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ON THE EFFICIENCY OF COLLOCATION METHOD FOR SOLUTION OF THE FALKNER-SKAN BOUNDARY-LAYER EQUATION

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Abstract. The current paper investigates a differential equation which has several applications in various areas of physics and engineering. In this paper some collocation methods have been implemented for solving the Falkner–Skan equation. Difficulties in finding the approximate solution of the problem are described and some modifications are proposed to overcome them. To see the efficiency of the presented methods some illustrative examples are given.

Keywords: spectral collocation methods; method of radial basis functions; Falkner–Skan equation; boundary–layer equation, Falkner–Skan wedge flow.

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1. Introduction

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A famous area of interest in the field of aerodynamics is the boundary layer theory. This problem involves the two-dimensional steady and laminar incompressible flows passing a semi-infinite wedge or a flat plate, that generally are expressed as a partial differential equation named Prandtl problem. We want to solve the Prandtl problem in a region including the parabolic boundary layer.

The most celebrated family of similarity solutions for these boundary layer flows was discovered by Falkner and Skan [15]. In fact Falkner and Skan developed a similarity transformation which reduced this partial differential equation to an one-dimensional third-order boundary value problem named Falkner–Skan equation. So the Falkner–Skan equation is an ordinary differential equation derived originally from a problem in boundary layer theory. This ordinary differential equation is given by

$$\frac{d^3f}{d\eta^3}+f(\eta)\frac{d^2f}{d\eta^2}+\beta(1-(\frac{df}{d\eta})^2)=0,\ \eta\in[0,\infty],$$

with boundary conditions

$$f(0)=f'(0)=0,\ f'(\infty)=1.$$

The diversity of application of Falkner–Skan equation in fluid mechanics caused to a strong interest to solve this equation. Generally this equation cannot be solved directly in a closed form. Therefore, it must be solved by numerical methods. The significance of solution of this equation has motivated many works, most related to the numerical nature of its solution.

First numerical works on this equation include [8, 18]. Numerical solution of this problem by employing finite difference scheme is given by [5, 7, 44, 50, 51]. Authors of [12, 13, 46] discussed the Falkner–Skan wedge flow by spectral and pseudo-spectral methods. Shooting method also was implemented [32, 41, 45, 48]. Other methods such as differential transformation method [23], homotopy perturbation method [53], and Adomian decomposition method [2, 14] were presented for solving this equation numerically. For more information about this equation see [1, 30].

For solving Falkner–Skan equation it is enough to find a value for $\frac{d^2f}{d\eta^2}(0)$. So in this work we try to find this value via spectral and Kansa's collocation methods.

The spectral collocation method emanated roughly in 1970. This method aries from the fundamental problem of approximation of a function by interpolation on an interval, and is very much successful for the numerical solution of ordinary or partial differential equations [6]. In fact the main advantage of this method lies in high accuracy even for a given small number of unknowns. Strong convergence estimates for spectral collocation method were applied to ordinary boundary value problems [20].

Also stability results are obtained with a fairly small number of collocation points. Although the spectral collocation method is remarkably useful to solve differential equations, there are a number of demerits associated with its use. Increasing N frequently causes that the corresponding differentiation matrices become ill-conditioned. Therefore, there has been remarkable interest in recent years in extending well-conditioned spectral collocation methods.

Mai–Duy [34] was introduced an approach to implement the spectral collocation method based on integration rather than conventional differentiation. This use of integration allows the multiple boundary conditions to be incorporated more efficiently. In [34] the author described this subject. Author of [12] used the conventional spectral collocation method to solve Falkner–Skan equation, but in the current work we use the approach of [34] to solve it. We also refer the interested reader to [3, 4, 19, 43] for more research works on Falkner–Skan equation.

Solving equations by traditional numerical methods such as finite difference, finite element and boundary element methods needs generation of a regular mesh in the domain of the problem which is computationally expensive.

In the last 15 years, radial basis functions have been very effective tools to approximate the solutions of equations. A radial basis function is a real-valued function whose values depend only on the distance from some points, called centers. The originator of the radial basis function is due to Hardy for interpolation problems [16]. Some well-known functions that generate radial basis functions are Hardy Multiquadric (MQ) functions, Inverse Multiquadric (IMQ) functions and Gaussian (Ga) functions that are defined as follows respectively

$$\phi(r) = \sqrt{r^2+c^2},$$

$$\phi(r) = (\sqrt{r^2+c^2})^{-1},$$

$$\phi(r) = e^{-cr^2}.$$

The concept of solving differential equations using radial basis functions was first introduced by Kansa [24, 25], who directly collocated the radial basis functions for the approximate solution of differential equations. This method is known as Kansa collocation method. The existence and uniqueness and convergence proofs in applying the radial basis functions were given in [39]. Madych and Nelson proved that MQ has exponential convergence [33].

Compared to numerical classical methods such as finite difference and finite element methods, there are several other advantages of Kansa collocation method [27]. Mainly Kansa collocation method is truly meshless in the sense that the collocation points need not have any connectivity requirement as needed in the finite difference and finite element methods [40]. Therefore, this method is very easy to implement, has good accuracy rather than finite difference method [25, 54] the classical collocation [26] and finite element method [28]. Also the use of univariate radial basis functions in principle saves the computational time for evaluating the approximation of the solution [21]. Because of dimensional independent property of the radial basis functions, they are very attractive to solve high-dimensional equations.

After introducing Kansa collocation method by Kansa it has received a great deal of attention from researchers. This method was implemented for various partial and ordinary differential equations. Heat transfer [54] 1-D and 2-D nonlinear Burger's equation [22, 29], Korteweg-de vries (Kdv) equation [11], 1-D parabolic partial differential equations subject to initial and nonlinear boundary conditions [49], and ill-posed boundary value problems [9] were solved by radial basis functions collocation method.

Motivated by the aforementioned works, we are interested in solving the Falkner–Skan equation by this method, so in this paper the Falkner–Skan equation is solved by Kansa collocation method.

Despite of merits of Kansa collocation method, there are some obstacles for it. Firstly most radial basis functions depend on a parameter c, named shape parameter. The optimal choice of the shape parameter is still an open question [49]. On the other hand, the big obstacle for Kansa collocation method is that the resultant coefficient matrix is usually full, nonsymmetric and highly ill-conditioned that restricts the applicability of Kansa collocation method [27]. Also as shown in [33] the accuracy for approximation a derivative by conventional radial basis functions is a decreasing order of the derivative. For the last difficulty authors of [35] have suggested the idea of constructing the radial basis functions approximations through integration. In this paper, this idea is used to solve the Falkner–Skan equation.

There are currently several ways to solve the ill-conditioning problem of radial basis functions for solving differential equations. In this work we propose a new technique to overcome this problem.

This paper is organized as follows:

In Section 2, some spectral collocation methods have been used for solving Falkner–Skan equation. This equation is solved via Kansa collocation method in Section 3. A new technique is used for solving Falkner–Skan equation in Section 4. By some illustrative examples, we will show and compare the efficiency of the new method presented in the current work. Some concluding remarks are given at the end of this paper.

2. Solving the boundary-layer problem via collocation method

After considering the classical two-dimensional incompressible boundary layer flow pasts a wedge and implementing similarity transforms on the Prandtl equation by Falkner and Skan, we have the 1-dimensional third order boundary value problem known as Falkner–Skan equation given by

$$\frac{d^3f}{d\eta^3}+f\frac{d^2f}{d\eta^2}+\beta(1-(\frac{df}{d\eta})^2)=0,\ \eta\in[0,\infty],$$

with boundary conditions

$$f(0)=0,\ \frac{df}{d\eta}(0)=0,\ \frac{df}{d\eta}(\infty)=1.$$

The third condition of boundary conditions is usually replaced by the condition

$$\frac{df}{d\eta}(\eta_\infty)=1,$$

for some sufficiently large values of η_∞ which are determined experimentally. So the domain of problem will be $[0,\eta_\infty]$.

Elbarbary used the traditional spectral collocation method to solve this equation [12]. He used Chebyshev polynomials as basis functions, next introduced algebraic mapping

$$\xi = \frac{2\eta}{\eta_\infty} - 1,$$

to transform the domain $[0, \eta_\infty]$ to domain $[-1, 1]$, so by this mapping Falkner–Skan equation becomes

$$\frac{d^3 f}{d\xi^3} + \frac{\eta_\infty}{2} f \frac{d^2 f}{d\xi^2} + \beta\left(\left(\frac{\eta_\infty}{2}\right)^3 - \frac{\eta_\infty}{2}\left(\frac{df}{d\xi}\right)^2\right) = 0,\ \xi \in [-1, 1], \tag{1}$$

$$f(-1) = 0,\ \frac{df}{d\xi}(-1) = 0,\ \frac{df}{d\xi}(1) = \frac{\eta_\infty}{2}. \tag{2}$$

In his method $f(\xi)$, the solution of (1) with conditions (2) is approximated by the linear combination of Chebyshev polynomials by unknown coefficients i.e

$$f(\xi) = \sum_{n=0}^{N} a_n T_n(\xi).$$

So f' and f'' and f''' are achieved by taking derivative of $f(\xi)$. After substituting f, f', f'' and f''' in (1) and (2) and then substituting Gauss-Lobatto points in the corresponding equation we have a nonlinear system that is solved by Newton iteration method. By this method author of [12] has solved the Falkner–Skan equation for some different values of β and has achieved $\frac{d^2 f}{d\eta^2}(0) = f''(0)$ for $N = 20$.

In [34] an indirect spectral collocation method that is more successful than the classical spectral collocation method has introduced. In this work, we use the method of [34] to solve the Falkner–Skan equation.

Suppose Falkner–Skan equation is given by (1) and (2). In this type of collocation method we set

$$\frac{d^3 f}{d\xi^3} = \sum_{n=0}^{N} a_n T_n(\xi), \tag{3}$$

so we have

$$\frac{d^2 f}{d\xi^2} = \int \sum_{n=0}^{N} a_n T_n(\xi) d\xi = \sum_{n=0}^{N} a_n I_{2,n}(\xi) + d_1, \tag{4}$$

and

$$\frac{df}{d\xi}=\int\sum_{n=0}^{N}(a_nI_{2,n}(\xi)+d_1)d\xi=\sum_{n=0}^{N}a_nI_{1,n}(\xi)+d_1\xi+d_2, \tag{5}$$

and

$$f=\int\sum_{n=0}^{N}(a_nI_{1,n}(\xi)+d_1\xi+d_2)d\xi=\sum_{n=0}^{N}a_nI_{0,n}(\xi)+d_1\frac{\xi^2}{2}+d_2\xi+d_3, \tag{6}$$

where $I_{2,n}(\xi)=\int_0^\xi T_n(x)dx$ and $I_{1,n}(\xi)=\int_0^\xi I_{2,n}(x)dx$ and $I_{0,n}(\xi)=\int_0^\xi I_{1,n}(x)dx$ and d_1 and d_2 and d_3 are integration constants. $I_{2,n}$ and $I_{1,n}$ and $I_{0,n}$ can be determined by integration of Chebyshev polynomials directly. They were computed in [34] such as

$$I_{2,n}(\xi)=\int_0^\xi T_n(x)dx=\frac{n}{2}\sum_{m=0}^{[\frac{n}{2}]}(-1)^m\frac{2^{n-2m}(n-m-1)!}{m!(n-2m+1)!}\xi^{n-2m+1},$$

$$I_{1,n}(\xi)=\int_0^\xi I_{2,n}(x)dx=\frac{n}{2}\sum_{m=0}^{[\frac{n}{2}]}(-1)^m\frac{2^{n-2m}(n-m-1)!}{m!(n-2m+2)!}\xi^{n-2m+2},$$

$$I_{0,n}(\xi)=\int_0^\xi I_{1,n}(x)dx=\frac{n}{2}\sum_{m=0}^{[\frac{n}{2}]}(-1)^m\frac{2^{n-2m}(n-m-1)!}{m!(n-2m+3)!}\xi^{n-2m+3}.$$

Now (3), (4), (5) and (6) are substituted to (1). So we get

$$\sum_{n=0}^{N}a_nT_n(\xi)+\frac{\eta_\infty}{2}(\sum_{n=0}^{N}a_nI_{0,n}(\xi)+d_1\frac{\xi^2}{2}+d_2\xi+d_3)(\sum_{n=0}^{N}a_nI_{2,n}(\xi)+d_1)$$

$$+\beta((\frac{\eta_\infty}{2})^3-\frac{\eta_\infty}{2}(\sum_{n=0}^{N}a_nI_{1,n}(\xi)+d_1\xi+d_2)^2)=0.$$

Then Gauss–Lobatto points $\xi_n=-\cos(\frac{n\pi}{N}),\ n=0,...,N$ are substituted to this equation separately, so we have $N+1$ equations with $N+4$ unknowns, because as it is seen apart from Chebyshev coefficients a_n, the integration constants are unknown, too. So we are forced to add three equations to this system. Boundary conditions are used to solve this problem. So we have to solve the following system to find the coefficients $a_0,a_1,...,a_n$ and constants d_1,d_2,d_3

$$\sum_{n=0}^{N}a_nT_n(\xi_j)+\frac{\eta_\infty}{2}(\sum_{n=0}^{N}a_nI_{0,n}(\xi_j)+d_1\frac{\xi_j^2}{2}+d_2\xi_j+d_3)(\sum_{n=0}^{N}a_nI_{2,n}(\xi_j)+d_1)$$

$$+\beta((\frac{\eta_\infty}{2})^3-\frac{\eta_\infty}{2}(\sum_{n=0}^{N}a_nI_{1,n}(\xi_j)+d_1\xi_j+d_2)^2)=0,\ j=0,1,...,N,$$

$$\sum_{n=0}^{N}a_nI_{0,n}(-1)+\frac{d_1}{2}-d_2+d_3=0,$$

$$\sum_{n=0}^{N} a_n I_{1,n}(-1) - d_1 + d_2 = 0,$$

$$\sum_{n=0}^{N} a_n I_{1,n}(1) + d_1 + d_2 - \frac{\eta_\infty}{2} = 0.$$

The resulting non-linear system is solved by Newton method.

3. Solving the equation via Kansa collocation method

In this section the Falkner–Skan equation is solved by the method that was introduced by Kansa [24]. The process is similar to spectral collocation method. The main difference between them is to select basis functions and collocation points.

The method of Kansa uses the radial basis functions as basis functions. Madych and Nelson [33] proved that multiquadric (MQ) mesh-independent radial basis functions enjoy exponential convergence [24].

The i-th shape parameter is introduced as

$$c_i = \min\{|\eta_i - \eta_{i+1}|, |\eta_i - \eta_{i-1}|\}, \tag{7}$$

where N collocation points in interval $[a,b]$ are chosen such as

$$\eta_i = a + (i-1)\frac{b-a}{N-1},\ i = 1,2,...,N. \tag{8}$$

In fact c_i is the distance from the i-th center to the nearest center [37]. Now the Falkner–Skan equation is given by

$$\frac{d^3 f}{d\eta^3} + f\frac{d^2 f}{d\eta^2} + \beta(1 - (\frac{df}{d\eta})^2) = 0,\ \eta \in [0, \eta_\infty], \tag{9}$$

$$f(0) = 0,\ \frac{df}{d\eta}(0) = 0,\ \frac{df}{d\eta}(\eta_\infty) = 1. \tag{10}$$

In the classical method of Kansa, solution of (9) is approximated by the linear combination of radial basis functions with unknown coefficients. It is shown in [33] that the order of accuracy of approximating a derivative by conventional radial basis function decreases by one [35]. For this difficulty authors of [35] suggested the idea of constructing the radial basis functions approximations through integration. So, we use this idea to solve (9) and (10). According to this method we set

$$\frac{d^3f}{d\eta^3} = \sum_{n=1}^{N} \lambda_n \phi_n(\eta), \tag{11}$$

where ϕ_n is Hardy Multiquadric function $\sqrt{(\eta-\eta_n)^2+{c_n}^2}$. So we have

$$\frac{d^2f}{d\eta^2} = \int \sum_{n=1}^{N} \lambda_n \phi_n(\eta) d\eta = \sum_{n=1}^{N} \lambda_n H_{2,n}(\eta) + d_1, \tag{12}$$

$$\frac{df}{d\eta} = \int \sum_{n=1}^{N} (\lambda_n H_{2,n}(\eta) + d_1) d\eta = \sum_{n=1}^{N} \lambda_n H_{1,n}(\eta) + d_1\eta + d_2, \tag{13}$$

$$f = \int \sum_{n=1}^{N} (\lambda_n H_{1,n}(\eta) + d_1\eta + d_2) d\eta = \sum_{n=1}^{N} \lambda_n H_{0,n}(\eta) + d_1\frac{\eta^2}{2} + d_2\eta + d_3, \tag{14}$$

where $H_{2,n}(\eta) = \int_0^\eta \phi_n(x)dx$ and $H_{1,n}(\eta) = \int_0^\eta H_{2,n}(x)dx$ and $H_{0,n}(\eta) = \int_0^\eta H_{1,n}(x)dx$ and d_1 and d_2 and d_3 are integration constants. Now we define

$$\begin{gathered} S(\eta,\lambda_1,\dots,\lambda_N,d_1,d_2,d_3) = f'''(\eta,\lambda_1,\dots,\lambda_N) \\ +f(\eta,\lambda_1,\dots,\lambda_N,d_1,d_2,d_3)f''(\eta,\lambda_1,\dots,\lambda_N,d_1) \\ +\beta(1-f'(\eta,\lambda_1,\dots,\lambda_N,d_1,d_2)^2), \end{gathered}$$

where primes denote differentiation with respect to η. The Eqs. (11), (12), (13) and (14) are substituted in S and according to (8) N collocation points are chosen in interval $[0,\eta_\infty]$ and are substituted in S. So we have a nonlinear system

$$G(\lambda_1,\dots,\lambda_N,d_1,d_2,d_3) = 0,$$

where

$$\begin{gathered} G_i(\lambda_1,\dots,\lambda_N,d_1,d_2,d_3) = S(\eta_i,\lambda_1,\dots,\lambda_N,d_1,d_2,d_3),\ i=1,\dots,N, \\ G_{N+1}(\lambda_1,\dots,\lambda_N,d_1,d_2,d_3) = f(0,\lambda_1,\dots,\lambda_N,d_1,d_2,d_3), \\ G_{N+2}(\lambda_1,\dots,\lambda_N,d_1,d_2,d_3) = f'(0,\lambda_1,\dots,\lambda_N,d_1,d_2), \\ G_{N+3}(\lambda_1,\dots,\lambda_N,d_1,d_2,d_3) = f'(\eta_\infty,\lambda_1,\dots,\lambda_N,d_1,d_2) - 1. \end{gathered}$$

The resulting non-linear system is solved by Newton method to find $\lambda_1,\lambda_2,...,\lambda_N,d_1,d_2$ and d_3.

4. The Kansa method combined with a finite difference method

As it was said before, despite of merits of Kansa method, there are some obstacles for it. The main demerit of this method is the resultant system which is generated from discretization using this method. The resultant coefficient matrix is usually full, nonsymmetric and highly ill-conditioned that restricts the applicability of Kansa method. In this method we can't increase N, because the method will be unstable. On the other hand, this method with small N can't achieve sufficiently accurate results. So this method needs an effective modification. In this section the Falkner–Skan equation is solved by Kansa method combined with a finite difference method.

This work uses the approach which was defined by [5]. Let $u = f'$, so the Falkner–Skan equation becomes

$$\begin{cases} f' = u, & f(0) = 0, \\ u'' + fu' + \beta(1-u^2) = 0, & u(0) = 0,\ u(\eta_\infty) = 1. \end{cases} \tag{15}$$

To determine u and f, we divide the interval $[0, \eta_\infty]$ to L subintervals of length $h = \frac{\eta_\infty}{L}$ and define

$$\eta_j = jh,\ j = 0, 1, \ldots, L.$$

So the boundary conditions will be

$$f_0 = 0,\ u_0 = 0,\ u_L = 1.$$

By discretization of (15) we have

$$\begin{cases} \frac{f_j - f_{j-1}}{h} - \frac{u_j + u_{j+1}}{2} = 0, & f_0 = 0, \\ \frac{u_{j+1} - 2u_j + u_{j-1}}{h^2} + f_j \frac{u_{j+1} - u_{j-1}}{2h} + \beta(1 - u_j^2) = 0, & u_0 = 0,\ u_L = 1. \end{cases} \tag{16}$$

Now the unknown vectors $u^T = [u_1, \ldots, u_{L-1}]$ and $f^T = [f_1, \ldots, f_L]$ are defined to construct a nonlinear system $G = 0$. Via Newton iteration method this system can be solved. So by a finite-difference method, solution of the Falkner–Skan equation can be obtained. Due to the importance of finding $f''(0)$, it is enough to concentrate on the $[\eta_0, \eta_1]$ to solve Falkner–Skan equation. So we solve the problem

$$\frac{d^3 f}{d\eta^3} + f\frac{d^2 f}{d\eta^2} + \beta(1 - (\frac{df}{d\eta})^2) = 0,\ \eta \in [0, \eta^1], \tag{17}$$

$$f(0)=0,\ \frac{df}{d\eta}(0)=0,\ \frac{df}{d\eta}(\eta^1)=y^1, \tag{18}$$

via Kansa method where y^1 is found by the finite difference scheme. According to (8), N collocation points are chosen in $[\eta_0,\eta_1]$ and the shape parameters are selected as introduced in (7). Now we can solve (17) with boundary conditions (18) using Kansa method (See the previous section).

Using this technique, the number of collocation points has been reduced . Therefore, the discretized problem is less ill-conditioned.

5. Numerical results and discussion

In this section the introduced collocation methods are implemented for some kinds of Falkner-Skan equation and results are shown for $f''(0)$. Table 1, compares the methods of [12, 7, 42] and the indirect spectral collocation method developed in the current paper, discretization by each of methods results a system of 21 unknowns. For a fair comparison, the method of [12] and the indirect spectral collocation method were implemented for less numbers of collocation points.

Also the condition numbers of the resulting systems are computed and results are shown in Table 2. These results demonstrate that the indirect spectral collocation method is more effective than the method of [12]. For example, suppose that the exact solution of $f''(0)$ for Falkner–Skan equation with $\beta=0.5$ and $\eta_\infty=3.7$ is 0.927805 that was achieved from method of [12] with 21 collocation points. The method of [12] attained the value 1.0594 with 13 collocation points, but the indirect spectral collocation method attained the value 0.927804 with only 8 collocation points. This shows that the convergence of the indirect spectral collocation method is faster. Moreover, in the first example with the exact value 1.67223 for $f''(0)$, the method of [12] has reported the value 1.5667 for $f''(0)$ with choice of 11 collocation points, but the indirect spectral collocation method has resulted the value 1.6872 with the same number of collocation points.

Because of larger condition number of the resulted system in the method of [12], it uses more CPU time than the indirect spectral collocation method. The condition numbers show the stability of the indirect spectral collocation method rather than the method of [12]. Kansa method can be used successfully to solve Falkner–Skan equation. Tables 3 and 4, show the values that are achieved from Kansa's method with Hardy multiquadric and inverse multiquadric radial basis functions, respectively. This method attains reasonable results by minimum number of collocation points. Modification of this method is done via a finite difference method.

Table 5, shows the results and represents the advantages of this modification. Although with small numbers of domain partition and large η^1 the results are the same as Kansa method, but when η^1 becomes smaller the results become more accurate in so far as with $\eta^1 = \frac{\eta_\infty}{24}$ and only 3 collocation points we have results the same as results of method presented in [12] with 21 collocation points. For example let us consider the last example with $\beta = 0$ and $\eta_\infty = 6.9$. The method of [12] attains the value 0.4696 for $f''(0)$ with 21 collocation points, the indirect spectral collocation method attains this value with 8 collocation points, Kansa method can't attain this value even with 5 collocation points, but the new technique with $\eta^1 = \frac{\eta_\infty}{6}$ and only 3 collocation points, produces this value. So the new method developed in the current paper based on this modification can be more effective than the Kansa method.

6. Conclusion

In this paper some collocation methods have been implemented for solving the Falkner–Skan equation. Our emphasis has been on the computational efficiency of these methods. A comparison between the presented methods shows that the indirect spectral collocation methods are more efficient, stable and more accurate than the direct one. Also a new technique based on a finite difference method is presented to improve the method of radial basis functions for solving the Faklner–Skan boundary-layer problem in larger domains. Results show that the new technique is more accurate and stable than the Kansa method.

Conflict of Interests

The authors declare that there is no conflict of interests.

REFERENCES

[1] W. A. Albarakati, An analytical solution of the stagnation point flow problem, J. Tech. Acoust. 21 (2007), 1-8.

[2] E. Alizadeh, M. Farhadi, K. Sedighi, H. R. Ebrahimi-Kebria, A. Ghafourian, Solution of the Falkner-Skan equation for wedge by Adomian decomposition method, Commun. Nonlinear Sci. Numer.Simulat. 14 (2009), 724-733.

[3] S. Abbasbandy, T. Hayat, Solution of the MHD Falkner–Skan flow by homotopy analysis method, Commun. Nonlinear. Sci. Numer. Simulat. 14 (2009), 3591-3598.

[4] S. Abbasbandy, T. Hayat, Solution of the MHD Falkner–Skan flow by Hankel–Pad'e method, Phys. Lett. A. 373 (2009), 731-734.

[5] A. Asaithambi, A finite-difference method for the Falkner-Skan equation, Appl. Math. Comput. 92 (1998), 135-141.

[6] E. Babolian, M. Bromilow, R. England, M. Saravi, A modification of pseudo-spectral method for solving a linear ODEs with singularity, Appl. Math. Comput. 188 (2007), 1260-1266.

[7] P. M. Beckett, Finite difference solution of boundary layer type equation, Int. J. Comput. Math. 14 (1983), 183-190.

[8] T. Cebeci, H. Keller, Shooting and parallel shooting methods for solving the Falkner-Skan boundary-layer equation, J. Comput. Phys. 7 (1971), 289-300.

[9] A. H. D. Cheng, J. J. S. P. Cabral, Direct solution of ill-posed boundary value problems by radial basis function collocation method, Int. J. Numer. Meth. Engng, 64 (2005), 45-64.

[10] H. Chlichting, Boundary layer theory, fourth edition, McGraw-Hill, New York, 1960.

[11] M. Dehghan, A. Shokri, A numerical method for KdV equation using collocation and radial basis functions, Nonlinear Dyn. 50 (2007), 111-120.

[12] M. E. Elbarbary, Chebyshev finite difference method for the solution of boundary–layer equations, Appl. Math. Comput. 160 (2005), 487-498.

[13] T. M. El-Gindy, H. M. El-Hawary, H. S. Hussein, An optimization technique for the Falkner-Skan equation, J. Optimization, 35 (1995), 357-366.

[14] N. S. Elgazery, Numerical solution for Falkner-Skan equation, Chaos, Solitons and Fractals 35 (2008), 738-746.

[15] V. M. Falkner, S. W. Skan, Some approximate solutions of the boundary layer equations, Phil. Mag. 12 (1931), 865-896.

[16] R. L. Hardy, Multiquadric equations of topography and other irregular surfaces, J. Geophys. Res. 176 (1971), 1905-1915.

[17] P. Hartman, Ordinary Differential Equations, Classics in Applied Mathematics 38, SIAM, Philadelphia, PA, 2002.

[18] D. R. Hartree, On an equation occurring in Falkner and Skan's approximate treatment of the equations of the boundary layer, Cambridge Phil. 33 (1937), 223-239.

[19] T. Hayat, Q. Hussain, T. Javed, The modified decomposition method and Pad'e approximants for the MHD flow over a non–linear stretching sheet, Nonlinear. Anal. 10 (2009), 966-973.

[20] W. Heinrichs, Strong convergence estimates for pseudo-spectral methods, Appl. Math. 37 (1992), 401-417.

[21] Y. C. Hon, Z. Wu, Additive Schwarz domain decomposition with radial basis approximation, Internat. J. Appl. Math. 4 (2000), 81-98.

[22] Y. C. Hon, X. Z. Mao, An efficient numerical scheme for Burger's equation, Appl. Math. Comput. 95 (1998), 37-50.

[23] B. L. Kuo, Application of the differential transformation method to the solutions of Falkner–Skan wedge flow, Acta Mechanica, 164 (2003), 161-174.

[24] E. J. Kansa, Multiquadrics—a scattered data approximation scheme with applications to computational fluid dynamics, I. Surface approximations and partial derivatives, Comput. Math. Appl. 19 (1992), 127-145.

[25] E. J. Kansa, Multiquadrics—a scattered data approximation scheme with applications to computational fluid dynamics, II. Solutions to parabolic, hyperbolic and elliptic partial differential equations, Comput. Math. Appl. 19 (1992), 147-161.

[26] E. Larsson, B. Fornberg, A numerical study of some radial basis function based solution methods for elliptic PDEs, preprint.

[27] J. Li, Y. C. Hon, Domain decomposition for radial basis meshless methods, Numer. Methods Partial Differential Equations, 20 (2004), 450-462.

[28] J. Li, A. H.-D. Cheng, C. S. Chen, On the efficiency and exponential convergence of multi–quadric collocation method compared to finite element method, Engng. Anal. Boundary Elements 27 (2003), 251-257.

[29] J. Li, Y. C. Hon, C. S. Chen, Numerical comparisons of two meshless methods using radial basis functions, Engng. Anal. Boundary Elements 26 (2002), 205-225.

[30] S. J. Liao, A uniformly valid analytic solution of two-dimensional viscous flow over a semi-infinite flat plate, J. Fluid Mech. 385 (1999), 10-128.

[31] L. Ling, E. J. Kansa, Preconditioning for radial basis functions with domain decomposition methods, Numer. Methods Partial Differential Equations, 20 (2003), 450-462.

[32] C. S. Liu, J. R. Chang, The Lie-group shooting method for multiple- solution of Falkner-Skan equation under suction-injection conditions, Int. J. Nonlinear Mech. 43 (2008), 844-851.

[33] W. R. Madych, S. A. Nelson, Multivariate interpolation and conditionally positive definite functions, II. Math Comput. 54 (1990), 211-230.

[34] N. Mai-Duy, An effective spectral collocation method for the direct solution of high-order ODEs, Commun. Numer. Meth. Engng, 22 (2006), 627-642.

[35] N. Mai-Duy, T. Tran–Cong, Integrated radial–basis–function networks for computing Newtonian and non-Newtonian fluid flows, Computers and Structures, 87 (2009), 642-650.

[36] N. Mai-Duy, T. Tran-Cong, A multidomain integrated-radial-basis-function collocation method for elliptic problems, Numer. Methods Partial Differential Equations, 24 (2008), 1301-1320.

[37] N. Mai-Duy, Solving high order ordinary differential equations with radial basis function networks, Int. J. Numer. Meth. Engng, 62 (2005), 824-852.

[38] N. Mai-Duy, T. Tran-Cong, An efficient domain-decomposition pseudo-spectral method for solving elliptic differential equations, Commun. Numer. Meth. Engng, 24 (2008), 795-806.

[39] C. A. Micchelli, Interpolation of scattered data: distance matrix and conditionally positive definite functions, Constr. Approx. 2 (1986), 11-22.

[40] J. A. Muñoz-Gómez, P. González-Casanova, G. Rodŕguez-Gómez, Domain decomposition by radial basis functions for time dependent partial differential equations, Advances in Computer Science and Technology, (2006), 105-109.

[41] T. Y. Na, Computational methods in engineering boundary value problems, Academic Press, New York, (1979).

[42] H. Nasr, I. A. Hassanien, H. M. El-Hawary, Chebyshev solution of laminar boundary layer flow, Int. J. Comput. Math. 33 (1990), 127-132.

[43] M.M. Rashidi, The modified differential transform method for solving MHD boundary-layer equations, Comput. Phys. Commun. 180 (2009), 2210-2217.

[44] L. Rosenhead, Laminar boundary layers, Oxford University Press, Oxford, (1963).

[45] A. A. Salama, Higher-order method for solving free boundary-value problems, Numer. Heat Transfer, Part B: Fundamentals, 45 (2004) 385-394 .

[46] H. T. Sharp, W. L. Harris, A pseudo-spectral method and parametric differentiation applied to the Falkner-Skan equation, J. Comput. Phys. 55 (1984), 377-386.

[47] I. Sher, A. Yakhot, New approach to the solution of the Falkner-Skan equation, AIAA J. 39 (2001), 965-967.

[48] A. M. O. Smith, Improved solutions of the Falkner and Skan boundary-layer equation, Fund Paper, J. Aero. Sci., Sherman M. Fairchild, (1954).

[49] M. Tatari, M. Dehghan, On the solution of the non–local parabolic partial differential equations via radial basis functions, Appl. Math. Modelling, 33 (2009), 1729-1738.

[50] A. R. Wadia, F. R. Payne, A boundary value technique for the analysis of laminar boundary layer flows, Inter. J. Comput. Math. 9 (1981), 163-172.

[51] F. M. White, Viscous fluid flow, McGraw–Hill, New York, NY, (1974).

[52] G. C. Yang, New results of Falkner–Skan equation arising in boundary layer theory, Appl. Math. Comput. 202 (2008), 406-412.

[53] B. Yao, J. Chen, Series solution to the Falkner-Skan equation with stretching boundary, Appl. Math. Comput. 208 (2009), 156-164.

[54] M. Zerroukat, H. Power, C. S. Chen, A numerical method for heat transfer problem using collocation and radial basis functions, Int. J. Numer. Meth. Engng. 42 (1998), 1263-1278.

TABLE 1. Comparison of $f''(0)$ obtained by various methods for some Falkner-Skan equations

η_∞	β	[12]	[42]	[7]	indirect spectral collocation method
3.1	2	1.68722586	1.6872	1.6872	1.68723238
3.7	0.5	0.92780539	0.9278	0.9278	0.92781320
4.4	0.3	0.77478274	0.7747	0.7747	0.77476620
6.9	0	0.46960012	0.4696	0.4696	0.46960004

TABLE 2. Comparison of $f''(0)$, number of collocation points, number of iterations and condition numbers obtained by various methods for some Falkner-Skan equations. Note that a(b) means a$\times 10^b$.

	Method	N	$f''(0)$	number of iterations	condition number
$\beta = 2$, $\eta_\infty = 3.1$	[12]	10	1.5667	11	8.2520(6)
	indirect spectral collocation method	7	1.687198	6	93.0221
$\beta = 0.5$, $\eta_\infty = 3.7$	[12]	12	1.0594	8	6.1663(9)
	indirect spectral collocation method	7	0.927804	6	65.5554
$\beta = 0.3$, $\eta_\infty = 4.4$	[12]	15	0.7818	9	1(5)
	inderct spectral collocation method	7	0.774767	5	124.8
$\beta = 0$, $\eta_\infty = 6.9$	[12]	10	0.06	7	5.0541(4)
	indirect spectral collocation method	7	0.469582	8	909.9324

TABLE 3. approximations of $f''(0)$ via Kansa collocation method using Hardy multiquadric functions

	N	$f''(0)$	time(second)
$\beta = 2$	3	1.6621	1.5377
	5	1.6519	3.0123
$\eta_\infty = 3.1$	7	1.6715	4.5880
$\beta = 0.5$	3	0.9005	5.6720
	5	0.9263	7.4523
$\eta_\infty = 3.7$	7	0.9270	12.2347
$\beta = 0.3$	3	0.7593	7.8463
	5	0.7784	10.0772
$\eta_\infty = 4.4$	7	0.7763	18.2180
$\beta = 0$	3	0.5939	13.9573
$\eta_\infty = 6.9$	5	0.4655	21.1732

TABLE 4. approximations of $f''(0)$ via Kansa collocation method using inverse multiquadric functions

	N	$f''(0)$	time(second)
$\beta = 2$	3	1.7381	2.2673
	5	1.6820	4.7788
$\eta_\infty = 3.1$	7	1.6877	7.3568
$\beta = 0.5$	3	0.9129	3.2311
	5	0.9314	5.3581
$\eta_\infty = 3.7$	7	0.9306	10.2935
$\beta = 0.3$	3	0.7618	4.1371
	5	0.7806	8.2122
$\eta_\infty = 4.4$	7	0.7780	11.5321
$\beta = 0$	3	0.5643	3.1333
	5	0.4937	7.2187
$\eta_\infty = 6.9$	7	0.4758	11.1032

TABLE 5. approximations of $f''(0)$ via Kansa collocation method and new technique using Hardy multiquadric and inverse multiquadric functions with $N = 3$.

	η^1	$f''(0)$[MQ]	$f''(0)$[IMQ]
$\beta = 2$	$\frac{\eta_\infty}{4}$	1.2226	1.1391
	$\frac{\eta_\infty}{6}$	1.7338	1.6809
$\eta_\infty = 3.1$	$\frac{\eta_\infty}{12}$	1.7034	1.6893
	$\frac{\eta_\infty}{24}$	1.6930	1.6899
$\beta = 0.5$	$\frac{\eta_\infty}{4}$	0.9394	0.9307
	$\frac{\eta_\infty}{6}$	0.9349	0.9334
$\eta_\infty = 3.7$	$\frac{\eta_\infty}{12}$	0.9314	0.9302
	$\frac{\eta_\infty}{24}$	0.9278	0.9278
$\beta = 0.3$	$\frac{\eta_\infty}{4}$	0.7798	0.7851
	$\frac{\eta_\infty}{6}$	0.7789	0.7844
$\eta_\infty = 4.4$	$\frac{\eta_\infty}{12}$	0.7753	0.7775
	$\frac{\eta_\infty}{4}$	0.7747	———
$\beta = 0$	$\frac{\eta_\infty}{4}$	0.4628	0.5201
$\eta_\infty = 6.9$	$\frac{\eta_\infty}{6}$	0.4696	0.5054