\mathscr{I}-dec}(A)\nsubseteq A$, and if $A=\{b,c\},\underline{R}_{\mathscr{I}-dec}(A)=\{b\}$. Then, $A\nsubseteq\underline{R}_{\mathscr{I}-dec}(A)$. In a similar way, we can add examples to part 4 and 5). Moreover, the converse of parts 2 and 3 is not necessarily true (i.e., $\overline{R}^{\mathscr{I}-inc}(A)\subseteq\overline{R}^{\mathscr{I}-inc}(B)\nRightarrow A\subseteq B$, take $A=\{a,c\},B=\{a,b\}$, then $\overline{R}^{\mathscr{I}-inc}(A)=\{a,c\},\overline{R}^{\mathscr{I}-inc}(B)=\{a,b,c\}$. Therefore, $\overline{R}^{\mathscr{I}-inc}(A)\subseteq\overline{R}^{\mathscr{I}-inc}(B)$ but $A\nsubseteq B$. In a similar way, we can add examples to show that $\overline{R}^{\mathscr{I}-dec}(A)\subseteq\overline{R}^{\mathscr{I}-dec}(B)\nRightarrow A\subseteq B$).

4. Generalized rough sets via filter by using increasing and decreasing sets

In this section, we introduce a new rough set in ordered topological filters. We consider the filter which is generated by the after sets that has a nonempty finite intersection. To construct the filter $\mathfrak{F}_R$, let $\xi=\{xR:x\in X\}$ be a subbase of a filter $\mathfrak{F}_R$ also we used partially order relation to construct the increasing and decreasing sets and hence define the lower and upper approximation by using the increasing and decreasing sets. The current approximations decrease the boundary region with the comparison of El-Shafei et al.'s approximations [3].

Definition 4.1. *A triple $(X, \mathfrak{F}_R, \rho)$, is said to be generalized order topological approximation space (GOTAS, for short), where $\mathfrak{F}_R$ is a filter generated by any relation R and ρ is a partially ordered relation.*

Definition 4.2. *Let $(X, \mathfrak{F}_R, \rho)$ be a GOTAS and $A \subseteq X$. Then, the lower, upper approximations, boundary region and accuracy respectively are given by:*

$$R_{*inc}(A) = \cup\{G \in \mathfrak{F}_R : G \text{ is increasing}, G \subseteq A\}.$$

$$R_{*dec}(A) = \cup\{G \in \mathfrak{F}_R : G \text{ is decreasing}, G \subseteq A\}.$$

$$R^{*inc}(A) = \begin{cases} \cap\{H \in \mathfrak{F}'_R : H \text{ is an increasing}, A \subseteq H\}. \\ X & \text{if not exists } H \in \mathfrak{F}'_R : H \text{ is an increasing}, A \subseteq H. \end{cases}$$

$$R^{*dec}(A) = \begin{cases} \cap\{H \in \mathfrak{F}'_R : H \text{ is a decreasing}, A \subseteq H\}. \\ X & \text{if not exists } H \in \mathfrak{F}'_R : H \text{ is a decreasing}, A \subseteq H. \end{cases}$$

$$BN_{*inc}(A) = R^{*inc}(A) \setminus R_{*inc}(A).$$

$$BN_{*dec}(A) = R^{*dec}(A) \setminus R_{*dec}(A).$$

$$\alpha^{*inc}(A) = \frac{|R_{*inc}(A)|}{|R^{*inc}(A)|}.$$

$$\alpha^{*dec}(A) = \frac{|R_{*dec}(A)|}{|R^{*dec}(A)|}.$$

The relationship between the topology τ_R which is generated by the subbase $\xi = \{xR : x \in X\}$ and the filter $\mathfrak{F}_R$ which is generated by the same subbase is presented in the following lemma.

Lemma 4.1. *In any GOTAS $(X, \mathfrak{F}_R, \rho)$, we have that $\tau_R \setminus \phi \subseteq \mathfrak{F}_R$.*

Proof. Straightforward.

The following proposition presents the relationship between the current approximations and the approximations in [3] (Definition 2.9).

Proposition 4.1. *Let $(X, \mathfrak{F}_R, \rho)$ be a GOTAS and $A \subseteq X$. Then,*

(1) $\underline{R}_{inc}(A) \subseteq R_{*inc}(A)$ $(\underline{R}_{dec}(A) \subseteq R_{*dec}(A))$.

(2) $R^{*inc}(A) \subseteq \overline{R}_{inc}(A)$ $(R^{*dec}(A) \subseteq \overline{R}_{dec}(A))$.

(3) $BN_{*inc}(A) \subseteq BN_{inc}(A)$ $(BN_{*dec}(A) \subseteq BN_{dec}(A))$.

(4) $\alpha^{*inc}(A) \geq \alpha^{inc}(A)$ $(\alpha^{*dec}(A) \geq \alpha^{dec}(A))$.

Proof. The proof is straightforward from Definitions 2.9, 4.2 and Lemma 4.1.

The following example is computed the lower, upper approximation, boundary region and accuracy for all subset of X by using El-Shafei et al.'s Definition 2.9[3] and the present method in Definition 4.2.

Example 4.1. *Let* $X = \{a,b,c,d\}$ $R = \triangle \cup \{(a,b),(a,c),(b,a),(b,c),(c,a),(c,b),(c,d),(d,c)\}$,
$\xi = \{X,\{c,d\},\{a,b,c\}\}, \beta = \{X,\{c\},\{c,d\},\{a,b,c\}\}$,
$\tau_R = \{X,\emptyset,\{c\},\{c,d\},\{a,b,c\}\}, \tau_R^{'} = \{X,\emptyset,\{d\},\{a,b\},\{a,b,d\}\}$,
$\mathfrak{F}_R = \{X,\{c\},\{a,c\},\{b,c\},\{c,d\},\{a,b,c\},\{a,c,d\},\{b,c,d\}\}$,
$\mathfrak{F}_R^{'} = \{\emptyset,\{a\},\{b\},\{d\},\{a,b\},\{a,d\},\{b,d\},\{a,b,d\}\}$,
$\rho = \Delta \cup \{(a,c),(a,d),(b,c),(d,c)\}$, *and* $\mathscr{I} = \{\phi,\{c\},\{d\},\{c,d\}\}$.

TABLE 3. Comparison between the boundary and accuracy by using El-Shafei et al.'s method in Definition 2.9[3] and the current method in Definition 4.2 in the case of increasing set.

A	El-Shafei et al.'s method in Definition 2.9[3]				The current method in Definition 4.2			
	$\underline{R}_{inc}(A)$	$\overline{R}^{inc}(A)$	$BN_{inc}(A)$	$\alpha^{inc}(A)$	$R_{*inc}(A)$	$R^{*inc}(A)$	$BN_{*inc}(A)$	$\alpha^{*inc}(A)$
ϕ	ϕ	ϕ	ϕ	0	ϕ	ϕ	ϕ	0
$\{a\}$	ϕ	X	X	0	ϕ	X	X	0
$\{b\}$	ϕ	X	X	0	ϕ	X	X	0
$\{c\}$	$\{c\}$	X	$\{a,b,d\}$	0.25	$\{c\}$	X	$\{a,b,d\}$	0.25
$\{d\}$	ϕ	X	X	0	ϕ	X	X	0
$\{a,b\}$	ϕ	X	X	0	ϕ	X	X	0
$\{a,c\}$	$\{c\}$	X	$\{a,b,d\}$	0.25	$\{c\}$	X	$\{a,b,d\}$	0.25
$\{a,d\}$	ϕ	X	X	0	ϕ	X	X	0
$\{b,c\}$	$\{c\}$	X	$\{a,b,d\}$	0.25	$\{b,c\}$	X	$\{a,d\}$	0.5
$\{b,d\}$	ϕ	X	X	0	ϕ	X	X	0
$\{c,d\}$	$\{c,d\}$	X	$\{a,b\}$	0.5	$\{c,d\}$	X	$\{a,b\}$	0.5
$\{a,b,c\}$	$\{c\}$	X	$\{a,b,d\}$	0.25	$\{b,c\}$	X	$\{a,d\}$	0.5
$\{a,b,d\}$	ϕ	X	X	0	ϕ	X	X	0
$\{a,c,d\}$	$\{c,d\}$	X	$\{a,b\}$	0.5	$\{a,c,d\}$	X	$\{b\}$	0.75
$\{b,c,d\}$	$\{c,d\}$	X	$\{a,b\}$	0.5	$\{b,c,d\}$	X	$\{a\}$	0.75
X	X	X	ϕ	1	X	X	ϕ	1

TABLE 4. Comparison between the boundary and accuracy by using El-Shafei et al.'s method in Definition 2.9[3] and the current method in Definition 4.2 in the case of decreasing set.

A	El-Shafei et al.'s method in Definition2.9[3]				The current method in Definition 4.2			
	$\underline{R}_{dec}(A)$	$\overline{R}^{dec}(A)$	$BN_{dec}(A)$	$\alpha^{dec}(A)$	$R_{*dec}(A)$	$R^{*dec}(A)$	$BN_{*dec}(A)$	$\alpha^{*dec}(A)$
ϕ	ϕ	ϕ	ϕ	0	ϕ	ϕ	ϕ	0
$\{a\}$	ϕ	$\{a,b\}$	$\{a,b\}$	0	ϕ	$\{a\}$	$\{a\}$	0
$\{b\}$	ϕ	$\{a,b\}$	$\{a,b\}$	0	ϕ	$\{b\}$	$\{b\}$	0
$\{c\}$	ϕ	X	X	0	ϕ	X	X	0
$\{d\}$	ϕ	X	X	0	ϕ	$\{a,d\}$	$\{a,d\}$	0
$\{a,b\}$	ϕ	$\{a,b\}$	$\{a,b\}$	0	ϕ	$\{a,b\}$	$\{a,b\}$	0
$\{a,c\}$	ϕ	X	X	0	$\{a,c\}$	X	$\{b,d\}$	0.5
$\{a,d\}$	ϕ	X	X	0	ϕ	$\{a,d\}$	$\{a,d\}$	0
$\{b,c\}$	ϕ	X	X	0	ϕ	X	X	0
$\{b,d\}$	ϕ	$\{a,b,d\}$	$\{a,b,d\}$	0	ϕ	$\{a,b,d\}$	$\{a,b,d\}$	0
$\{c,d\}$	ϕ	X	X	0	ϕ	X	X	0
$\{a,b,c\}$	ϕ	X	X	0	$\{a,c\}$	X	$\{b,d\}$	0.5
$\{a,b,d\}$	ϕ	$\{a,b,d\}$	$\{a,b,d\}$	0	ϕ	$\{a,b,d\}$	$\{a,b,d\}$	0
$\{a,c,d\}$	ϕ	X	X	0	$\{a,c\}$	X	$\{b,d\}$	0.5
$\{b,c,d\}$	ϕ	X	X	0	ϕ	X	X	0
X	X	X	ϕ	1	X	X	ϕ	1

Proposition 4.2. *Let $(X,\mathfrak{F}_R,\rho)$ be a GOTAS and $A,B\subseteq X$. Then,*

(1) $R_{*inc}(A)\subseteq A\subseteq R^{*inc}(A)$ $(R_{*dec}(A)\subseteq A\subseteq R^{*dec}(A))$, *equality hold if $A=\phi$ or X.*

(2) $A\subseteq B\Rightarrow R^{*inc}(A)\subseteq R^{*inc}(B)$ $(R^{*inc}(A)\subseteq R^{*dec}(B))$.

(3) $A\subseteq B\Rightarrow R_{*inc}(A)\subseteq R_{*inc}(B)$ $(R_{*dec}(A)\subseteq R_{*dec}(B))$.

(4) $R^{*inc}(A\cap B)\subseteq R^{*inc}(A)\cap R^{*inc}(B)$ $(R^{*dec}(A\cap B)\subseteq R^{*dec}(A)\cap R^{*dec}(B))$.

(5) $R_{*inc}(A\cup B)\supseteq R_{*inc}(A)\cup R_{*inc}(B)$ $(R_{*dec}(A\cup B)\supseteq R_{*dec}(A)\cap R_{*dec}(B))$.

(6) $R^{*inc}(A\cup B)=R^{*inc}(A)\cup R^{*inc}(B)$ $(R^{*dec}(A\cup B)=R^{*dec}(A)\cup R^{*dec}(B))$.

(7) $R_{*inc}(A\cap B)=R_{*inc}(A)\cap R_{*inc}(B)$ $(R_{*dec}(A\cap B)=R_{*dec}(A)\cap R_{*dec}(B))$.

(8) $R^{*inc}(R^{*inc}(A))\supseteq R^{*inc}(A)$ $(R^{*dec}(R^{*dec}(A))\supseteq R^{*dec}(A))$.

(9) $R_{*inc}(R_{*inc}(A))\subseteq R_{*inc}(A)$ $(R_{*dec}(R_{*dec}(A))\subseteq R_{*dec}(A))$.

Proof. The proof is similar to Proposition 3.2.

Example 4.1 shows that the inclusion in Proposition 4.2 parts 1,4 and 5 can not be replaced by equality relation (for part 1, if $A=\{d\}, R^{*inc}(A)=X, R_{*inc}(A)=\phi$, also if $A=\{d\}, R^{*dec}(A)=X, R_{*dec}(A)=\phi$. Then, $R^{*inc}(A)\nsubseteq A\nsubseteq R_{*inc}(A)$ and also $R^{*dec}(A)\nsubseteq A\nsubseteq R_{*dec}(A)$). In a similar way, we can add examples to part 4 and 5). Moreover, the converse of parts 2 and 3 is not necessarily true (i.e., $R_{*inc}(A) \subseteq R_{*inc}(B) \nRightarrow A \subseteq B$, take $A = \{a,b,c\}, B = \{b,c,d\}$, then $R_{*inc}(A) = \{b,c\}, R_{*inc}(B) = \{b,c,d\}$. Therefore, $R_{*inc}(A) \subseteq R_{*inc}(B)$ but $A \nsubseteq B$. In a similar way, we can add examples to show that $R_{*dec}(A) \subseteq R_{*dec}(B)$ but $A \nsubseteq B$).

5. Generalized rough sets via filter by using $\mathscr{I}$-increasing and $\mathscr{I}$-decreasing sets

The main purpose of this section is to use a subbase of a filter to define the lower and upper approximations of any set with respect to any relation by using the notion of $\mathscr{I}$-increasing and $\mathscr{I}$-decreasing sets instead of increasing and decreasing sets. Moreover, comparisons between the current approximations in this section, Sections 3, 4 and the previous approximations in [3] are introduced.

Definition 5.1. *A quadrable $(X, \mathfrak{F}_R, \rho, \mathscr{I})$ is said to be generalized ideal order topological approximation space (GIOTAS, for short), where $\mathfrak{F}_R$ is a filter generated by any relation R and ρ is a partially ordered relation.*

Definition 5.2. *Let $(X, \mathfrak{F}_R, \rho, \mathscr{I})$ be a GIOTAS and $A \subseteq X$. Then, the lower, upper approximations, boundary region and accuracy of a set A with respect to a relation R by using the notion of $\mathscr{I}$-increasing and $\mathscr{I}$-decreasing sets are given by:*

$$R_{*\mathscr{I}-inc}(A) = \cup\{G \in \mathfrak{F}_R : G \; is \; \mathscr{I} - increasing, G \subseteq A\}.$$

$$R_{*\mathscr{I}-dec}(A) = \cup\{G \in \mathfrak{F}_R : G \; is \; \mathscr{I} - decreasing, G \subseteq A\}.$$

$$R^{*\mathscr{I}-inc}(A) = \begin{cases} \cap\{H \in \mathfrak{F}'_R : H \; is; \mathscr{I} - increasing, A \subseteq H\}. \\ X \qquad \text{if not exists } H \in \mathfrak{F}'_R : H \; is \; \mathscr{I} - increasing, A \subseteq H. \end{cases}$$

$$R^{*\mathscr{I}-dec}(A) = \begin{cases} \cap\{H \in \mathfrak{F}'_R : H \; is \; \mathscr{I} - decreasing, A \subseteq H\}. \\ X \qquad \text{if not exists } H \in \mathfrak{F}'_R : H \; is \; \mathscr{I} - decreasing, A \subseteq H. \end{cases}$$

$$BN_{*\mathscr{I}-inc}(A) = R^{*\mathscr{I}-inc}(A) \setminus R_{*\mathscr{I}-inc}(A).$$

$$BN_{*\mathscr{I}-dec}(A) = R^{*\mathscr{I}-dec}(A) \setminus R_{*\mathscr{I}-dec}(A).$$

$$\alpha^{*\mathscr{I}-inc}(A) = \frac{|R_{*\mathscr{I}-inc}(A)|}{|R^{*\mathscr{I}-inc}(A)|}.$$

$$\alpha^{*\mathscr{I}-dec}(A) = \frac{|R_{*\mathscr{I}-dec}(A)|}{|R^{*\mathscr{I}-dec}(A)|}.$$

The following proposition presents the relationship between El-Shafei et al.'s method in Definition 2.9[3] and the current approximations in Definition 5.2.

Proposition 5.1. *Let $(X, \mathfrak{F}_R, \rho, \mathscr{I})$ be a GIOTAS and $A \subseteq X$. Then,*

(1) $\underline{R}_{inc}(A) \subseteq R_{*\mathscr{I}-inc}(A)$ $(\underline{R}_{dec}(A) \subseteq R_{*\mathscr{I}-dec}(A))$.
(2) $R^{*\mathscr{I}-inc}(A) \subseteq \overline{R}^{inc}(A)$ $(R^{*\mathscr{I}-dec}(A) \subseteq \overline{R}^{dec}(A))$.
(3) $BN_{*\mathscr{I}-inc}(A) \subseteq BN_{inc}(A)$ $(BN_{*\mathscr{I}-dec}(A) \subseteq BN_{dec}(A))$.
(4) $\alpha^{*\mathscr{I}-inc}(A) \geq \alpha^{inc}(A)$ $(\alpha^{*\mathscr{I}-dec}(A) \geq \alpha^{dec}(A))$.

Proof. The proof is straightforward from Definitions 2.9, 5.2, Propositions 2.1 and Lemma 4.1.

The following proposition presents the relationship between the current approximations in Definitions 3.2 and 5.2.

Proposition 5.2. *Let $(X, \mathfrak{F}_R, \rho, \mathscr{I})$ be a GIOTAS and $A \subseteq X$. Then,*

(1) $\underline{R}_{\mathscr{I}-inc}(A) \subseteq R_{*\mathscr{I}-inc}(A)$ $(\underline{R}_{\mathscr{I}-dec}(A) \subseteq R_{*\mathscr{I}-dec}(A))$.
(2) $R^{*\mathscr{I}-inc}(A) \subseteq \overline{R}^{\mathscr{I}-inc}(A)$ $(R^{*\mathscr{I}-dec}(A) \subseteq \overline{R}^{\mathscr{I}-dec}(A))$.
(3) $BN_{*\mathscr{I}-inc}(A) \subseteq BN_{\mathscr{I}-inc}(A)$ $(BN_{*\mathscr{I}-dec}(A) \subseteq BN_{\mathscr{I}-dec}(A))$.
(4) $\alpha^{*\mathscr{I}-inc}(A) \geq \alpha^{\mathscr{I}-inc}(A)$ $(\alpha^{*\mathscr{I}-dec}(A) \geq \alpha^{\mathscr{I}-dec}(A))$.

Proof. The proof is straightforward from Definitions 3.2, 5.2 and Lemma 4.1.

The following proposition presents the relationship between the current approximations in Definitions 4.2 and 5.2.

Proposition 5.3. *Let $(X, \mathfrak{F}_R, \rho, \mathscr{I})$ be a GIOTAS and $A \subseteq X$. Then,*

(1) $R_{*inc}(A) \subseteq R_{*\mathscr{I}-inc}(A)$ $(R_{*dec}(A) \subseteq R_{*\mathscr{I}-dec}(A))$.
(2) $R^{*\mathscr{I}-inc}(A) \subseteq R^{*}{}_{inc}(A)$ $(R^{*\mathscr{I}-dec}(A) \subseteq R^{*}{}_{dec}(A))$.
(3) $BN_{*\mathscr{I}-inc}(A) \subseteq BN_{*inc}(A)$ $(BN_{*\mathscr{I}-dec}(A) \subseteq BN^{*}{}_{dec}(A))$.

(4) $\alpha_{*\mathscr{I}-inc}(A) \geq \alpha_{*inc}(A)$ $(\alpha_{*\mathscr{I}-dec}(A) \geq \alpha_{*dec}(A))$.

Proof. The proof is straightforward from Definitions 4.2, 5.2 and Proposition 2.1.

Proposition 5.4. *Let $(X, \mathfrak{F}_R, \rho, \mathscr{I})$ be a GIOTAS and $A, B \subseteq X$. Then,*

(1) $R_{*\mathscr{I}-inc}(A) \subseteq A \subseteq R^{*\mathscr{I}-inc}(A)$ $(R_{*\mathscr{I}-dec}(A) \subseteq A \subseteq R^{*\mathscr{I}-dec}(A))$, *equality hold if* $A = \phi$ *or* X.

(2) $A \subseteq B \Rightarrow R^{*\mathscr{I}-inc}(A) \subseteq R^{*\mathscr{I}-inc}(B)$ $(R^{*\mathscr{I}-inc}(A) \subseteq R^{*\mathscr{I}-dec}(B))$.

(3) $A \subseteq B \Rightarrow R_{*\mathscr{I}-inc}(A) \subseteq R_{*\mathscr{I}-inc}(B)$ $(R_{*\mathscr{I}-dec}(A) \subseteq R_{*\mathscr{I}-dec}(B))$.

(4) $R^{*\mathscr{I}-inc}(A \cap B) \subseteq R^{*\mathscr{I}-inc}(A) \cup R^{*\mathscr{I}-inc}(B)$ $(R^{*\mathscr{I}-dec}(A \cap B) \subseteq R^{*\mathscr{I}-dec}(A) \cup R^{*\mathscr{I}-dec}(B))$.

(5) $R_{*\mathscr{I}-inc}(A \cup B) \supseteq R_{*\mathscr{I}-inc}(A) \cap R_{*\mathscr{I}-inc}(B)$ $(R_{*\mathscr{I}-dec}(A \cup B) \supseteq R_{*\mathscr{I}-dec}(A) \cap R_{*\mathscr{I}-dec}(B))$.

(6) $R^{*\mathscr{I}-inc}(A \cup B) = R^{*\mathscr{I}-inc}(A) \cup R^{*\mathscr{I}-inc}(B)$ $(R^{*\mathscr{I}-dec}(A \cup B) = R^{*\mathscr{I}-dec}(A) \cup R^{*\mathscr{I}-dec}(B))$.

(7) $R_{*\mathscr{I}-inc}(A \cap B) = R_{*\mathscr{I}-inc}(A) \cap R_{*\mathscr{I}-inc}(B)$ $(R_{*\mathscr{I}-dec}(A \cap B) = R_{*\mathscr{I}-dec}(A) \cap R_{*\mathscr{I}-dec}(B))$.

(8) $R^{*\mathscr{I}-inc}(R^{*\mathscr{I}-inc}(A)) \supseteq R^{*\mathscr{I}-inc}(A)$ $(R^{*\mathscr{I}-dec}(R^{*\mathscr{I}-dec}(A)) \supseteq R^{*\mathscr{I}-dec}(A))$.

(9) $R_{*\mathscr{I}-inc}(R_{*\mathscr{I}-inc}(A)) \subseteq R_{*\mathscr{I}-inc}(A)$ $(R_{*\mathscr{I}-dec}(R_{*\mathscr{I}-dec}(A)) \subseteq R_{*\mathscr{I}-dec}(A))$.

Proof. The proof is similar to Proposition 3.2.

Example 4.1 shows that the inclusion in Proposition 5.4 parts 1,4 and 5 can not be replaced by equality relation. Moreover, the converse of parts 3 and 2 is not necessarily true.

By using Example 4.1 we calculate the lower, upper approximation, boundary region and and accuracy by using El-Shafei et al.'s method 2.9[3] and the current approximations in Definitions 3.2, 4.2 and 5.2 as shown in the following tables.

TABLE 5. Comparison between the boundary and accuracy by using El-Shafei et al.'s method in Definition 2.9[3] and the current approximations in Definitions 3.2, 4.2 and 5.2 in the case of increasing ($\mathscr{I}$-increasing) set.

A	El-Shafei et al.'s method 2.9[3]				The current method in Definition 3.2				The current method in Definition 4.2				The current method in Definition 5.2			
	$\underline{R}_{inc}(A)$	$\overline{R}^{inc}(A)$	$BN_{inc}(A)$	$\alpha^{inc}(A)$	$\underline{R}_{\mathscr{I}-inc}(A)$	$\overline{R}^{\mathscr{I}-inc}(A)$	$BN_{\mathscr{I}-inc}(A)$	$\alpha^{\mathscr{I}-inc}(A)$	$R_{*inc}(A)$	$R^{*inc}(A)$	$BN_{*inc}(A)$	$\alpha^{*}{}_{inc}(A)$	$R_{*\mathscr{I}-inc}(A)$	$R^{*\mathscr{I}-inc}(A)$	$BN_{*\mathscr{I}-inc}(A)$	$\alpha^{*}{}_{\mathscr{I}-inc}(A)$
ϕ	ϕ	ϕ	ϕ	0	ϕ	ϕ	ϕ	0	ϕ	ϕ	ϕ	0	ϕ	ϕ	ϕ	0
$\{a\}$	ϕ	X	X	0	ϕ	$\{a,b\}$	$\{a,b\}$	0	ϕ	X	X	0	ϕ	$\{a\}$	$\{a\}$	0
$\{b\}$	ϕ	X	X	0	ϕ	$\{a,b\}$	$\{a,b\}$	0	ϕ	X	X	0	ϕ	$\{b\}$	$\{b\}$	0
$\{c\}$	$\{c\}$	X	$\{a,b,d\}$	0.25	$\{c\}$	X	$\{a,b,d\}$	0.25	$\{c\}$	X	$\{a,b,d\}$	0.25	$\{c\}$	X	$\{a,b,d\}$	0.25
$\{d\}$	ϕ	X	X	0	ϕ	$\{d\}$	$\{d\}$	0	ϕ	X	X	0	ϕ	$\{d\}$	$\{d\}$	0
$\{a,b\}$	ϕ	X	X	0	ϕ	$\{a,b\}$	$\{a,b\}$	0	ϕ	X	X	0	ϕ	$\{a,b\}$	$\{a,b\}$	0
$\{a,c\}$	$\{c\}$	X	$\{a,b,d\}$	0.25	$\{c\}$	X	$\{a,b,d\}$	0.25	$\{c\}$	X	$\{a,b,d\}$	0.25	$\{a,c\}$	X	$\{b,d\}$	0.5
$\{a,d\}$	ϕ	X	X	0	ϕ	$\{a,b,d\}$	$\{a,b,d\}$	0	ϕ	X	X	0	ϕ	$\{a,d\}$	$\{a,d\}$	0
$\{b,c\}$	$\{c\}$	X	$\{a,b,d\}$	0.25	$\{c\}$	X	$\{a,b,d\}$	0.25	$\{b,c\}$	X	$\{a,d\}$	0.5	$\{b,c\}$	X	$\{a,d\}$	0.5
$\{b,d\}$	ϕ	X	X	0	ϕ	$\{a,b,d\}$	$\{a,b,d\}$	0	ϕ	X	X	0	ϕ	$\{b,d\}$	$\{b,d\}$	0
$\{c,d\}$	$\{c,d\}$	X	$\{a,b\}$	0.5	$\{c,d\}$	X	$\{a,b\}$	0.5	$\{c,d\}$	X	$\{a,b\}$	0.5	$\{c,d\}$	X	$\{a,b\}$	0.5
$\{a,b,c\}$	$\{c\}$	X	$\{a,b,d\}$	0.25	$\{a,b,c\}$	X	$\{d\}$	0.75	$\{b,c\}$	X	$\{a,d\}$	0.5	$\{a,b,c\}$	X	$\{d\}$	0.75
$\{a,b,d\}$	ϕ	X	X	0	ϕ	$\{a,b,d\}$	$\{a,b,d\}$	0	ϕ	X	X	0	ϕ	$\{a,b,d\}$	$\{a,b,d\}$	0
$\{a,c,d\}$	$\{c,d\}$	X	$\{a,b\}$	0.5	$\{c,d\}$	X	$\{a,b\}$	0.5	$\{a,c,d\}$	X	$\{b\}$	0.75	$\{a,c,d\}$	X	$\{b\}$	0.75
$\{b,c,d\}$	$\{c,d\}$	X	$\{a,b\}$	0.5	$\{c,d\}$	X	$\{a,b\}$	0.5	$\{b,c,d\}$	X	$\{a\}$	0.75	$\{b,c,d\}$	X	$\{a\}$	0.75
X	X	X	ϕ	1	X	X	ϕ	1	X	X	ϕ	1	X	X	ϕ	1

Table 5 shows that the current method in Definition 5.2 reduces the boundary region by increasing the lower approximation and decreasing the upper approximation with the comparison of El-Shafei et al.'s method in Definition 2.9[3] and the current method in Definition 3.2, 4.2 and 5.2.

TABLE 6. Comparison between the boundary and accuracy by using El-Shafei et al.'s method in Definition 2.9[3] and the current approximations in Definitions 3.2, 4.2 and 5.2 in the case of decreasing ($\mathscr{I}$-decreasing) set.

A	El-Shafei et al.'s method 2.9[3]				The current method in Definition 3.2				The current method in Definition 4.2				The current method in Definition 5.2			
	$\underline{R}_{dec}(A)$	$\overline{R}^{dec}(A)$	$BN_{dec}(A)$	$\alpha^{dec}(A)$	$\underline{R}_{\mathscr{I}-dec}(A)$	$\overline{R}^{\mathscr{I}-dec}(A)$	$BN_{\mathscr{I}-dec}(A)$	$\alpha^{\mathscr{I}-dec}(A)$	$R_{*dec}(A)$	$R^{*dec}(A)$	$BN_{*dec}(A)$	$\alpha^{*dec}(A)$	$R_{*\mathscr{I}-dec}(A)$	$R^{*\mathscr{I}-dec}(A)$	$BN_{*\mathscr{I}-dec}(A)$	$\alpha^{*\mathscr{I}-dec}(A)$
ϕ	ϕ	ϕ	ϕ	0	ϕ	ϕ	ϕ	0	ϕ	ϕ	ϕ	0	ϕ	ϕ	ϕ	0
$\{a\}$	ϕ	$\{a,b\}$	$\{a,b\}$	0	ϕ	$\{a,b\}$	$\{a,b\}$	0	ϕ	$\{a\}$	$\{a\}$	0	ϕ	$\{a\}$	$\{a\}$	0
$\{b\}$	ϕ	$\{a,b\}$	$\{a,b\}$	0	ϕ	$\{a,b\}$	$\{a,b\}$	0	ϕ	$\{b\}$	$\{b\}$	0	ϕ	$\{b\}$	$\{b\}$	0
$\{c\}$	ϕ	X	X	0	ϕ	X	X	0	ϕ	X	X	0	ϕ	X	X	0
$\{d\}$	ϕ	X	X	0	ϕ	$\{a,b,d\}$	$\{a,b,d\}$	0	ϕ	$\{a,d\}$	$\{a,d\}$	0	ϕ	$\{a,d\}$	$\{a,d\}$	0
$\{a,b\}$	ϕ	$\{a,b\}$	$\{a,b\}$	0	ϕ	$\{a,b\}$	$\{a,b\}$	0	ϕ	$\{a,b\}$	$\{a,b\}$	0	ϕ	$\{a,b\}$	$\{a,b\}$	0
$\{a,c\}$	ϕ	X	X	0	ϕ	X	X	0	$\{a,c\}$	X	$\{b,d\}$	0.5	$\{a,c\}$	X	$\{b,d\}$	0.5
$\{a,d\}$	ϕ	X	X	0	ϕ	$\{a,b,d\}$	$\{a,b,d\}$	0	ϕ	$\{a,d\}$	$\{a,d\}$	0	ϕ	$\{a,d\}$	$\{a,d\}$	0
$\{b,c\}$	ϕ	X	X	0	ϕ	X	X	0	ϕ	X	X	0	ϕ	X	X	0
$\{b,d\}$	ϕ	$\{a,b,d\}$	$\{a,b,d\}$	0	ϕ	$\{a,b,d\}$	$\{a,b,d\}$	0	ϕ	$\{a,b,d\}$	$\{a,b,d\}$	0	ϕ	$\{a,b,d\}$	$\{a,b,d\}$	0
$\{c,d\}$	ϕ	X	X	0	ϕ	X	X	0	ϕ	X	X	0	ϕ	X	X	0
$\{a,b,c\}$	ϕ	X	X	0	$\{a,b,c\}$	X	$\{d\}$	0.75	$\{a,c\}$	X	$\{b,d\}$	0.5	$\{a,b,c\}$	X	$\{d\}$	0.75
$\{a,b,d\}$	ϕ	$\{a,b,d\}$	$\{a,b,d\}$	0	ϕ	$\{a,b,d\}$	$\{a,b,d\}$	0	ϕ	$\{a,b,d\}$	$\{a,b,d\}$	0	ϕ	$\{a,b,d\}$	$\{a,b,d\}$	0
$\{a,c,d\}$	ϕ	X	X	0	ϕ	X	X	0	$\{a,c\}$	X	$\{b,d\}$	0.5	$\{a,c\}$	X	$\{b,d\}$	0.5
$\{b,c,d\}$	ϕ	X	X	0	ϕ	X	X	0	ϕ	X	X	0	ϕ	X	X	0
X	X	X	ϕ	1	X	X	ϕ	1	X	X	ϕ	1	X	X	ϕ	1

Table 6 shows that the current method in Definition 5.2 reduces the boundary region by increasing the lower approximation and decreasing the upper approximation with the comparison of El-Shafei et al.'s method in Definition 2.9[3] and the current method in Definitions 3.2, 4.2 and 5.2.

6. Conclusion

The information systems contains data about objects of interest, characterized by a finite set of attributes [5, 7, 14, 20]. It is often interesting to discover some dependency relationships (patterns) among attributes. For a long time, many mathematicians believed that abstract topological structures are far from application fields in general and specially computer sciences and developments of rough set theory as a new mathematical tool to deal with vagueness and uncertainty in information. Lower and upper approximations are the main rough set tools for defining uncertain concepts in information systems. Most constructions of these approximations and their generalizations depend only on one relation resulted from the available information. In rough set theory basic concepts are based on a special type of topological structures known by partition (clo-open quasi discrete) topology. However, the original rough set theory does not consider attributes with preference-ordered domains, that is, criteria. In fact, in many real-world situations, we are often faced with the problems in which the ordering of properties of the considered attributes plays a crucial role. In this paper, we initiated an application for ordered topological space in the context of rough set approximation. The approximation space approached depend on general binary relation, partially order relation, ideal and filter concepts. The approximations suggested in this paper are based simultaneously on an order relation on the collection of information system objects and the topology generated on these objects by the relation resulted from information system. These approximations can play a significant role in the problem of decision making and optimizations, where order of objects is essential in such problems.

Conflict of Interests

The authors declare that there is no conflict of interests.

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