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CERTAIN PROPERTIES OF COUNTABLY Q – COMPACT FUZZY SETS

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Abstract: In this paper, we introduced the countably Q – compact fuzzy sets (in the sense of Q – compact fuzzy sets given in [2]) and study their certain properties.

Keywords: Fuzzy topological spaces; countably Q – compact fuzzy sets.

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1. Introduction

The concept of fuzzy sets and fuzzy set operations were first introduced by L. A. Zadeh [12] in 1965, describing fuzzyness mathematically for the first time. C. L. Chang [4] developed the theory of fuzzy topological spaces in 1968. The concept of Q – compact fuzzy sets was introduced by D. M. Ali and M. A. M. Talukder in [2]. The purpose of this paper is to introduce and study the concept of countably Q – compact fuzzy sets and to obtain their several properties. In doing this, we have used the idea of q – coincident of a fuzzy singleton with a fuzzy set or the same between two fuzzy sets. We find that this concept has many tangible flavors.

2. Preliminaries

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In this section, we recall some fundamental definitions which are essential in our study and can be found in the papers referred to.

Definition 2.1 [12]: Let X be a non-empty set and I is the closed unit interval [0, 1]. A fuzzy set in X is a function u: X $\rightarrow$ I which assigns to every element x $\in$ X. u(x) denotes a degree or the grade of membership of x. The set of all fuzzy sets in X is denoted by I^X. A member of I^X may also be called a fuzzy subset of X.

Definition 2.2 [12]: Let X be a non-empty set and A $\subseteq$ X. Then the characteristic function 1_A(x): X $\rightarrow$ {0, 1 } defined by $1_A(x) = \begin{cases} 1 \ \ if \ \ x \in A \\ 0 \ \ if \ \ x \notin A \end{cases}$

Thus we can consider any subset of a set X as a fuzzy set whose range is { 0, 1 }.

Definition 2.3 [10]: A fuzzy set is empty iff its grade of membership is identically zero. It is denoted by 0 or ϕ.

Definition 2.4 [10]: A fuzzy set is whole iff its grade of membership is identically one in X . It is denoted by 1 or X.

Definition 2.5 [4]: Let u and v be two fuzzy sets in X. Then we define

(i) u = v iff u(x) = v(x) for all x $\in$ X

(ii) u $\subseteq$ v iff u(x) $\leq$ v(x) for all x $\in$ X

(iii) λ = u $\cup$ v iff λ(x) = (u $\cup$ v) (x) = max [u(x), v(x)] for all x $\in$ X

(iv) μ = u $\cap$ v iff μ(x) = (u $\cap$ v) (x) = min [u(x), v(x)] for all x $\in$ X

(v) $\gamma = u^c$ iff γ(x) = 1 – u(x) for all x $\in$ X.

Remark: Two fuzzy sets u and v are disjoint iff u $\cap$ v = 0.

Definition 2.6 [4]: In general, if { u_i : i $\in$ J } is family of fuzzy sets in X, then union $\cup u_i$ and intersection $\cap u_i$ are defined by

$\cup u_i$(x) = sup { u_i(x): i $\in$ J and x $\in$ X }

$\cap u_i$(x) = inf { u_i(x): i $\in$ J and x $\in$ X }, where J is an index set.

Definition 2.7 [4]: Let f: X $\rightarrow$ Y be a mapping and u be a fuzzy set in X. Then the image of u, written $f(u)$, is a fuzzy set in Y whose membership function is given by

$$f(u)(y) = \begin{cases} \sup\{u(x) : x \in f^{-1}(y)\} & if \ f^{-1}(y) \neq \phi \\ 0 & if \ f^{-1}(y) = \phi \end{cases}.$$

Definition 2.8 [4]: Let f: X $\rightarrow$ Y be a mapping and v be a fuzzy set in Y. Then the inverse of v, written f^{-1}(v), is a fuzzy set in X whose membership function is given by f^{-1}(v) (x) = v(f(x)).

Distributive laws 2.9 [12]: Distributive laws remain valid for fuzzy sets in X i.e. if u, v and w are fuzzy sets in X, then

(i) u $\cup$ (v $\cap$ w) = (u $\cup$ v) $\cap$ (u $\cup$ w)

(ii) u $\cap$ (v $\cup$ w) = (u $\cap$ v) $\cup$ (u $\cap$ w).

Definition 2.10 [4]: Let X be a non-empty set and t $\subseteq$ I^X i.e. t is a collection of fuzzy set in X. Then t is called a fuzzy topology on X if

(i) 0, 1 $\in$ t

(ii) $u_i \in$ t for each $i \in$J, then $\bigcup_i u_i \in$ t

(iii) u, v $\in$ t, then u $\cap$ v $\in$t

The pair (X, t) is called a fuzzy topological space and in short, fts. Every member of t is called a t-open fuzzy set. A fuzzy set is t-closed iff its complements is t-open. In the sequel, when no confusion is likely to arise, we shall call a t-open (t-closed) fuzzy set simply an open (closed) fuzzy set.

Definition 2.11 [4]: Let (X, t) and (Y, s) be two fuzzy topological spaces. A mapping f: $(X, t) \rightarrow (Y, s)$ is called an fuzzy continuous iff the inverse of each s-open fuzzy set is t-open.

Definition 2.12 [10]: Let (X, t) be an fts and A $\subseteq$ X. Then the collection t_A = { u|A: u $\in$ t } = { u $\cap$ A: u $\in$ t } is fuzzy topology on A, called the subspace fuzzy topology on A and the pair (A, t_A) is referred to as a fuzzy subspace of (X, t).

Definition 2.13 [5]: Let (A, t_A) and (B, s_B) be fuzzy subspaces of fuzzy topological spaces (X, t) and (Y, s) respectively and f is a mapping from (X, t) to (Y, s), then we say that f is a mapping from (A, t_A) to (B, s_B) if f(A) $\subseteq$ B.

Definition 2.14 [5]: Let (A, t_A) and (B, s_B) be fuzzy subspaces of fts's (X, t) and (Y, s) respectively. Then a mapping f: $(A, t_A) \rightarrow (B, s_B)$ is relatively fuzzy continuous iff for each v $\in$ s_B, the intersection f^{-1}(v) $\cap$ A $\in$ t_A.

Definition 2.15 [1]: Let λ be a fuzzy set in X, then the set { x $\in$ X: λ(x) > 0 } is called the support of λ and is denoted by λ_0 or supp λ .

Definition 2.16 [1]: Let (X, T) be a topological space. A function f: X $\rightarrow$ **R** (with usual topology) is called lower semi-continuous (l . s . c.) if for each a $\in$ **R**, the set f^{-1}(a, ∞) $\in$ T. For a topology T on a set X, let ω(T) be the set of all l . s . c. functions from (X, T) to I (with usual topology); thus ω(T) = { u $\in$ I^X : u^{-1}(a, 1] $\in$ T, a $\in$ I_1 } . It can be shown that ω(T) is a fuzzy topology on X.

Let P be a property of topological spaces and FP be its fuzzy topology analogue. Then FP is called a ' good extension' of P " iff the statement (X, T) has P iff $(X, \omega(T))$has FP" holds good for every topological space (X, T). Thus characteristic functions are l . s. c.

Definition 2.17 [11]: An fts (X, t) is said to be fuzzy – T_1 space iff for every x, y $\in$ X, x $\neq$ y, there exist u, v $\in$ t such that u(x) = 1, u(y) = 0 and v(x) = 0, v(y) = 1.

Definition 2.18 [6]: An fts (X, t) is said to be fuzzy Hausdorff iff for all x, y $\in$ X, x $\neq$ y, there exist u, v $\in$ t such that u(x) = 1, v(y) = 1 and u $\cap$ v = 0.

Definition 2.19 [9]: An fts (X, t) is said to be fuzzy Hausdorff iff for all x, y $\in$ X, x $\neq$ y, there exist u, v $\in$ t such that u(x) = 1, v(y) = 1 and u $\subseteq$ 1 – v.

Definition 2.20 [9]: An fts (X, t) is said to be fuzzy regular iff for each x $\in$ X and u $\in t^c$ with u(x) = 0, there exist v, w $\in$ t such that v(x) = 1, u $\subseteq$ w and v $\subseteq$ 1 – w.

Definition 2.21[3]: Let $\lambda \in I^X$ and $\mu \in I^Y$. Then $(\lambda \times \mu)$ is a fuzzy set in X $\times$Y for which $(\lambda \times \mu)$ (x, y) = min { λ(x), μ(y) }, for every (x, y) $\in$ X $\times$Y.

Definition 2.22 [2]: Let (X, t) be an fts and λ be a fuzzy set in X . Let M = { u_i : i$\in$ J } $\subseteq$ I^X be a family of fuzzy sets . Then M = {u_i } is called a Q – cover of λ iff λ(x) + u_i(x) $\geq$ 1 for all x $\in$ X and for some u_i $\in$ M . If each u_i is open, then M = { u_i } is called an open Q – cover of λ .

3. Main results

In this section, we found certain tangible properties of countably Q – compact fuzzy sets. Also, A$\subset$X means that A is a proper subset of X.

Definition 3.1: A fuzzy set λ in X is said to be countably Q – compact iff every countable open Q – cover of λ has a finite Q – subcover.

Theorem 3.2: Let λ be a fuzzy set in an fts (X, t), A $\subset$ X and $\lambda_0 \subseteq$ A. Then the following are equivalent:

(i) λ is countably Q – compact fuzzy set with respect to t.

(ii) λ is countably Q – compact fuzzy set with respect to the subspace fuzzy topology t_A on A.

Proof: Suppose λ is countably Q – compact fuzzy set with respect to t. Let { u_k : $k \in \mathbf{N}$ } be a countable open Q – cover of λ with respect to t_A. Then by definition of subspace fuzzy topology, there exists $v_k \in$ t such that u_k = A $\cap$ v_k $\subseteq v_k$. Hence λ(x) + u_k(x) $\geq$ 1 for all x $\in$ X and consequently λ(x) + v_k(x) $\geq$ 1 for all x $\in$ X. Therefore { v_k : $k \in$ **N** } is a countable open Q – cover of λ with respect to t. Since λ is countably Q – compact fuzzy set in $(X\ , t)$, then λ has finite Q – subcover i.e. there exist $v_{k_r} \in \{v_k\}$ (r = 1, 2,…, n) such that λ(x) + v_{k_r}(x) $\geq$ 1 for all x $\in$ X . But then λ(x) + (A $\cap$ v_{k_r}) (x) $\geq$ 1 for all x $\in$ X and therefore λ(x) + u_{k_r}(x) $\geq$ 1 for all x $\in$ X . Thus { u_k : $k \in$ **N** } contains a finite Q – subcover { u_{k_r} } (r = 1, 2,…, n) and hence λ is countably Q – compact fuzzy set with respect to t_A .

Conversely, suppose λ is countably Q – compact fuzzy set with respect to t_A. Let { v_k : $k \in$ **N** } be a countable open Q – cover of λ with respect to t. Set u_k = A $\cap$ v_k, then λ(x) + v_k(x) $\geq$ 1 for all x $\in$ X . Therefore λ(x) + (A $\cap$ v_k) (x) $\geq$ 1 for all x $\in$ X and consequently λ(x) + u_k(x) $\geq$ 1 for all x $\in$ X . But $u_k \in t_A$, then { u_k : $k \in$ **N** } is a countable open Q – cover of λ with respect to t_A. Since λ is countably Q – compact fuzzy set with respect to t_A, then λ has a finite Q – subcover i.e. there exist $u_{k_r} \in$ { u_k } (r = 1, 2, ……, n) such that λ(x) + u_{k_r}(x) $\geq$ 1 for all x $\in$ X . This implies that λ(x) + (A $\cap$ v_{k_r}) (x) $\geq$ 1 for all x $\in$ X and hence λ(x) + v_{k_r}(x) $\geq$ 1 for all x $\in$ X . Thus { v_k : $k \in$ **N** } contains a finite Q – subcover { v_{k_r} } (r = 1, 2, ……, n) and therefore λ is countably Q – compact fuzzy set with respect to t .

Theorem 3.3: Let $(X\ , t)$ and $(Y\ , s)$ be two fuzzy topological spaces and f: $(X\ , t) \rightarrow (Y\ , s)$ be fuzzy continuous, one – one and onto. If λ is countably Q – compact fuzzy set in $(X\ , t)$, then f(λ) is also countably Q – compact fuzzy set in $(Y\ , s)$.

Proof: Let $\{u_k : k \in \mathbf{N}\}$ be a countable open Q – cover of f(λ) in (Y, s) i.e. (f(λ)) (x) + u_k (x) $\geq$ 1 for all x $\in$ Y . Since f is fuzzy continuous, then $f^{-1}(u_k) \in$ t and hence $\{ f^{-1}(u_k): k \in \mathbf{N} \}$ is a countable open Q – cover of λ in (X, t). As λ is countably Q – compact fuzzy set in (X, t), then λ has a finite Q – subcover i.e. there exist $u_{k_r} \in \{ u_k \}$ (r = 1, 2, ……, n) such that λ(x) + ($f^{-1}(u_{k_r})$) (x) $\geq$ 1 for all x $\in$ X . Again, let u be any fuzzy set in Y. Since f is onto, then for any y $\in$Y, we have f(f^{-1}(u)) (y) = sup { f^{-1}(u) (z): z $\in$ f^{-1}(y), f^{-1}(y) $\neq$ ϕ } = sup {u(f(z)): f(z) = y } = sup {u(y) } = u(y) i.e. f(f^{-1}(u)) = u . This is true for any fuzzy set in Y. As f is one – one and onto, so f(1) = 1 . Hence f(λ(x) + ($f^{-1}(u_{k_r})$) (x)) $\geq$ f(1) implies that (f (λ)) (x) + (u_{k_r})) (x) $\geq$ 1 for all x $\in$ Y . Therefore f(λ) is countably Q – compact fuzzy set in (Y, s) .

Theorem 3.4: Let (X, t) and (Y, s) be two fuzzy topological spaces and f: $(X, t) \rightarrow (Y, s)$ be open and bijective. If λ is countably Q – compact fuzzy set in (Y, s), then f^{-1}(λ) is countably Q – compact fuzzy set in (X, t).

Proof: Let $\{ u_k : k \in \mathbf{N} \}$ be a countable open Q – cover of f^{-1}(λ) in (X, t) i.e. (f^{-1}(λ)) (x) + u_k (x) $\geq$ 1 for all x $\in$ X. Then { f(u_k): $k \in \mathbf{N}$ } is a countable open Q – cover of λ in (Y, s). Since λ is countably Q – compact fuzzy set in (Y, s), then λ has a finite Q – subcover i.e. there exist f(u_{k_r}) $\in$ { f(u_k) } (r = 1, 2, ……, n) such that λ(x) + f(u_{k_r}) (x) $\geq$ 1 for all x $\in$. Again, let u be any fuzzy set in X. Since f is bijective, then for any x $\in$ X, we have f^{-1}(f(u)) (x) = f(u) (f(x)) = u (f^{-1}(f(x))) = u(x). Thus f^{-1}(f(u)) = u and this is true for any fuzzy set in X. As f is bijective, so f(1) = 1. Hence f^{-1}(λ(x) + f(u_{k_r})

(x)) ≥ f(1) implies that ($f^{-1}(\lambda)$) (x) + u_{k_r} (x) ≥ 1. Therefore $f^{-1}(\lambda)$ is countably Q – compact fuzzy set in (X, t).

Theorem 3.5: Let (X, t) be an fts and (A, t_A) be subspace of (X, t) and f : $(X, t) \rightarrow (A, t_A)$ be continuous and onto. If λ is countably Q – compact fuzzy set in (X, t), then f (λ) is countably Q – compact fuzzy set in (A, t_A).

Proof: Let { u_k : $k \in$ **N** } be a countable open Q – cover of f (λ) in (A, t_A) i.e. (f (λ)) (x) + u_k (x) ≥ 1 for all x ∈ A. Since f is fuzzy continuous, then $f^{-1}(u_k) \in$ t and hence { $f^{-1}(u_k)$: $k \in$ **N** } is a countable open Q – cover of λ in (X, t). As λ is countably Q – compact fuzzy set in (X, t), then λ has a finite Q – subcover i.e. there exist $f^{-1}(u_{k_r}) \in$ { $f^{-1}(u_k)$ } (r = 1, 2, ……, n) such that λ (x) + $f^{-1}(u_{k_r})$ (x) ≥ 1 for all x ∈ X. Again, let u be any fuzzy set in A. Since f is bijective, then for any y ∈A, we have f (f^{-1}(u)) (y) = sup { f^{-1}(u) (z): z ∈ f^{-1}(y), f^{-1}(y) ≠ ϕ } = sup {u(f(z)): f(z) = y } = sup {u(y) } = u(y) i.e. f (f^{-1}(u)) = u . This is true for any fuzzy set in A and also f (1) = 1. Hence f (λ (x) + $f^{-1}(u_{k_r})$ (x))≥ f (1) ⇒ (f (λ)) (x) + (u_{k_r}) (x) ≥ 1 for all x ∈ A. Therefore f (λ) is countably Q – compact fuzzy set in (A, t_A).

Theorem 3.6: Let (A, t_A) and (B, s_B) be fuzzy subspaces of fuzzy topological spaces (X, t) and (Y, s) respectively. Let λ be a countably Q – compact fuzzy set in (A, t_A) and f: $(A, t_A) \rightarrow (B, s_B)$ be relatively fuzzy continuous and onto. Then f (λ) is also countably Q – compact fuzzy set in (B, s_B).

Proof: Let { v_k : $k \in$ **N** } be a countable open Q – cover of f (λ) in (B, s_B) i.e. (f (λ)) (x) + v_k (x) ≥ 1 for all x ∈ B. Since $v_k \in s_B$, then there exists $u_k \in$ s such that $v_k = u_k \cap$ B.

Hence (f (λ)) (x) + v_k (x) ≥ 1 for all x ∈ B. As f is fuzzy relatively continuous, then $f^{-1}(v_k)$ ∩ A ∈ t_A and hence { $f^{-1}(v_k)$ ∩ A: k ∈ **N** } is a countable open Q – cover of λ in (A, t_A) i.e. { $f^{-1}(u_k \cap$ B) ∩ A: k ∈ **N** } = { $f^{-1}(u_k)$ ∩ f^{-1}(B) ∩ A: k ∈ **N** } = { $f^{-1}(u_k)$ ∩ A: k ∈ **N** } is a countable open Q – cover of λ in (A, t_A) i.e λ (x) + ($f^{-1}(u_k)$ ∩ A) (x) ≥ 1 for all x ∈ A. Since λ is countably Q – compact fuzzy set in (A, t_A), so λ has a finite Q – subcover i.e. there exist $f^{-1}(u_{k_r})$ ∩ A ∈ { $f^{-1}(u_{k_r})$ ∩ A } (r = 1, 2, ……, n) such that λ (x) + ($f^{-1}(u_{k_r})$ ∩ A) (x) ≥ 1 for all x ∈ A. Again, let v be any fuzzy set in B. As f is onto, then for any y ∈ B, we have f (f^{-1}(v)) (y) = sup { f^{-1}(v) (z): z ∈ f^{-1}(y), f^{-1}(y) ≠ ϕ } = sup { v (f(z)): f(z) = y } = sup { v(y) } = v(y) i.e. f (f^{-1}(v)) = v and this is true for any fuzzy set in B and also f (1) = 1. Hence f (λ (x) + ($f^{-1}(u_{k_r})$ ∩ A) (x)) ≥ f (1) ⇒ (f (λ)) (x) + ((u_{k_r}) ∩ f (A)) (x)) ≥ 1 ⇒ ⇒ (f (λ)) (x) + ((u_{k_r}) ∩ B) (x)) ≥ 1 ⇒ (f (λ)) (x) + (v_{k_r}) (x)) ≥ 1 for all x ∈ B. Therefore f (λ) is countably Q – compact fuzzy set in (B, s_B).

Theorem 3.7: Let λ be a countably Q – compact fuzzy set in a fuzzy – T_1 space (X, t) with $\lambda_0 \subset$ X (proper subset). Let x ∉ λ_0 (λ (x) = 0), then there exist u, v ∈ t such that u(x) = 1 and $\lambda_0 \subseteq v^{-1}(0, 1]$.

Proof: Let y ∈ λ_0. Then clearly x ≠ y. As (X, t) is fuzzy – T_1 space, then there exist u_y, v_y ∈ t such that u_y(x) = 1, u_y(y) = 0 and v_y(x) = 0, v_y(y) = 1. Therefore λ (x) + u_y(x) ≥ 1, x ∈ X and λ (y) + v_y(y) ≥ 1, y ∈ λ_0 i.e. { u_y, v_y : y ∈ λ_0 } is a countable open Q – cover of λ . Since λ is countably Q – compact fuzzy set in (X, t), then λ has a finite Q – subcover i.e. there exist u_{y_k} ∈ { u_y } and v_{y_k} ∈ { v_y } (k = 1, 2, ……, n) such that λ (x) +

u_{y_k} (x) ≥ 1 for all x ∈ X when λ(x) = 0 and some u_{y_k} ∈ { u_y} and λ(y) + v_{y_k} (y) ≥ 1 for all y ∈ X when λ(y) > 0 and some v_{y_k} ∈ { v_y } . Now, let v = $v_{y_1} \cup v_{y_2} \cup \ldots\ldots \cup v_{y_n}$ and u = $u_{y_1} \cap u_{y_2} \cap \ldots\ldots \cap u_{y_n}$. Hence v and u are open fuzzy sets, as they are the union and finite intersection of open fuzzy sets respectively i.e. v, u ∈ t. Furthermore, $\lambda_0 \subseteq v^{-1}$(0, 1] and u(x) = 1, as u_{y_k} (x) = 1 for each k .

Theorem 3.8: Let λ and μ be disjoint countably Q – compact fuzzy sets in a fuzzy – T_1 space (X, t) with λ_0, μ_0 ⊂ X (proper subsets). Then there exist u, v ∈ t such that $\lambda_0 \subseteq u^{-1}$(0, 1] and $\mu_0 \subseteq v^{-1}$(0, 1].

Proof: Let y ∈ λ_0. Then y ∉ μ_0, as λ and μ are disjoint. Since μ is countably Q – compact fuzzy set in (X, t), then by theorem 3.7, there exist u_y, v_y ∈ t such that u_y(y) = 1 and $\mu_0 \subseteq v_y^{-1}$(0, 1]. As u_y(y) = 1, then λ(x) + v_y(x) ≥ 1, x ∈ X and λ(y) + u_y(y) ≥ 1, y ∈ λ_0 i.e. {v_y, u_y: y ∈ λ_0 } is a countable open Q – cover of λ . Since λ is countably Q – compact fuzzy set in (X, t), then λ has a finite Q – subcover i.e. there exist v_{y_k} ∈ { v_y } and u_{y_k} ∈ { u_y} (k = 1, 2, ……, n) such that λ(x) + v_{y_k} (x) ≥ 1 for all x ∈ X when λ(x) = 0 and some v_{y_k} ∈ { v_y } and λ(y) + u_{y_k} (y) ≥ 1 for all y ∈ X when λ(y) > 0 and some u_{y_k} ∈ { u_y}. Furthermore, μ(x) + v_{y_k} (x) ≥ 1 for all x ∈ X when μ(x) > 0 and some v_{y_k} ∈ { v_y } and μ (y) + u_{y_k} (y) ≥ 1 for all y ∈ X when μ (y) = 0 and some u_{y_k} ∈ { u_y}. Now, let u = $u_{y_1} \cup u_{y_2} \cup \ldots\ldots \cup u_{y_n}$ and v = $v_{y_1} \cap v_{y_2} \cap \ldots\ldots \cap v_{y_n}$. Thus we see that $\lambda_0 \subseteq u^{-1}$(0, 1] and $\mu_0 \subseteq v^{-1}$(0, 1], as $\mu \subseteq v_{y_k}$ for each k . Hence u and v are open fuzzy sets, as they are the union and finite intersection of open fuzzy sets respectively i.e. u, v ∈ t.

Remark: If $\lambda(x) \neq 0$ for all x $\in$ X i.e. λ_0 = X, then the above two theorems are not at all true.

The following Example will show that the countably Q – compact fuzzy set in a fuzzy – T_1 space need not be closed.

Example – 3.9: Let X = { a, b } and I = [0, 1]. Let u_1, $u_2 \in I^X$ with $u_1(a) = 1$, $u_1(b) = 0$ and $u_2(a) = 0$, $u_2(b) = 1$. Now, take t = { 0, 1, u_1, u_2 }, then (X, t) is a fuzzy – T_1 space. Let $\lambda \in I^X$ defined by $\lambda(a) = 0.7$, $\lambda(b) = 0.9$. Thus we see that $\lambda(x) + u_k(x) \geq 1$ for all x $\in$ X and for some u_k (k = 1, 2). Then clearly λ is countably Q – compact fuzzy set in (X, t). But λ is not closed, as its complement λ^c is not open in (X, t).

Theorem 3.10: Let λ be a countably Q – compact fuzzy set in a fuzzy Hausdorff space (X, t) (in the sense of Definition 2.18) with $\lambda_0 \subset$ X (proper subset). Let x $\notin \lambda_0$ ($\lambda(x)$ = 0), then there exist u, v $\in$ t such that u(x) = 1, $\lambda_0 \subseteq v^{-1}(0, 1]$ and u $\cap$ v = 0.

Proof: Suppose y $\in \lambda_0$. Then clearly x $\neq$ y. Since (X, t) is fuzzy Hausdorff, then there exist u_y, $v_y \in$ t such that $u_y(x) = 1$, $v_y(y) = 1$ and $u_y \cap v_y = 0$. Hence $\lambda(x) + u_y(x) \geq 1$, x $\in$ X and $\lambda(y) + v_y(y) \geq 1$, y $\in \lambda_0$ i.e. { u_y, v_y : y $\in \lambda_0$ } is a countable open Q – cover of λ . As λ is countably Q – compact fuzzy set in (X, t), then λ has a finite Q – subcover i.e. there exist $u_{y_k} \in \{ u_y \}$ and $v_{y_k} \in \{ v_y \}$ (k = 1, 2, ……, n) such that $\lambda(x) + u_{y_k}(x) \geq$ 1 for all x $\in$ X when $\lambda(x) = 0$ and some $u_{y_k} \in \{ u_y \}$ and $\lambda(y) + v_{y_k}(y) \geq 1$ for all y $\in$ X when $\lambda(y) > 0$ and some $v_{y_k} \in \{ v_y \}$. Now, let v = $v_{y_1} \cup v_{y_2} \cup \ldots\ldots \cup v_{y_n}$ and u = $u_{y_1} \cap u_{y_2} \cap \ldots\ldots \cap u_{y_n}$. Thus we see that v and u are open fuzzy sets, as they are the union and finite intersection of open fuzzy sets respectively i.e. v, u $\in$ t. Furthermore, $\lambda_0 \subseteq v^{-1}(0, 1]$ and u(x) = 1, as u_{y_k} (x) = 1 for each k.

Finally, we have to show that u $\cap$ v = 0. First, we observe that $u_{y_k} \cap v_{y_k} = 0$ implies u $\cap\, v_{y_k} = 0$, by distributive law, we see that u $\cap$ v = u $\cap$ ($v_{y_1} \cup v_{y_2} \cup \ldots\ldots \cup v_{y_n}$) = 0.

Theorem 3.11: Let λ and μ be disjoint countably Q – compact fuzzy sets in a fuzzy Hausdorff space (X, t) (in the sense of Definition 2.18) with λ_0, $\mu_0 \subset$ X (proper subsets). Then there exist u, v $\in$ t such that $\lambda_0 \subseteq u^{-1}(0, 1]$, $\mu_0 \subseteq v^{-1}(0, 1]$ and u $\cap$ v = 0.

Proof: Suppose y $\in \lambda_0$. Then y $\notin \mu_0$, as λ and μ are disjoint. Since μ is countably Q – compact fuzzy set in (X, t), then by theorem 3.10, there exist u_y, $v_y \in$ t such that u_y(y) = 1, $\mu_0 \subseteq v_y^{-1}(0, 1]$ and $u_y \cap v_y = 0$. As u_y(y) = 1, then λ(x) + v_y(x) $\geq$ 1, x $\in$ X and λ(y) + u_y(y) $\geq$ 1, y $\in \lambda_0$ i.e. $\{v_y, u_y : y \in \lambda_0\}$ is a countable open Q – cover of λ. Since λ is countably Q – compact fuzzy set in (X, t), then λ has a finite Q – subcover i.e. there exist $v_{y_k} \in \{v_y\}$ and $u_{y_k} \in \{u_y\}$ (k = 1, 2, ……, n) such that λ(x) + v_{y_k} (x) $\geq$ 1 for all x $\in$ X when λ(x) = 0 and some $v_{y_k} \in \{v_y\}$ and λ(y) + u_{y_k} (y) $\geq$ 1 for all y $\in$ X when λ(y) > 0 and some $u_{y_k} \in \{u_y\}$. Furthermore, μ(x) + v_{y_k} (x) $\geq$ 1 for all x $\in$ X when μ(x) > 0 and some $v_{y_k} \in \{v_y\}$ and μ (y) + u_{y_k} (y) $\geq$ 1 for all y $\in$ X when μ (y) = 0 and some $u_{y_k} \in \{u_y\}$. Now, let u = $u_{y_1} \cup u_{y_2} \cup \ldots\ldots \cup u_{y_n}$ and v = $v_{y_1} \cap v_{y_2} \cap \ldots\ldots \cap v_{y_n}$. Hence we see that $\lambda_0 \subseteq u^{-1}(0, 1]$ and $\mu_0 \subseteq v^{-1}(0, 1]$, as $\mu \subseteq v_{y_k}$ for each k. Hence u and v are open fuzzy sets, as they are the union and finite intersection of open fuzzy sets respectively i.e. u, v $\in$ t.

Finally, we have to show that u $\cap$v = 0 . As $u_{y_k} \cap v_{y_k}$ = 0 implies $u_{y_k} \cap$v= 0, by distributive law, we have u $\cap$v = ($u_{y_1} \cup u_{y_2} \cup \ldots\ldots \cup u_{y_n}$)$\cap$v = 0.

Remark: If λ(x) $\neq$ 0 for all x $\in$ X i.e. λ_0 = X, then the above two theorems are not at all true.

Note: Example – 3.9 is also valid for remain that countably Q – compact fuzzy set in a fuzzy Hausdorff space (in the sense of Definition 2.18) need not be closed.

Theorem 3.12: Let λ be a countably Q – compact fuzzy set in a fuzzy Hausdorff space (X, t) (in the sense of Definition 2.19) with $\lambda_0 \subset$ X (proper subset). Let x $\notin \lambda_0$ (λ(x) = 0), then there exist u, v $\in$ t such that u(x) = 1, $\lambda_0 \subseteq v^{-1}$(0, 1] and u $\subseteq$ 1 – v.

Proof: Let y $\in \lambda_0$. Then clearly x $\neq$ y. Since (X, t) is fuzzy Hausdorff, then there exist u_y, $v_y \in$ t such that u_y(x) = 1, v_y(y) = 1 and $u_y \subseteq 1 - v_y$. Hence λ(x) + u_y(x) $\geq$ 1, x $\in$ X and λ(y) + v_y(y) $\geq$ 1, y $\in \lambda_0$ i.e. { u_y, v_y : y $\in \lambda_0$ } is a countable open Q – cover of λ. As λ is countably Q – compact in (X, t), then λ has a finite Q – subcover i.e. there exist $u_{y_k} \in$ { u_y} and $v_{y_k} \in$ { v_y } (k = 1, 2, ……, n) such that λ(x) + u_{y_k}(x) $\geq$ 1 for all x $\in$ X when λ(x) = 0 and some $u_{y_k} \in$ { u_y} and λ(y) + v_{y_k}(y) $\geq$ 1 for all y $\in$ X when λ(y) > 0 and some $v_{y_k} \in$ { v_y }. Now, let v = $v_{y_1} \cup v_{y_2} \cup \ldots\ldots \cup v_{y_n}$ and u = $u_{y_1} \cap u_{y_2} \cap \ldots\ldots \cap u_{y_n}$. Thus we see that v and u are open fuzzy sets, as they are the union and finite intersection of open fuzzy sets respectively i.e. v, u $\in$ t . Furthermore, $\lambda_0 \subseteq v^{-1}$(0, 1] and u(x) = 1, as u_{y_k}(x) = 1 for each k.

Finally, we have to show that u $\subseteq$ 1 – v. As $u_y \subseteq 1 - v_y$ implies that u $\subseteq 1 - v_y$. Since u_{y_k}(x) $\leq 1 - v_{y_k}$(x) for all x $\in$ X and for each k, then u $\subseteq$ 1 – v . If not, then there exist x $\in$ X, such that u_y(x) >1 $- v_y$(x). But we have u_y(x) $\leq u_{y_k}$(x) for each k. Then for some k,

u_{y_k} (x) >1 $-v_{y_k}$ (x). This is a contradiction, as u_{y_k} (x) $\leq$ 1 $-v_{y_k}$ (x) for each k. Hence u $\subseteq$ 1 – v.

Theorem 3.13: Let λ and μ be disjoint countably Q – compact fuzzy sets in a fuzzy Hausdorff space (X, t) (in the sense of Definition 2.19) with λ_0, μ_0 $\subset$ X (proper subsets). Then there exist u, v $\in$ t such that $\lambda_0 \subseteq u^{-1}$(0, 1], $\mu_0 \subseteq v^{-1}$(0, 1] and u $\subseteq$ 1 – v.

Proof: Let y $\in \lambda_0$. Then y $\notin \mu_0$, as λ and μ are disjoint . As μ is countably Q – compact fuzzy set in (X, t), then by theorem 3.12, there exist u_y, v_y $\in$ t such that u_y(y) = 1, $\mu_0 \subseteq v_y^{-1}$(0, 1] and $u_y \subseteq 1 - v_y$. Since u_y(y) = 1, then λ(x) + v_y(x) $\geq$ 1, x $\in$ X and λ(y) + u_y(y) $\geq$ 1, y $\in \lambda_0$ i.e. { v_y , u_y: y $\in \lambda_0$ } is a countable open Q – cover of λ . As λ is countably Q – compact fuzzy set in (X, t), then λ has a finite Q – subcover i.e. there exist $v_{y_k} \in$ { v_y } and $u_{y_k} \in$ { u_y } (k = 1, 2, ……, n) such that λ(x) + v_{y_k} (x) $\geq$ 1 for all x $\in$ X when λ(x) = 0 and for some $v_{y_k} \in$ { v_y } and λ(y) + u_{y_k} (y) $\geq$ 1 for all y $\in$ X when λ(y) > 0 and for some $u_{y_k} \in$ { u_y }. Furthermore, μ(x) + v_{y_k} (x) $\geq$ 1 for all x $\in$ X when μ(x) > 0 and for some $v_{y_k} \in$ { v_y } and μ (y) + u_{y_k} (y) $\geq$ 1 for all y $\in$ X when μ (y) = 0 and some $u_{y_k} \in$ { u_y }. Now, let u = $u_{y_1} \cup u_{y_2} \cup \ldots\ldots \cup u_{y_n}$ and v = $v_{y_1} \cap v_{y_2} \cap \ldots\ldots \cap v_{y_n}$. Thus we see that $\lambda_0 \subseteq u^{-1}$(0, 1] and $\mu_0 \subseteq v^{-1}$(0, 1], as $\mu \subseteq v_{y_k}$ for each k. Hence u and v are open fuzzy sets, as they are the union and finite intersection of open fuzzy sets respectively i.e. u, v $\in$ t.

Finally, we have to show hat u $\subseteq$ 1 – v . As $u_{y_k} \subseteq 1 - v_{y_k}$ for each k implies that $u_{y_k} \subseteq$1– v for each k and it is clear that u $\subseteq$ 1 – v.

Remark: If λ(x) $\neq$ 0 for all x $\in$ X i.e. λ_0 = X, then the above two theorems are not at all true.

Note: Example – 3.9 is also valid for remain that countably Q – compact fuzzy set in a fuzzy Hausdorff space (in the sense of Definition 2.19) need not be closed.

Theorem 3.14: Let λ be a countably Q – compact fuzzy set in a fuzzy regular space (X, t) with $\lambda_0 \subset$ X (proper subset). If for each x $\in$ λ_0, there exist u $\in t^c$ with u(x) = 0, we have v, w $\in$ t such that v(x) = 1, u $\subseteq$ w, $\lambda_0 \subseteq v^{-1}$(0, 1] and v $\subseteq$ 1 – w.

Proof: Let (X, t) be a fuzzy regular space and λ be a countbly Q – compact fuzzy set in X. Then for each x $\in$ λ_0, there exists u $\in t^c$ with u(x) = 0. As (X, t) is fuzzy regular, we have v_x, w_x $\in$ t such that v_x(x) = 1, $u_x \subseteq w_x$ and $v_x \subseteq 1 - w_x$. Thus we see that λ(x) + v_x(x) $\geq$ 1 for all x $\in$ X i.e. { v_x : x $\in$ λ_0 } is a countable open Q – cover of λ. As λ is countably Q – compact fuzzy set in (X, t), so λ has a finite subcover i.e. there exist $v_{x_k} \in \{ v_x \}$ (k = 1, 2, ……, n) such that λ(x) + v_{x_k} (x) $\geq$ 1 for all x $\in$ X. Now, let v = $v_{x_1} \cup v_{x_2} \cup \ldots\ldots \cup v_{x_n}$ and w = $w_{x_1} \cap w_{x_2} \cap \ldots\ldots \cap w_{x_n}$. Hence we see that v and w are open fuzzy sets, as they are the union and finite intersection of open fuzzy sets respectively i.e. v, w $\in$ t. Furthermore, $\lambda_0 \subseteq v^{-1}$(0, 1], v(x) = 1and u $\subseteq$ w, as u $\subseteq$ w_{x_k} for each k.

Finally, we have to show that v $\subseteq$ 1 – w. As $v_{x_k} \subseteq 1 - w_{x_k}$ for each k implies that $v_{x_k} \subseteq$ 1 – w for each k and hence it is clear that v $\subseteq$ 1 – w.

The “ good extension property” does not remain valid in countably Q – compact fuzzy sets.

Example 3.15: Let X = { a, b, c } and T = { {b}, {c}, {b, c}, ϕ, X }, then (X, T) is a topological space. Let u_1, u_2, u_3 $\in$ I^X with u_1(a) = 0, u_1(b) = 0.4, u_1(c) = 0 ; u_2(a) = 0, u_2(b) = 0, u_2(c) = 0.6 and u_3(a) = 0, u_3(b) = 0.4, u_3(c) = 0.6. Then ω(T) = { u_1, u_2, u_3, 0, 1 } and $\left(X\ , \omega\,(T\ ) \right)$ is an fts. Now, let G = { b, c }. Then clearly G is countably compact in $\left(X\ , T\ \right)$. But 1_G is not countably Q – compact fuzzy set in $\left(X\ , \omega\,(T\ ) \right)$ as there do not exist $u_k \in \{\ \omega\,(\mathrm{T})\ \}$ (k = 1, 2, 3) such that 1_G(a) + u_k(a) ≥ 1 for a ∈ X.

Again, let $v_1, v_2, v_3, v_4 \in I^X$ with v_1(a) = 0, v_1(b) = 0.3, v_1(c) = 0 ; v_2(a) = 0, v_2(b) = 0, v_2(c) = 0.8 ; v_3(a) = 0, v_3(b) = 1, v_3(c) = 1 and v_4(a) = 0, v_4(b) = 0.3, v_4(c) = 0.8. Then ω(T) = { v_1, v_2, v_3, v_4, 0, 1 } and $\left(X\ , \omega\,(T\ ) \right)$ is an fts.

Now, let $\lambda \in I^X$ defined by λ(a) = 1, λ(b) = 0.7, λ(c) = 0. We observe that, λ(x) + v_3(x) ≥ 1 for all x ∈ X. Then clearly λ is countably Q – compact fuzzy set in $\left(X\ , \omega\,(T\ ) \right)$, but λ^{-1}(0, 1] = { a, b } is not countably compact in $\left(X\ , T\ \right)$. It is, therefore, observe that " good extension property" does not hold good for countably Q – compact fuzzy sets.

Theorem 3.16: Let λ and μ be countably Q – compact fuzzy sets in an fts $\left(X\ , t \right)$. Then ($\lambda \times \mu$) is also countably Q – compact in $\left(X \times X\ , t \times t \right)$.

Proof : Let M = { a_k : $a_k \in$ t × t and $k \in \mathbf{N}$ } be a countable Q – cover of ($\lambda \times \mu$) in $\left(X \times X\ , t \times t \right)$. Then ($\lambda \times \mu$) (x , y) + a_k(x , y) ≥ 1 for all (x ,y) ∈ X×X. Now, we can write $a_k = u_k \times v_k$, where u_k, $v_k \in$ t. Thus we have ($\lambda \times \mu$) (x ,y) + ($u_k \times v_k$) (x ,y) ≥ 1 for all (x ,y) ∈ X×X. Hence it is clear that λ(x) + u_k(x) ≥ 1 for all x ∈X and μ(y) + v_k(y) ≥ 1 for all y ∈X. Therefore, { u_k : $k \in \mathbf{N}$ } and { v_k : $k \in \mathbf{N}$ } are countable Q – covers of λ and μ respectively. Since λ and μ are countably Q – compact, then { u_k : $k \in \mathbf{N}$ } and { v_k : $k \in \mathbf{N}$ } have finite Q – subcovers i.e. there exist $u_{k_r} \in \{ u_k \}$ (r = 1, 2,…, n) and $v_{k_r} \in$ { v_k } (r = 1, 2,…, n) such that λ(x) + u_{k_r}(x) ≥ 1 for all x ∈X and μ(y) + v_{k_r}(y) ≥ 1 for all y $\in$X respectively. Thus we can write ($\lambda \times \mu$) (x ,y) + ($u_{k_r} \times v_{k_r}$) (x ,y) $\geq$ 1 for all (x ,y) $\in$ X$\times$X. Hence ($\lambda \times \mu$) is countably Q – compact in $\left(X \times X , t \times t \right)$.

Conflict of Interests

The author declares that there is no conflict of interests.

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