Available online at http://scik.org
J. Math. Comput. Sci. 4 (2014), No. 2, 298-305
ISSN: 1927-5307

SPECTRAL RADIUS OF SLANT HANKEL OPERATORS

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Abstract. A slant Hankel operator S_ϕ with ϕ in $L^\infty(\partial D)$ is an operator on $L^2(\partial D)$ whose representing matrix $M = (\alpha_{ij})$ is given by $\alpha_{ij} = \langle \phi, z^{-2i-j} \rangle$ where $\langle , \rangle$ is the usual inner product on $L^2(\partial D)$. In this paper, the bounds of spectral radius of the operator are determined. Also the point spectrum of the adjoint of the operator is identified.

Keywords: adjoint operator; slant Hankel operator; spectral radius.

2010 AMS Subject Classification: 47B35, 47A10.

1. Introduction

Let D be the unit disc$\{z : |z| < 1\}$ in the complex plane and let $\{z : |z| = 1\}$, the unit circle, be the boundary ∂D of D. Let $\phi(z) = \sum_{i=-\infty}^{\infty} a_i z^i$ be a bounded measurable function on the unit circle.Then $\phi \in L^\infty(\partial D)$. The slant Hankel operator S_ϕ is the operator on $L^2(\partial D)$ given by the following matrix w.r.t. the usual basis $\{z^i : i \in Z\}$ of $L^2(\partial D)$:

$$
\begin{pmatrix}
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & a_2 & a_1 & a_0 & \cdots & \cdots \\
\cdots & a_0 & a_{-1} & a_{-2} & \cdots & \cdots \\
\cdots & \cdots & a_{-3} & a_{-4} & a_{-5} & \cdots \\
\cdots & \cdots & \cdots & a_{-6} & a_{-7} & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{pmatrix} \tag{1.1}
$$

Received Jan 23, 2014

The motivation of the construction of the matrix of the slant Hankel operator [4] has come from the matrix of slant Toeplitz operator given in [2].

2. Spectral Radius of S_ϕ

In this section, we attempt to determine an expression for the spectral radius of the operator S_ϕ. We begin by proving the following.

Theorem 2.1. *Let ϕ in $L^\infty(\partial D)$ be essentially bounded away from 0. If $<\phi_n>$ is a sequence in $L^\infty(\partial D)$ such that $\|\phi_n - \phi\|_\infty \to 0$, then $\lim_{n\to\infty}(S_{\phi_n}) = r(S_\phi)$.*

Proof: Take $\rho > 1$ and $\delta > 1$. Using lemma 3.1 [3], there exists an $\varepsilon > 0$ such that $\delta|\phi| < |\psi| < \rho|\phi|$ a.e. whenever $\|\psi - \phi\|_\infty < \varepsilon$ for any $L^\infty(\partial D)$ function ψ. We can choose large integer N such that $\|\phi_n - \phi\| < \varepsilon$ whenever $n > N$. Therefore, we have for each $n > N$,

$$\delta|\phi| < |\phi_n| < \rho|\phi|.$$

From [5], we have

$$r(S_\phi) = \lim_{n\to\infty} \|\psi_n\|^{\frac{1}{n}}, \tag{1.2}$$

where $\psi_n = S^n_{|\phi|^2}(1)$. We can write for each $n > N$,

$$r(S_{\delta\phi}) \leq r(S_{\phi_n}) \leq r(S_{\rho\phi}).$$

It gives

$$\delta r(S_\phi) \leq \liminf_{n\to\infty} r(S_{\phi_n}) \leqslant \limsup_{n\to\infty} \leq \rho r(S_\phi).$$

Making $\rho, \delta \to 1$, we get $\lim_{n\to\infty} r(S_{\phi_n}) = r(S_\phi)$. For trigonometric polynomial ϕ, it is shown in [5], that

$$r(S_\phi) = \lim_{n\to\infty} \left(\int_0^{2\pi} \prod_{k=0}^{n-1} |\phi((-2)^k\theta)|^2 \frac{d\theta}{2\pi}\right)^{\frac{1}{2n}}.$$

We can also have the same result for continuous ϕ. This completes the proof.

Corrollary 2.2. *Let ϕ be a non-zero continuous function on ∂D. Then*

$$r(S_\phi) = \lim_{n\to\infty} \Big(\int_0^{2\pi} \prod_{k=0}^{n-1} |\phi((-2)^k\theta)|^2 \frac{d\theta}{2\pi}\Big)^{\frac{1}{2n}}$$

Now, we make use of Ergodic theory results. Consider the positive measure space $(\partial D, A, \mu)$, where μ is the normalized lebesgue measure $\frac{d\theta}{2\pi}$. Let ν be a function on ∂D defined as $\nu(e^{i\theta}) = e^{-2i\theta}$. Then ν is a measure preserving continuous map. Also consider the composition operator T on $L^p(\partial D), 1 \le p < \infty$ induced by ν i.e. $Tf = fo\nu$ for any $f\varepsilon L^p(\partial D)$. It is also easy to see that ν is ergodic. Then by Birkhoff ergodic theorem, we have that
for any $f\varepsilon L^p(\partial D), 1 \le p < \infty, \frac{1}{n}\sum_{k=0}^{n-1} T^k f \to l$ a.e.
in L^p and for some constant l i.e.

$$\|\frac{1}{n}\sum_{k=0}^{n-1} T^k - l\|_p \to 0, as, n \to \infty.$$

But for each n,

$$\begin{aligned}
&\textstyle\int_0^{2\pi} \frac{1}{n}\sum_{k=0}^{n-1} T^k f \frac{d\theta}{2\pi} \\
&= \int_0^{2\pi} \frac{1}{n}\sum_{k=0}^{n-1} f((-2)^k\theta)\frac{d\theta}{2\pi} \\
&= \frac{1}{n}\sum_{k=0}^{n-1}\int_0^{2\pi} f((-2)^k\theta)\frac{d\theta}{2\pi} \\
&= \int_0^{2\pi} f\frac{d\theta}{2\pi}.
\end{aligned}$$

Also,

$$\begin{aligned}
|\int_0^{2\pi} \frac{1}{n}\sum_{k=0}^{n-1} T^k \frac{d\theta}{2\pi} - l| &\le \|\frac{1}{n}\sum_{k=0}^{n-1} T^k f - l\|_1 \\
&\le \|\frac{1}{n}\sum_{k=0}^{n-1} T^k f - l\|_p \to \infty
\end{aligned}$$

as $n \to \infty$. Therefore, $l = \int_0^{2\pi} f\frac{d\theta}{2\pi}$.

Theorem 2.3. *For any $\phi\varepsilon L^{\infty(\partial D)}$, $e^{\int_0^{2\pi}\log|\phi|\frac{d\theta}{2\pi}} \le r(S_\phi)$.*

Proof. Since ϕ is bounded above,

$$-\infty \le \int_0^{2\pi} \log|\phi|\frac{d\theta}{2\pi} < \infty.$$

Without loss of generality, we can take

$$-\infty < \int_0^{2\pi} \log|\phi| \frac{d\theta}{2\pi} < \infty.$$

Therefore, $\log|\phi|$ is integrable. So applying Birkhoff ergodic theorem to the L^1 function $\log|\phi|$, we have

$$\frac{1}{n}\sum_{k=0}^{n-1} \log|\phi((-2)^k\theta)| \to \int_0^{2\pi} \log|\phi|\frac{d\theta}{2\pi}$$

a.e. on∂D. Therefore,

$$[\prod_{k=0}^{n-1} |\phi((-2)^k\theta)|]^{\frac{1}{n}} = e^{\frac{1}{n}\sum_{k=0}^{n-1}\log|\phi((-2)^k\theta)|} \tag{1.3}$$

$$\to e^{\int_0^{2\pi}\log|\phi|\frac{d\theta}{2\pi}} \tag{1.4}$$

a.e. on ∂D. Now from (1), we get

$$\begin{aligned} r(S_\phi) &\geq \limsup_{n\to\infty} \|\psi_{(n)}\|_1^{\frac{1}{2n}} \\ &= \limsup_{n\to\infty}[\int_0^{2\pi} S^n_{|\phi|^2}(1)\frac{d\theta}{2\pi}]^{\frac{1}{2n}} \\ &= \limsup_{n\to\infty}[\int_0^{2\pi}\prod_{k=0}^{n-1}|\phi((-2)^k\theta)|^2\frac{d\theta}{2\pi}]^{\frac{1}{2n}}. \end{aligned}$$

As $f(x) = x^{\frac{1}{2n}}$ is a concave function on $[0,\infty)$, therefore using Lebesgue dominated convergence theorem, we have for each n,

$$e^{\int_0^{2\pi}\log|\phi|\frac{d\theta}{2\pi}} \leq r(S_\phi)$$

Theorem 2.4. *For any $\phi \varepsilon L^\infty(\partial D)$, such that $\log|\phi|$ is integrable,*

$$\sigma_p(S^*_\phi) \subseteq \{\lambda : |\lambda| = e^{\int_0^{2\pi}\log|\phi|\frac{d\theta}{2\pi}}\}.$$

Proof. Let λ be an eigenvalue of S^*_ϕ. We show that $\lambda \neq 0$. If possible, let $\lambda = 0$. Then there exists $f \neq 0$ in $L^2(\partial D)$ such that $S^*_\phi f = 0$. It gives $\overline{\phi}(\theta)f((-2)) = 0$ a.e. θ. Therefore, we have for almost every θ in $[0,2\pi), f(-2\theta) = 0$ whenever $\overline{\phi}(\theta) \neq 0$. But $\log|\phi|$ is integrable and so $\phi \neq 0$ a.e. on ∂D. Let $E = \theta\varepsilon[0,2\pi) : f(\theta) = 0$. Then E is invariant under ν and ν is ergodic. So, $m(E) = 0$ or $m(E) = 1$ where m is the normalized lebesgue measure on ∂D. Using the fact that ν is onto in $[0,2\pi)$, we have $m(E) \neq 0$. Therefore, $m(E) = 1$. It implies $f = 0$ a.e. on ∂D, a contradiction. Therefore $\lambda \neq 0$. Corresponding to the eigenvalue λ,there exists a unit vector $g \neq o$ in $L^2(\partial D)$ such that $S_\phi^* g = \lambda g$. It gives $\overline{\phi}(\theta) g((-2)\theta) = \lambda g(\theta)$ a.e. $\theta[0,2\pi) : g(\theta) \neq 0$.

Consider $E_1 = \{\theta \varepsilon [0,2\pi) : g(\theta) \neq 0\}$. Then E_1 is also invariant under ν and ν is ergodic. Therefore,$m(E_1) = 0$ or $m(E_1) = 1$. But $g \neq 0$ and so $m(E_1 \neq 0)$. Hence $m(E_1) = 1$. Thus, we get $g \neq 0$ a.e. on ∂D. As (E_1) is invariant under $\nu, go\nu^n \neq 0$ a.e. on ∂D for every n.

Now we have for each n,

$$[\prod_{k=0}^{n-1} |\phi((-2)^k\theta)|] |g((-2)^n\theta)| = |\lambda|^n |g(\theta)|$$

a.e. θ.

But $go\nu^n \neq 0$ a.e. on ∂D and therefore

$$[\prod_{k=0}^{n-1} |\phi((-2)^k\theta)|]^{\frac{1}{n}} = |\lambda| \frac{|g(\theta)|^{\frac{1}{n}}}{|g((-2)^n\theta)|^{\frac{1}{n}}} \tag{1.5}$$

a.e. θ. Now we show that

$$\frac{|g(\theta)|^{\frac{1}{n}}}{|g((-2)^n\theta)|^{\frac{1}{n}}} \to l$$

a.e. θ on ∂D. By the definition of g, $\log \frac{|g|}{|go\nu|} = \log|\phi| - \log|\lambda|$ and so $\log|\phi| - \log|\lambda|$ is integrable. Therefore by Birkhoff ergodic theorem, we have,

$$\frac{1}{n}\sum_{k=0}^{n-1} \log \frac{|g((-2)^k\theta)|}{|g((-2)^{k+1}\theta)|} \to \int_{\partial D} \log \frac{|g|}{|go\nu|} = 0$$

a.e. θ. So,

$$\frac{|g(\theta)|^{\frac{1}{n}}}{|g((-2)^n\theta)|^{\frac{1}{n}}} = e^{\frac{1}{n}\sum_{k=0}^{n-1} \log \frac{|g((-2)^k\theta)|}{|g((-2)^{k+1}\theta)|}} \to l$$

a.e. θ. Therefore, we get

$$|\lambda| = e^{\int_0^{2\pi} \log|\phi| \frac{d\theta}{2\pi}}.$$

Hence,

$$\sigma_p(S_\phi^*) \subseteq \{\lambda : |\lambda| = e^{\int_0^{2\pi} \log|\phi| \frac{d\theta}{2\pi}}\}.$$

Let us consider the mapping $\tau(e^{i\theta}) = e^{2i\theta}$ for θ in $[0,2\pi)$. Using Theorem $[3,4.1]$ we can have the following consequences.

Corollary 2.5. *Let ϕ be an invertible $L^\infty(\partial D)$ function such that $\log|\phi|$ is integrable. If $\sigma_p(A_\phi^*)\cap\sigma_p(S_\phi^*)\neq\emptyset$, then there exists functions f,g in $L^2(\partial D)$ such that*

$$g(fo\tau)=f(go\nu) a.e., on \partial D$$

Corollary 2.6. *Let ϕ be an invertible $L^\infty(\partial D)$ function such that $\log|\phi|$ is integrable. Then for any λ such that $|\lambda|\neq e^{\int_0^{2\pi}\log|\phi|\frac{d\theta}{2\pi}}$, $S_\phi-\lambda$ is invertible iff $S_\phi^*-\overline{\lambda}$ is onto.*

3. Isometry of S_ϕ^*

In this section, we assume that S_ϕ^* is an isometry. Here is one characterization for ϕ.

Theorem 3.1. *For a $L^\infty(\partial D)$ function ϕ, S_ϕ^* is an isometry iff $|\phi(\theta)|^2+|\phi(\theta+\pi)|^2=2$ a.e. θ in $[0,2\pi)$.*

Proof. Let f be any $L^2(\partial D)$ function. Then

$$\begin{aligned}
&\|S_\phi^* f\|_2^2\\
&= \int_0^{2\pi}|\phi(\theta)|^2|f(-2\theta)|^2\frac{d\theta}{2\pi}\\
&= \int_0^{2\pi}[\frac{|\phi(\frac{\theta}{2})|^2+|\phi(\frac{\theta}{2}+\pi)|^2}{2}]|f(-2\theta)|^2\frac{d\theta}{2\pi}\\
&= \|M_\psi g\|_2^2,
\end{aligned}$$

where $g(\theta)=f(-\theta)$ and $\psi(\theta)=[\frac{|\phi(\frac{\theta}{2})|^2+|\phi(\frac{\theta}{2}+\pi)|^2}{2}]^{\frac{1}{2}}$ a.e. θ in $[0,2\pi)$. But, $\|M_\psi g\|_2=\|g\|_2$ iff $|\psi|=1$ a.e. on ∂D. Also, $\|f\|_2=\|g\|_2$. Using the above theorem, we can easily check that $S_{\phi_\alpha}^*$ is an isometry for each complex number α where $\phi_\alpha(\theta)=\frac{\alpha+e^{i\theta}}{\sqrt{1+|\alpha|^2}}$.

Theorem 3.2. *Let ϕ be an invertible $L^\infty(\partial D)$ function such that $\log|\phi|$ is integrable. Then $\sigma_p(S_\phi^*)\cap\{\mu:|\mu|=1\}\neq\emptyset$ if and only if $\phi=\lambda[\frac{f}{fo\nu}]$ for some $|\lambda|=1$ and f in $L^2(\partial D)$ with $|f|=1$ a.e. on ∂D and $S_\phi^* f=\overline{\lambda}f$.*

Proof.

$$\begin{aligned}\int_0^{2\pi}\log|\phi(\theta)|\frac{d\theta}{2\pi} &= \frac{1}{2}\{\int_0^{2\pi}\log|\phi(\frac{\theta}{2})|\frac{d\theta}{2\pi}+\int_0^{2\pi}\log|\phi(\frac{\theta}{2}+\pi)|\frac{d\theta}{2\pi}\} \\ &= \frac{1}{2}\int_0^{2\pi}\log|\phi(\frac{\theta}{2})||\phi(\frac{\theta}{2}+\pi)|\frac{d\theta}{2\pi} \\ &\leq \frac{1}{2}\int_0^{2\pi}\log[\frac{|\phi(\frac{\theta}{2})|^2+|\phi(\frac{\theta}{2}+\pi)|^2}{2}]\frac{d\theta}{2\pi}\end{aligned}$$

But S_ϕ^* is an isometry and using Theorem 2.1, we get

$$\int_0^{2\pi}\log|\phi(\theta)|\frac{d\theta}{2\pi}\leq\frac{1}{2}\int_0^{2\pi}\log 1\frac{d\theta}{2\pi}=0.$$

Therefore,

$$e^{\int_0^{2\pi}\log|\phi(\theta)|\frac{d\theta}{2\pi}}\leq 1.$$

Suppose $\sigma_p(S_\phi^*)\cap\{\mu:|\mu|=1\}\neq\emptyset$. Then by Theorem 1.4, we have

$$e^{\int_0^{2\pi}\log|\phi(\theta)|\frac{d\theta}{2\pi}}=1.$$

Therefore, the set $\{\theta\varepsilon[0,2\pi):|\phi(\theta)\phi(\theta+\pi)|=\frac{|\phi(\theta)|^2+|\phi(\theta)+\pi|^2}{2}\}$ has measure 1. So, we have $|\phi(\theta)|=|\phi(\theta+\pi)|$ a.e. θ. It gives $|\phi|=1$ a.e. on ∂D. Let $\mu\varepsilon\sigma_p(S_\phi^*)$be such that $|\mu|=1$. Then, there exists non-zero g in $L^2(\partial D)$ such that $S_\phi^*g=\mu g$. So, we have $|go\nu|=|g|$ a.e. and so $|g|o\nu=|g|$. Since ν is ergodic, we can take without loss of generality that $|g|=1$ a.e. on ∂D. Hence, we have $\phi=\lambda[\frac{f}{fo\nu}]$ where $\lambda=\overline{\mu}$ and $f=\overline{g}$.

Conflict of Interests

The author declares that there is no conflict of interests.

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