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SOME I-LACUNARY DIFFERENCE DOUBLE SEQUENCES IN n-NORMED SPACES DEFINED BY SEQUENCE OF ORLICZ FUNCTIONS

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Abstract. The notion of ideal convergence was introduced first by Kostryko et al [16] as a generalization of statistical convergence. In this paper we introduce a new class of generalized difference double sequence spaces using the concept of ideal, lacunary convergence and sequence of Orlicz functions in n-normed space. Further we obtain various inclusion relations involving these sequence spaces.

Keywords: Difference double sequence spaces, Ideal, Lacunary convergence, n-norm, Sequence of Orlicz Functions.

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1. Introduction

The notion of ideal convergence was introduced first by Kostyrko et.al.[16] as an interesting generalization of statistical convergence which was further studied in topological spaces. A family $I \subset 2^Y$ of subsets a nonempty set Y is said to be an ideal in Y if

(1) $\emptyset \in I$;

(2) $A, B \in I$ imply $A \cup B \in I$;

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(3) $A \in I, B \subset A$ imply $B \in I$,

while an admissible ideal I further satisfies $\{x\} \in I$ for each $x \in Y$.
Given $I \subset 2^{\mathbb{N}}$ be a nontrivial ideal in $\mathbb{N}$. The sequence (x_j) in X is said to be $I-convergent$ to $\xi \in X$, if for each $\varepsilon > 0$ the set $A(\varepsilon) = \{j \in \mathbb{N} : \|x_j - \xi\| \geq \varepsilon\}$ belongs to I.

The concept of 2-normed spaces was initially introduced by G̈ahler[4,6] in the mid of 1960's as an interesting nonlinear generalization of a normed linear space which was subsequently studied by many authors see for instance[6,10,12,14].

Definition 1.1.[3] Let $n \in \mathbb{N}$ and X be real vector space of dimension d, where $n \leq d$. An n-norm on X is a function $\|., ..., .\| : X \times X \times ... \times X \to \mathbb{R}$ on X^n which satisfy the following four conditions:

(1) $\|x_1, x_2, ..., x_n\| = 0$ if and only if $x_1, x_2, ..., x_n$ are linearly dependent;
(2) $\|x_1, x_2, ..., x_n\|$ is invariant under permutation:
(3) $\|\alpha x_1, x_2, ..., x_n\| = |\alpha| \|x_1, x_2, ..., x_n\|$, for any $\alpha \in R$:
(4) $\|x + x', x_2, ..., x_n\| \leq \|x, x_2, ..., x_n\| + \|x', x_2, ..., x_n\|$

is called an $n-norm$ on X, and the pair $(X, \|., ., ..., .\|)$ is then called an n-normed space.

Example 1.1. As a standard example of an n-normed space we may take R^n being equipped with the n-norm $\|x_1, x_2, ..., x_n\|_E =$ the volume of the n-dimensional parallelop-iped spaned by the vectors $x_1, x_2, ..., x_n$ which may be given explicitly by the formula

$$\|x_1, x_2, ..., x_n\|_E = |\det(x_{ij})|,$$

where $x_i = (x_{i1}, x_{i2}, ..., x_{in}) \in R^n$ for each $i = 1, 2, ..., n$.

Example 1.2. Let $(X, \|., ., ..., .\|)$ be an n-normed space of dimension $d \geq n \geq 2$ and $\{a_1, a_2, ..., a_n\}$ be a linearly independent set in X. Then the following function $\|., ., ..., .\|_\infty$

defined by

$$\|x_1, x_2, ..., x_n\|_\infty = \max\{\|x_1, x_2, ..., x_{n-1}, a_i\| : i = 1, 2, ..., n\}$$

defines an $(n-1)-$norm on X with respect to $\{a_1, a_2, ..., a_n\}$.

Example 1.3. Let $n \in \mathbb{N}$ and $(X, \langle ., . \rangle)$ be a real inner product space of dimension $d \geq n$, then the following function $\|., ., ..., .\|_S$ on $X \times ... \times X$(n factor) defined by

$$\|x_1, x_2, ..., x_n\|_S = [\det(\langle x_i, x_j \rangle)]^{\frac{1}{2}}$$

is an n-norm on X. Let w, l_∞, c and c_0 denote the spaces of all, bounded, convergent and null sequences $x = (x_k)$ with complex terms, respectively normed by

$$\|x\| = \sup_k |x_k|.$$

Kizmaz [15], defined the difference sequences $l_\infty(\Delta), c(\Delta)$ and $c_0(\Delta)$ as follows:

$$Z(\Delta) = \{x = (x_k) : (\Delta x_k) \in Z\},$$

for $Z = l_\infty, c$ and c_0, where $\Delta x = (\Delta x_k) = (x_k - x_{k+1})$, for all $k \in \mathbb{N}$.
The above spaces are Banach spaces, normed by

$$\|x\|_\Delta = |x_1| + \sup_k \|\Delta x_k\|.$$

The notion of difference sequence spaces was generalized by Et. and Colak[1] as follows:

$$Z(\Delta^n) = \{x = (x_k) : (\Delta^n x_k) \in Z\},$$

for $Z = l_\infty, c$ and c_0, where $n \in \mathbb{N}, (\Delta^n x_k) = (\Delta^{n-1} x_k - \Delta^{n-1} x_{k+1})$ and so that

$$\Delta^n x_k = \sum_{v=0}^{n} (-1)^v \binom{n}{v} x_{k+v}.$$

An *Orlicz Function* is a function $M : [0, \infty) \to [0, \infty)$ which is continuous, nondecreasing and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \to \infty$, as $x \to \infty$.

Lindenstrauss and Tzafriri [17] used the idea of Orlicz sequence space;

$$l_M := \left\{ x \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \text{ for some } \rho > 0 \right\}$$

which is Banach space with the norm

$$\|x\|_M = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\}.$$

Orlicz function has been studied by V.A.Khan[7,8,9], V.A.Khan and S.Tabassum[10,11,12,13,14] and many others.

Throughout, a double sequence $x = (x_{jk})$ is a double infinite array of elements x_{jk} for $j, k \in \mathbb{N}$. Double sequences have been studied by V.A.Khan[9], V.A.Khan and S. Tabassum[10,11,12,13,14], Moricz and Rhoades[18] and many others.

By a lacunary sequence $\theta = (k_r)$, r=0,1,2,... where $k_o = 0$, we mean an increasing sequence of non negative integers $h_r = (k_r - k_{r-1}) \to \infty (r \to \infty)$. The intervals determined by θ are denoted by $I_r = (k_{r-1}, k_r]$ and ratio $\frac{k_r}{k_{r-1}}$ will be denoted by q_r.
The space of lacunary strongly convergent sequence N_θ was defined by Freedman et al.[2] as follows

$$N_\theta = \left\{ x = (x_j) : \lim_{r \to \infty} \frac{1}{h_r} \sum_{j \in I_r} |x_j - L| = 0, \text{ for some } L \right\}.$$

The double lacunary sequence was defined by E.Savas and R.F.Patterson[20] as follows:
The double sequence $\theta_{r,s} = \{(k_r, l_s)\}$ is called double lacunary if there exist two increasing sequence of integers such that

$$k_0 = 0, h_r = k_r - k_{r-1} \to \infty \text{ as } r \to \infty$$

and

$$l_0 = 0, h_s^- = l_s - l_{s-1} \to \infty \text{ as } s \to \infty.$$

The following intervals are determined by θ.

$$I_r = \{(k) : k_{r-1} < k < k_r\}, I_s = \{(l) : l_{s-1} < l < l_s\}$$

$$I_{r,s} = \{(k,l) : k_{r-1} < k < k_r \text{ and } l_{s-1} < l < l_s\},$$

$q_r = \frac{k_r}{k_{r-1}}, q_s^- = \frac{l_s}{l_{s-1}}$ and $q_{r,s} = q_r q_s^-$. We will denote the set of all lacunary sequences by $N_{\theta_{r,s}}$.

Let $x = (x_{jk})$ be a double sequence that is a double infinite array of elements x_{jk}. The space of double lacunary strongly convergent sequence is defined as follows:

$$N_{\theta_{r,s}} = \left\{ x = (x_{jk}) : \lim_{r,s} \frac{1}{h_{r,s}} \sum_{(j,k) \in I_{r,s}} |x_{jk} - L| = 0 \text{ for some } L \right\}.$$

2. Preliminaries

Let I be an admissible ideal, $M = (M_k)$ be the sequence of Orlicz functions, $(X, \|., ..., .\|)$ be an n-normed space, and $p = (p_{jk})$ be a sequence of positive real numbers. Let ${}_2W(n-X)$ be the space of all double sequences defined over an n-normed space $(X, \|., ..., .\|)$. We define

$${}_2W[N_{\theta_{r,s}}, M, \Delta^m, p, \|., ..., .\|]^I = \Bigg\{ (x_{jk}) \in W(n-X) : \forall \epsilon > 0 \Bigg\{ (j,k) \in N \times N : \lim_{r,s} \frac{1}{h_{rs}} \sum_{(j,k) \in I_{r,s}} \left[M_k \left(\left\| \frac{\Delta^m x_{jk} - L}{\rho}, z_1, z_2, .., z_{n-1} \right\| \right) \right]^{p_{jk}} \geq \varepsilon \Bigg\}, \text{for some } \rho > 0, L \in X, \text{ each } z_1, z_2, ., z_{n-1} \in X \Bigg] \in I \Bigg\}$$

$${}_2W[N_{\theta_{r,s}}, M, \Delta^m, p, \|., ..., .\|]^I_\circ = \Bigg\{ (x_{jk}) \in W(n-X) : \forall \epsilon > 0 \Bigg\{ (j,k) \in N \times N : \lim_{r,s} \frac{1}{h_{rs}} \sum_{(j,k) \in I_{r,s}} \left[M_k \left(\left\| \frac{\Delta^m x_{jk}}{\rho}, z_1, z_2, ..., z_{n-1} \right\| \right) \right]^{p_{jk}} \geq \varepsilon \Bigg\}, \text{for some } \rho > 0, \text{ and each } z_1, z_2, .., z_{n-1} \in X \Bigg] \in I \Bigg\}.$$

When $m = 0$ we obtain the following sequence spaces:

$$_2W[N_{\theta_{r,s}}, M, p, \|., ..., .\|]^I = \Bigg\{(x_{jk}) \in W(n-X) : \forall\epsilon > 0\Bigg\{(j,k) \in N \times N : \lim_{r,s} \frac{1}{h_{rs}}$$

$$\sum_{(j,k)\in I_{r,s}} \left[M_k\left(\left\|\frac{x_{jk}-L}{\rho}, z_1, z_2, .., z_{n-1}\right\|\right)\right]^{p_{jk}} \geq \epsilon\Bigg\}\text{for some } \rho > 0, L \in X \text{ and each } z_1, z_2, ., z_{n-1} \in X\Bigg] \in I\Bigg\}$$

$$_2W[N_{\theta_{r,s}}, M, p, \|., ..., .\|]^I_\circ = \Bigg\{(x_{jk}) \in W(n-X) : \forall\epsilon > 0\Bigg\{(j,k) \in N \times N : \lim_{r,s} \frac{1}{h_{rs}}$$

$$\sum_{(j,k)\in I_{r,s}} \left[M_k\left(\left\|\frac{x_{jk}}{\rho}, z_1, z_2, ..., z_{n-1}\right\|\right)\right]^{p_{jk}} \geq \epsilon\Bigg\}, \text{ for some } \rho > 0, \text{ and each } z_1, z_2, .., z_{n-1} \in X\Bigg] \in I\Bigg\}.$$

When $m = 1$, we obtain the following difference sequence spaces:

$$_2W[N_{\theta_{r,s}}, M, \Delta, p, \|., ..., .\|]^I = \Bigg\{(x_{jk}) \in W(n-X) : \forall\epsilon > 0\Bigg\{(j,k) \in N \times N : \lim_{r,s} \frac{1}{h_{rs}}$$

$$\sum_{(j,k)\in I_{r,s}} \left[M_k\left(\left\|\frac{\Delta x_{jk}-L}{\rho}, z_1, z_2, .., z_{n-1}\right\|\right)\right]^{p_{jk}} \geq \epsilon\Bigg\}, \text{ some } \rho > 0, L \in X\text{and each } z_1, z_2, ., z_{n-1} \in X\Bigg] \in I\Bigg\}.$$

$$_2W[N_{\theta_{r,s}}, M, \Delta, p, \|., ..., .\|]^I_\circ = \Bigg\{(x_{jk}) \in W(n-X) : \forall\epsilon > 0\Bigg\{(j,k) \in N \times N : \lim_{r,s} \frac{1}{h_{rs}}$$

$$\sum_{(j,k)\in I_{r,s}} \left[M_k\left(\left\|\frac{\Delta x_{jk}}{\rho}, z_1, z_2, ..., z_{n-1}\right\|\right)\right]^{p_{jk}} \geq \epsilon\Bigg\}, \text{ for some } \rho > 0, \text{ and each } z_1, z_2, .., z_{n-1} \in X\Bigg] \in I\Bigg\}.$$

where

$$(\Delta^m x_{jk}) = (\Delta^{m-1}x_{jk} - \Delta^{m-1}x_{j+1,k} - \Delta^{m-1}x_{j,k+1} + \Delta^{m-1}x_{j+1,k+1})$$

$$(\Delta^1 x_{jk}) = (\Delta x_{jk}) = (x_{jk} - x_{j+1,k} - x_{j,k+1} + x_{j+1,k+1})$$

and

$$(\Delta^0 x) = (x_{jk})$$

and also this generalized difference double notion has the following binomial representation:

$$\Delta^m x_{jk} = \sum_{k=0}^{m}\sum_{l=0}^{m}(-1)^{k+l}\binom{m}{k}\binom{m}{l}x_{i+k,j+l}.$$

The following inequality will be used througout the paper. Let $p = (p_{jk})$ be a double sequence of real numbers with $0 < p_{jk} \leq \sup p_{jk} = H$ and let $K = \max[1, 2^{H-1}]$. Then for the factorable sequences (a_{jk}) and (b_{jk}) in the complex plane we have:

$$|a_{jk} + b_{jk}|^{p_{jk}} \leq H\{|a_{jk}|^{p_{jk}} + |b_{jk}|^{p_{jk}}\}, \tag{2.1}$$

3. Main results

Theorem 3.1. *Let $M = (M_k)$ be the sequence of Orlicz functions and $p = (p_{jk})$ be a bounded sequence of strictly positive real numbers, then ${}_2W[N_{\theta_{r,s}}, M, \Delta^m, \|., ..., .\|]^I$ and ${}_2W[N_{\theta_{r,s}}, M, \Delta^m, \|., ..., .\|]^I_\circ$ are linear spaces over the complex field $\mathbb{C}$.*

Proof. Let (x_{jk}) and $(y_{jk}) \in {}_2W[N_{\theta_{r,s}}, M, \Delta^m, p, \|., ..., .\|]^I_\circ$ and $\alpha, \beta \in \mathbb{C}$. Then there exist some $\rho_1, \rho_2 > 0$ such that

$$\lim_{r,s} \tfrac{1}{h_{rs}} \sum_{(j,k)\in I_{r,s}} \left[M_k\left(\left\|\tfrac{\Delta^m(\alpha x_{jk}+\beta y_{jk})}{|\alpha|\rho_1+|\beta|\rho_2}, z_1, z_2, ..., z_{n-1}\right\|\right)\right]^{p_{jk}}$$

$$\leq K \lim_{r,s} \frac{1}{h_{rs}} \sum_{(j,k)\in I_{r,s}} \left[\frac{|\alpha|}{|\alpha|\rho_1+|\beta|\rho_2} M_k\left(\left\|\frac{\Delta^m x_{jk}}{\rho_1}, z_1, z_2, ..., z_{n-1}\right\|\right)\right]^{p_{jk}}$$

$$+K \lim_{r,s} \frac{1}{h_{rs}} \sum_{(j,k)\in I_{r,s}} \left[\frac{|\beta|}{|\alpha|\rho_1+|\beta|\rho_2} M_k\left(\left\|\frac{\Delta^m y_{jk}}{\rho_2}, z_1, z_2, ..., z_{n-1}\right\|\right)\right]^{p_{jk}}$$

$$\leq KF \lim_{r,s} \frac{1}{h_{rs}} \sum_{(j,k)\in I_{r,s}} \left[M_k\left(\left\|\frac{\Delta^m x_{jk}}{\rho_1}, z_1, z_2, ..., z_{n-1}\right\|\right)\right]^{p_{jk}}$$

$$+KF \lim_{r,s} \frac{1}{h_{rs}} \sum_{(j,k)\in I_{r,s}} \left[M_k\left(\left\|\frac{\Delta^m y_{jk}}{\rho_2}, z_1, z_2, ..., z_{n-1}\right\|\right)\right]^{p_{jk}}$$

Where $F = \max\left[1, \left(\frac{|\alpha|}{|\alpha|\rho_1+|\beta|\rho_2}\right)^H, \left(\frac{|\beta|}{|\alpha|\rho_1+|\beta|\rho_2}\right)^H\right]$

On the other hand from the above inequality we get

$$\left\{(j,k) \in N \times N : \lim_{r,s} \tfrac{1}{h_{rs}} \sum_{(j,k)\in I_{r,s}} \left[M_k\left(\left\|\tfrac{\Delta^m(\alpha x_{jk}+\beta y_{jk})}{\rho}, z_1, z_2, ..., z_{n-1}\right\|\right)\right]^{p_{jk}} \geq \epsilon\right\}$$

$$\subseteq \left\{(j,k) \in N \times N : KF \lim_{r,s} \frac{1}{h_{rs}} \sum_{(j,k)\in I_{r,s}} \left[M_k\left(\left\|\frac{\Delta^m x_{jk}}{\rho_1}, z_1, z_2, ..., z_{n-1}\right\|\right)\right]^{p_{jk}} \geq \frac{\epsilon}{2}\right\}$$

$$\cup\left\{(j,k)\in N\times N: KF\lim_{r,s}\frac{1}{h_{rs}}\sum_{(j,k)\in I_{r,s}}\left[M_k\left(\left\|\frac{\Delta^m y_{jk}}{\rho_2},z_1,z_2,...,z_{n-1}\right\|\right)\right]^{p_{jk}}\geq\frac{\epsilon}{2}\right\}$$

The two sets on the right side belongs to I,

This completes the proof.

Lemma 3.2. *Let M be an Orlicz function which satisfies Δ_2-condition and $0<\delta<1$. Then for each $x\geq\delta$ and some constant $K>0$ we have*

$$M(x)\leq K\delta^{-1}M(2).$$

Theorem 3.3. *Let $M=(M_k)$ be the sequence of Orlicz functions which satifies Δ_2-condition and $0<\inf_{j,k}p_{jk}=h\leq p_{jk}\leq\sup_{j,k}p_{jk}=H<\infty$, then*
${}_2W[N_{\theta_{r,s}},\Delta^m,p,\|.,...,.\|]^I\subset{}_2W[N_{\theta_{r,s}},M,\Delta^m,p,\|.,...,.\|]^I$
and
${}_2W[N_{\theta_{r,s}},\Delta^m,p,\|.,...,.\|]^I_\circ\subset{}_2W[N_{\theta_{r,s}},M,\Delta^m,p,\|.,...,.\|]^I_\circ$

Proof. Let $(x_{jk})\in{}_2W[N_{\theta_{r,s}},\Delta^m,p,\|.,...,.\|]^I$ then for some $L>0$ and for every $z_1,z_2,...,z_{n-1}\in X$

$$\left\{(j,k)\in N\times N:\lim_{r,s}\frac{1}{h_{rs}}\sum_{(j,k)\in I_{r,s}}\left[\left(\left\|\Delta^m x_{jk}-L,z_1,z_2,...,z_{n-1}\right\|\right)\right]^{p_{jk}}\geq\epsilon\right\}.$$

Now let $\epsilon>0$ be given. We can choose $0<\delta<1$ such that for every t with $0\leq t\leq\delta$ we have $M_k(t)<\epsilon$ for all k. Now using lemma we get

$$\left\{(j,k)\in N\times N:\lim_{r,s}\frac{1}{h_{rs}}\sum_{(j,k)\in I_{r,s}}\left[M_k\left(\left\|\frac{\Delta^m x_{jk}-L}{\rho},z_1,z_2,...,z_{n-1}\right\|\right)\right]^{p_{jk}}\geq\epsilon\right\}$$

$$=\left\{(j,k)\in N\times N:\lim_{r,s}\frac{1}{h_{rs}}(h_{rs}\max\{\epsilon^h,\epsilon^H\})\geq\epsilon\right\}$$

$$\cup\left\{(j,k)\in N\times N:\lim_{r,s}\frac{1}{h_{rs}}\max\{(K\delta^{-1}M_k(2))^h,(K\delta^{-1}M_k(2))^H\right\}$$

$$\sum_{(k,l)\in I_{r,s}}(\|\Delta^m x_{jk}-L,z_1,z_2,...,z_{n-1}\|)^{p_{jk}}.$$

This completes the proof. The other can be proved similarly.

Theorem 3.4. *Let $M=(M_k)$ be the sequence of Orlicz functions . If*

$$\limsup_x \frac{M_k(x)}{x} = \gamma > 0, \quad \text{for all } k,$$

then

$$_2W[N_{\theta_{r,s}}, \Delta^m, p, \|., ..., .\|]^I_\circ = {}_2W[N_{\theta_{r,s}}, M, \Delta^m, p, \|., ..., .\|]^I_\circ$$

and

$$_2W[N_{\theta_{r,s}}, \Delta^m, p, \|., ..., .\|]^I = {}_2W[N_{\theta_{r,s}}, M, \Delta^m, p, \|., ..., .\|]^I$$

Proof. In Theorem[3.3], it was shown that

$$_2W[N_{\theta_{r,s}}, \Delta^m, p, \|., ..., .\|]^I \subset {}_2W[N_{\theta_{r,s}}, M, \Delta^m, p, \|., ..., .\|]^I$$

Now let $\gamma > 0$ and let $x \in {}_2W[N_{\theta_{r,s}}, M, \Delta^m, p, \|., ..., .\|]^I$

Now since $\gamma > 0$, for every $x > 0$ we write $M_k(x) \geq \gamma x$ for all k. From this inequality

$$\frac{1}{h_{rs}} \sum_{(j,k)\in I_{r,s}} \left[M_k\left(\left\|\frac{\Delta^m x_{jk}-L}{\rho}, z_1, z_2, ..., z_{n-1}\right\|\right)\right]^{p_{jk}}$$
$$\geq \gamma^H \frac{1}{h_{rs}} \sum_{(j,k)\in I_{r,s}} \left[\left(\left\|\Delta^m x_{jk} - L, z_1, z_2, ..., z_{n-1}\right\|\right)\right]^{p_{jk}}$$

and this inequality gives the result.

Corollary 3.5. *Let $M=(M_k)$ and $N=(N_k)$ be the sequence of Orlicz functions.*

If

$$\limsup_x \frac{M_k(x)}{N_k(x)} < \infty,$$

then

$$ {}_2W[N_{\theta_{r,s}}, M, \Delta^m, p, \|., ..., .\|]^I_\circ \subset {}_2W[N_{\theta_{r,s}}, N_k, \Delta^m, p, \|., ..., .\|]^I_\circ $$

and

$$ {}_2W[N_{\theta_{r,s}}, M_k, \Delta^m, p, \|., ..., .\|]^I \subset {}_2W[N_{\theta_{r,s}}, M, \Delta^m, p, \|., ..., .\|]^I $$

Theorem 3.6. *Let $M = (M_k)$ and $N = (N_k)$ be the sequence of Orlicz functions which satifies Δ_2-condition and $0 < \inf_{j,k} p_{jk} = h \leq p_{jk} \leq \sup_{j,k} p_{jk} = H < \infty$, then*

$$ {}_2W[N_{\theta_{r,s}}, M, \Delta^m, p, \|., ..., .\|]^I_\circ \subset {}_2W[N_{\theta_{r,s}}, M \circ N, \Delta^m, p, \|., ..., .\|]^I_\circ $$

and

$$ {}_2W[N_{\theta_{r,s}}, M, \Delta^m, p, \|., ..., .\|]^I \subset {}_2W[N_{\theta_{r,s}}, M \circ N, \Delta^m, p, \|., ..., .\|]^I $$

and

$$ {}_2W[N_{\theta_{r,s}}, M, \Delta^m, p, \|., ..., .\|]^I_\circ \cap {}_2W[N_{\theta_{r,s}}, N, \Delta^m, p, \|., ..., .\|]^I_\circ \subset {}_2W[N_{\theta_{r,s}}, M+N, \Delta^m, p, \|., ..., .\|]^I_\circ $$

and

$$ {}_2W[N_{\theta_{r,s}}, M, \Delta^m, p, \|., ..., .\|]^I \cap {}_2W[N_{\theta_{r,s}}, N, \Delta^m, p, \|., ..., .\|]^I \subset {}_2W[N_{\theta_{r,s}}, M+N, \Delta^m, p, \|., ..., .\|]^I $$

Proof. Let $x = x_{jk} \in {}_2W[N_{\theta_{r,s}}, M, \Delta^m, p, \|., ..., .\|]^I_\circ$

Let $\epsilon > o$ and choose δ with $0 < \delta < 1$ such that $M(t) < \epsilon$ for $0 \leq t \leq \delta$.
Let $y_{jk} = N_k\left(\left\|\frac{\Delta^m x_{jk} - L}{\rho}, z_1, z_2, ..., z_{n-1}\right\|\right)$ for all $j, k \in \mathbb{N}$
We can write

$$ \frac{1}{h_{rs}} \sum_{(j,k) \in I_{r,s}} [M_k(y_{jk})]^{p_{jk}} = \frac{1}{h_{rs}} \sum_{(j,k) \in I_{r,s}, y_{jk} \leq \delta} [M_k(y_{jk})]^{p_{jk}} + \frac{1}{h_{rs}} \sum_{(j,k) \in I_{r,s}, y_{jk} > \delta} [M_k(y_{jk})]^{p_{jk}} $$

Then

$$\frac{1}{h_{rs}}\sum_{(j,k)\in I_{r,s},y_{jk}\leq\delta}[M_k(y_{jk})]^{p_{jk}}\leq\epsilon^H \text{ for } t\leq\delta$$

Since M_k is continuous and $M(t)<\epsilon$ for $t\leq\delta$.

Now for $y_{jk}>\delta$, we use the fact that

$$y_{jk}<\frac{y_{jk}}{\delta}<1+\frac{y_{jk}}{\delta}$$

Since M_k is non decreasing and convex, it follows that

$$M(y_{jk})<M(1+\delta^{-1}y_{jk})=M\left(\frac{2}{2}+\frac{2}{2}\delta^{-1}y_{jk}\right)$$
$$<\frac{1}{2}M(2)+\frac{1}{2}M(2\delta^{-1}y_{jk})$$

Since M satisfies Δ_2-condition, there is a constant $K>0$ such that

$$M(2\delta^{-1}y_{jk})\leq\frac{1}{2}K\delta^{-1}y_{jk}M(2)$$

Hence

$$\frac{1}{h_{rs}}\sum_{(j,k)\in I_{r,s},y_{jk}>\delta}[M_k(y_{jk})]^{p_{jk}}\leq\max\left(1,\left(\frac{KM(2)}{\delta}\right)^H\right)\frac{1}{h_{rs}}\sum_{(j,k)\in I_{r,s},y_{jk}>\delta}[(y_{jk})]^{p_{jk}}$$

Which together with

$$\frac{1}{h_{rs}}\sum_{(j,k)\in I_{r,s},y_{jk}\leq\delta}[M_k(y_{jk})]^{p_{jk}}\leq\epsilon^H \quad \text{yields}$$

$$\frac{1}{h_{rs}}\sum_{(j,k)\in I_{r,s}}[M_k(y_{jk})]^{p_{jk}}\leq\epsilon^H+\left(1,\left(\frac{KM(2)}{\delta}\right)^H\right)\frac{1}{h_{rs}}\sum_{(j,k)\in I_{r,s},y_{jk}>\delta}[(y_{jk})]^{p_{jk}}$$

This completes the proof.

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