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A COMMON FIXED POINT THEOREM IN G-METRIC SPACE BY USING SUB-COMPATIBLE MAPS

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Abstract. In this paper, we introduce the new concepts of subcompatibility and subsequencial continuity which are respectively weaker than occasionally weak compatibility and reciprocal continuity. With them, we establish a common fixed point theorem for four maps. We introduce an example to support our results. Our results extend the results of [1].

Keywords: G-metric space; Subcompatibility; Subsequencial continuity; Common fixed point theorem.

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1. Introduction

In 1992, Dhage[2] introduced the concept of D - metric space. Recently, Mustafa and Sims[5] shown that most of the results concerning Dhage's D - metric spaces are invalid. Therefore, they introduced a improved version of the generalized metric space structure, which they called it as G - metric spaces. For more details on G - metric spaces, one can referred to the papers [5]- [8],[10].

Now we give basic definitions and some basic results ([5]-[8]) which are helpful for proving

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our main result.

In 2006, Mustafa and Sims [6] introduced the concept of G-metric spaces as follows:

Definition 1.1.[6] Let X be a nonempty set, and let $G : X \times X \times X \to R^+$ be a function satisfying the following axioms:

(G1) $G(x, y, z) = 0$ if $x = y = z$,

(G2) $0 < G(x, x, y)$, for all $x, y \in X$ with $x \neq y$,

(G3) $G(x, x, y) \leq G(x, y, z)$, for all $x, y, z \in X$ with $z \neq y$,

(G4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots$ (symmetry in all three variables)

(G5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$, (rectangle inequality)

then the function G is called a generalized metric, or, more specifically a G - metric on X and the pair (X, G) is called a G - metric space.

Definition 1.2.[6] Let (X, G) be a G-metric space, and let $\{x_n\}$ be a sequence of points in X, a point $x \in X$ is said to be the limit of the sequence $\{x_n\}$ if $\lim_{m,n\to\infty} G(x, x_n, x_m) = 0$ and one says that sequence $\{x_n\}$ is G-convergent to x.
Thus, that if $x_n \to x$ or $\lim_{n\to\infty} x_n \to x$ as n $\to \infty$ in a G-metric space (X, G) then for each $\epsilon > 0$, there exists a positive integer N such that $G(x, x_n, x_m) < \epsilon$ for all $m, n \geq$ N.

Now we state some results from the papers ([7]-[9]) which are helpful for proving our main results.

Proposition 1.1.[7] Let (X, G) be a G - metric space. Then the following are equivalent:

(1) $\{x_n\}$ is G-convergent to x,

(2) $G(x_n, x_n, x) \to 0$ as $n \to \infty$,

(3) $G(x_n, x, x) \to 0$ as $n \to \infty$,

(4) $G(x_m, x_n, x) \to 0$ as $m, n \to \infty$.

Definition 1.3.[7] Let (X, G) be a G - metric space. A sequence $\{x_n\}$ is called G - Cauchy if, for each $\epsilon > 0$ there exists a positive integer N such that $G(x_n, x_m, x_l) < \epsilon$ for all $n, m, l \geq N$;

i.e. if $G(x_n, x_m, x_l) \to 0$ as $n, m, l \to N$.

Proposition 1.2.[7] If (X, G) is a G - metric space then the following are equivalent:

(1) The sequence $\{x_n\}$ is G - Cauchy,

(2) for each $\epsilon > 0$, there exist a positive integer N such that $G(x_n, x_m, x_m) < \epsilon$ for all $n, m \geq N$.

Proposition 1.3.[7] Let (X, G) be a G - metric space. Then the function $G(x, y, z)$ is jointly continuous in all three of its variables.

Definition 1.4.[7] A G - metric space (X, G) is called a symmetric G - metric space if $G(x, y, y) =$ G(y, x, x) for all $x, y \in X$.

Proposition 1.4.[7] Every G - metric space (X, G) will defines a metric space (X, d_G) by

(i)$d_G(x, y) = G(x, y, y) + G(y, x, x)$ for all $x, y \in X$.

If (X, G) is a symmetric G - metric space, then

(ii) d_G(x, y) $= 2G(x, y, y)$ for all $x, y \in X$.

However,if (X, G) is not symmetric, then it follows from the G - metric properties that

(iii) 3/2 $G(x, y, y) \leq$ d_G(x, y)$\leq$ $3G(x, y, y)$ for all $x, y \in X$.

Proposition 1.5.[6] Let (X, G) be a G - metric space. Then the function G(x, y, z) is jointly continuous in all three of its variables.

Definition 1.5.[6] A G - metric space (X, G) is said to be G-complete if every G-Cauchy sequence in (X, G) is G-convergent in X.

Proposition 1.6.[6] A G - metric space (X, G) is G - complete if and only if (X, d_G) is a complete metric space.

Proposition 1.7.[6] Let (X, G) be a G - metric space. Then, for any $x, y, z, a \in X$ it follows that:

(i) If $G(x, y, z) = 0$, then $x = y = z$,

(ii) $G(x,y,z) \leq G(x,x,y) + G(x,x,z)$,

(iii) $G(x,y,y) \leq 2G(y,x,x)$,

(iv) $G(x,y,z) \leq G(x,a,z) + G(a,y,z)$,

(v) $G(x,y,z) \leq \frac{2}{3}\left(G(x,y,a) + G(x,a,z) + G(a,y,z)\right)$,

(vi) $G(x,y,z) \leq (G(x,a,a) + G(y,a,a) + G(z,a,a))$.

Definition 1.6. A pair of self mappings (f,g) of a G-metric space (X,G) is said to be compatible if $\lim_{n\to\infty} G(fgx_n, gfx_n, gfx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z$, where $z \in X$.

Definition 1.7.[1] Let f and g be self maps on X, then a point $x \in X$ is called a coincidence point of f and g iff $fx = gx$. In this case, $w = fx = gx$ is called a point of coincidence of f and g.

Definition 1.8.[1] Two self mappings f and g on a metric space are said to be weakly compatible if they commute at the coincidence points i.e., if $fu = gu$ for some $u \in X$, then $fgu = gfu$.

It is easy to see that two compatible maps are weakly compatible but converse is not true.

Definition 1.9[1] Two self mappings f and g of a metric space are said to be occasionally weakly compatible (owc) iff there is a point $x \in X$ which is coincidence point of f and g at which f and g commute.

In this paper, we weaken the above notion by introducing a new concept called subcompatibility just as defined by H. Bouhadjera[1] in metric space, as follows:

Definition 1.10. Let (X,G) be a G-metric space. Self maps f and g on X are said to be subcompatible iff there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z$, where $z \in X$ and satisfy

$$\lim_{n\to\infty} G(fgx_n, gfx_n, gfx_n) = 0.$$

Obviously, two owc maps are subcompatible, however the converse is not true in general. The example below shows that there exist subcompatible maps which are not owc.

Example 1.1. Let $X = [0, \infty)$ and $G : X \times X \times X \to R^+$ be the G - metric defined as follows: $G(x, y, z) = (|x - y| + |y - z| + |z - x|)$, for all $x, y, z \in X$. Define f and g as follows: $f(x) = x^2$, $g(x) = x + 2$ if $x \in [0, 4]$ or $(9, \infty)$ and $g(x) = x + 12$ if $x \in (4, 9]$. Let $\{x_n\}$ be a sequence in X defined by $\{x_n\} = \{2 + 1/n\}$ for $n = 1, 2, 3, ?$ Then, $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = 4$,where $4 \in X$ and $\lim_{n\to\infty} fgx_n = \lim_{n\to\infty} gfx_n = 16$.

Thus, $\lim_{n\to\infty} G(fgx_n, gfx_n, gfx_n) = 0$. i.e. f and g are subcompatible. On the other hand, we have $fx = gx$ iff $x = 2$ and $fg(2) \neq gf(2)$, hence f and g are not owc.

Now, our second objective is to introduce subsequential continuity in G- metric space which weaken the concept of reciprocal continuity which was introduced by Pant[9] just as introduced by H. Bouhadjera[1] in metric space, as follows:

Definition 1.11. Let (X, G) be a G-metric space. Self maps f and g on X are said to be reciprocally continuous iff $\lim_{n\to\infty} fgx_n = ft$ and $\lim_{n\to\infty} gfx_n = gt$, whenever sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$, where $t \in X$.

Clearly, any continuous pair is reciprocally continuous but the converse is not true in general.

Definition 1.12. Let Let (X, G) be a G-metric space. Self maps f and g on X are said to be subsequentially continuous iff there exist a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$, where $t \in X$ and satisfy $\lim_{n\to\infty} fgx_n = ft$ and $\lim_{n\to\infty} gfx_n = gt$.

Clearly, if f and g are continuous or reciprocally continuous then they are obviously subsequentially continuous. The next example shows that there exist subsequential continuous pairs of maps which are neither continuous nor reciprocally continuous.

Example 1.2. Let and $G : X \times X \times X \to R^+$ be the G - metric defined as follows: X by $G(x, y, z) = (|x - y| + |y - z| + |z - x|)$, for all $x, y, z \in X$. Define f and $g$as follows:

$f(x) = 1 + x$ if $x \in [0, 1]$, $f(x) = 2x - 1$ if $x \in (1, \infty)$

and $g(x) = -x + 1$ if $x \in [0, 1)$, $g(x) = 3x - 2$ if $x \in [1, \infty)$

Clearly f and g are discontinuous at $x = 1$. Let $\{x_n\}$ be a sequence in X defined by $x_n = \{1/n\}$ for $n = 1, 2, 3, ...$ Then, $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = 1$, $1 \in X$

and $\lim_{n\to\infty} fgx_n = 2 = f(1)$, $\lim_{n\to\infty} gfx_n = g(1)$.

Therefore, f and g are subsequential continuous. Now, let $\{x_n\}$ be a sequence in X defined by $x_n = \{1 + 1/n$ for $n = 1, 2, 3, ...$ Then, $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = 1$, $1 \in X$

and $\lim_{n\to\infty} fgx_n = 1 \neq 2 = f(1)$,

so f and g are not reciprocally continuous.

In this paper, we establish a common fixed point theorem for four maps. Our results extend the results of [1].

2. Main results

Now, we prove our main theorem using definition of subcompatible and subsequential continuous maps as follows:

Theorem 2.1. Let f, g, h and k be four self maps of a G-metric space (X, G). If the pairs (f, h) and (g, k) are subcompatible and subsequentially continuous, then

(a) f and h have a coincidence point;

(b) g and k have a coincidence point.

Further, let $\Phi : (\Re^+)^6 \to \Re$ be an upper semi-continuous function satisfying the following condition:

(i) $\Phi(u, u, 0, 0, u, u) > 0$, for all u > 0 .

We suppose that (f, h) and (g, k) satisfy,

(ii)

$$\Phi(G(fx, gy, gy), G(hx, ky, ky), G(fx, hx, hx), G(gy, ky, ky), G(hx, gy, gy), G(fx, ky, ky)) \leq 0$$

for all $x, y \in X$.

Then, f, g, h and k have a unique common fixed point.

Proof. Since, the pairs (f, h) and (g, k) are subcompatible and subsequentially continuous, then, there exists two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} hx_n = z$, where $z \in X$ and which satisfy

$\lim_{n\to\infty} G(fhx_n, hfx_n, hfx_n) = G(fz, hz, hz) = 0$;

$\lim_{n\to\infty} gy_n = \lim_{n\to\infty} ky_n = z'$, where $z' \in X$ and which satisfy

$\lim_{n\to\infty} G(gkx_n, kgx_n, kgx_n) = G(gz', kz', kz') = 0$.

Therefore, $fz = hz$ and $gz' = kz'$; that is, z is a coincidence point of f and h and z' is a coincidence point of g and k.

Now, we prove that $z = z'$. Indeed, by inequality (ii), we have

$$\Phi(G(fx_n, gy_n, gy_n), G(hx_n, ky_n, ky_n), G(fx_n, hx_n, hx_n), G(gy_n, ky_n, ky_n), G(hx_n, gy_n, gy_n), G(fx_n, ky_n, ky_n)) \leq 0$$

Since, Φ is upper semi-continuous, taking the limit as $n \to \infty$ yields

$$\Phi(G(z, z', z'), G(z, z', z'), G(z, z, z), G(z', z', z'), G(z, z', z'), G(z, z', z')) \leq 0$$

which contradicts (i) if $z \neq z'$. Hence, $z = z'$.

Also, we claim that $fz = z$. If $fz \neq z$, using (ii), we get

$$\Phi(G(fz, gy_n, gy_n), G(hz, ky_n, ky_n), G(fz, hz, hz), G(gy_n, ky_n, ky_n), G(hz, gy_n, gy_n), G(fz, ky_n, ky_n)) \leq 0$$

Since, Φ is upper semi-continuous, taking the limit as $n \to \infty$ yields

$\Phi(G(fz,z,z),G(fz,z,z),G(fz,fz,fz),G(z,z,z),G(fz,z,z),G(fz,z,z))\leq 0$

$\Phi(G(fz,z,z),G(fz,z,z),0,0,G(fz,z,z),G(fz,z,z))\leq 0$

This contradicts (i). Hence $z = fz = hz$.

Again, suppose that $gz \neq z$, using (ii), we get

$\Phi(G(fz,gz,gz),G(hz,kz,kz),G(fz,hz,hz),G(gz,kz,kz),G(hz,gz,gz),G(fz,kz,kz))\leq 0$

$\Phi(G(z,gz,gz),G(z,gz,gz),G(z,z,z),G(gz,gz,gz),G(z,gz,gz),G(z,gz,gz)) \leq 0$

$\Phi(G(z,gz,gz),G(z,gz,gz),0,0,G(z,gz,gz),G(z,gz,gz)) \leq 0$

this contradicts (i), hence $z = gz = kz$.

Therefore, $z = fz = gz = hz = kz$; i.e. z is a common fixed point of f, g, h and k.

For Uniqueness: Suppose that there exist another fixed point w of f, g, h and k such that $z \neq w$. Then, by condition (ii), we have

$\Phi(G(fz,gw,gw),G(hz,kw,kw),G(fz,hz,hz),G(gw,kw,kw),G(hz,gw,gw),G(fz,kw,kw))\leq 0$

$\Phi(G(z,w,w),G(z,w,w),G(z,z,z),G(w,w,w),G(z,w,w),G(z,w,w)) \leq 0$

$\Phi(G(z,w,w),G(z,w,w),0,0,G(z,w,w),G(z,w,w)) \leq 0$

This contradicts condition (i). Hence, $z = w$. Therefore, uniqueness follows.

If we put $f = g$ and $h = k$ in Theorem 2.1, we get the next corollary.

Corollary 2.1. Let f and h be self maps of a G-metric space (X, G) such that the pairs (f, h) is subcompatible and subsequentially continuous, then f and h have a coincidence point; Further, let $\Phi : (\Re^+)^6 \to \Re$ be an upper semi-continuous function satisfying the following condition:

(i) $\Phi(u, u, 0, 0, u, u) > 0$ for all u > 0 .

We suppose that (f, h) satisfy,

(ii)

$\Phi(G(fx,fy,fy),G(hx,hy,hy),G(fx,hx,hx),G(fy,hy,hy),G(hx,fy,fy),G(fx,hy,hy))\leq 0$

for all $x, y \in X$.

Then, f and h have a unique common fixed point.

If we put $h = k$, in Theorem 2.1, we get the following result:

Corollary 2.2. Let f, g and h be three self maps of a G-metric space (X, G). If the pairs (f, h) and (g, h) are subcompatible and subsequentially continuous, then

(c) f and h have a coincidence point;

(d) g and h have a coincidence point.

Further, let $\Phi : (\Re^+)^6 \to \Re$ be an upper semi-continuous function satisfying the following condition:

(i)$\Phi(u, u, 0, 0, u, u) > 0$ for all u > 0.

We suppose that (f, h) and (g, h) satisfy:

(ii)

$$\Phi(G(fx, gy, gy), G(hx, hy, hy), G(fx, hx, hx), G(gy, hy, hy), G(hx, gy, gy), G(fx, hy, hy)) \leq 0$$

for all $x, y \in X$

Then, f, g and h have a unique common fixed point.

Example 2.1. Define $\Phi : (\Re^+)^6 \to \Re$ by $\Phi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 + t_2 + t_5 + t_6 - 10t_3 - 10t_4$. Let $X = [0, 1/2]$, define $f, h : X \to X$ by $f(x) = x$ and $h(x) = x^2$. Also, define a G-metric on X by $G(x, y, z) = (|x - y| + |y - z| + |z - x|)$. Then Φ, f and h satisfy all the hypotheses of Corollary 2.2. Thus, f and h have a unique common fixed point. Here, $z = 0$ is the only common fixed point.

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