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EXPANSION THEOREMS FOR DISCONTINUOUS STURM-LIOUVILLE PROBLEMS

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Abstract. In this paper, we investigate a discontinuous Sturm-Liouville problem which has several discontinuities inside a finite interval and eigenparameter dependent on one of the boundary conditions. We construct Green's function and the resolvent operator for this problem and prove theorems about the eigenfunction expansion for Green's function and the modified Parseval equality in the special Hilbert space.

Keywords: Sturm-Liouville problems; transmission conditions; Green's function; expansion theorems.

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1. Introduction

In this paper, we consider the following Sturm-Liouville equation

(1) $$\tau(u) := -u'' + q(x)u = \lambda u,\ x \in I$$

with boundary conditions

(2) $$B_0(u) := \beta_1 u(\theta_0) + \beta_2 u'(\theta_0) = 0,$$

(3) $$B_{m+1}(u) := \lambda\left(\alpha_1' u(\theta_{m+1}) - \alpha_2' u'(\theta_{m+1})\right) + \alpha_1 u(\theta_{m+1}) - \alpha_2 u'(\theta_{m+1}) = 0$$

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and transmission conditions at points of discontinuities $\theta_k, k=\overline{1,m}$

$$T_k(u) := \begin{pmatrix} u(\theta_k+0) \\ u'(\theta_k+0) \end{pmatrix} - D_k \begin{pmatrix} u(\theta_k-0) \\ u'(\theta_k-0) \end{pmatrix} = 0, \ \ k=\overline{1,m}, \tag{4}$$

where $I := [\theta_0,\theta_1)\cup(\theta_1,\theta_2)\cup\ldots\cup(\theta_m,\theta_{m+1}]$; λ is a complex eigenparameter; the potential $q(x)$ is given real-valued function which is continuous in each of the intervals $[\theta_0,\theta_1),(\theta_1,\theta_2),\ldots,$ $(\theta_m,\theta_{m+1}]$ and has finite limit $q(\pm\theta_k) = \lim\limits_{x\to\pm\theta_k} q(x)$, $\left(k=\overline{1,m}\right)$; $\beta_i, \alpha_i, \alpha_i'$ $(i=1,2)$ are real numbers and $|\beta_1|+|\beta_2|\neq 0$; $\rho := (\alpha_1'\alpha_2-\alpha_1\alpha_2') > 0$; $D_k = \begin{pmatrix} \gamma_{1k} & \gamma_{2k} \\ \gamma_{3k} & \gamma_{4k} \end{pmatrix}$ $\gamma_{jk}\in\mathbb{R}$ $\left(j=\overline{1,4}\right)$, $|D_k|>0$ for $k=\overline{1,m}$ and D_0 be the 2×2 identity matrix.

Some boundary value problems with transmission conditions arise in heat and mass transfer problems (see [10]) and thermal conduction problems for a thin laminated plate (i.e., plate composed by materials with different characteristics piled in the thickness). In these class of problems, transmission condition across the interface should be added since the plate is laminated (see [16]). Also some problems with transmission conditions which arise in diffraction problems [17] and in vibrating string problems when the string loaded additionaly with point masses [14].

The case for Sturm-Liouville problems with one point of discontinuity and eigenparameter dependent boundary conditions have been investigated in [5,21,2,12,19], respectively. Even more recently, these results were extended to transmission conditions at two and a finite points of discontinuity (see [20,8,7,22] and [4,5,13,18,23], respectively). In [1,11], Green's function and resolvent operator were constructed and derived asymptotic approximation formulae for Green's function. Eigenfunction expansions problem for Sturm-Liouville problems with one point of discontinuity have been investigated in [9,3,15]. By this paper, these expansion theorems extend to a Sturm-Liouville problem which has several discontinuities inside a finite interval.

Operator theoretic interpretation, asymptotic formulas for the eigenvalues and the eigenfunctions for the same problem were given in [6]. We state the results briefly which will use in this paper. Then we construct Green's function and the resolvent operator for the problem (1)-(4)

and prove theorems about the eigenfunction expansion for Green's function and the modified Parseval equality in the special Hilbert space.

We construct a special fundamental system of solutions of the equation (1). By virtue of Theorem 1.5 in [15] we can define two solutions $\phi_\lambda(x) := \phi(x,\lambda)$ and $\chi_\lambda(x) := \chi(x,\lambda)$ as follows:

$$\phi_\lambda(x) = \begin{cases} \phi_{1\lambda}(x), & x \in [\theta_0, \theta_1) \\ \phi_{2\lambda}(x), & x \in (\theta_1, \theta_2) \\ \vdots & \\ \phi_{(m+1)\lambda}(x), & x \in (\theta_m, \theta_{m+1}] \end{cases}, \chi_\lambda(x) = \begin{cases} \chi_{1\lambda}(x), & x \in [\theta_0, \theta_1) \\ \chi_{2\lambda}(x), & x \in (\theta_1, \theta_2) \\ \vdots & \\ \chi_{(m+1)\lambda}(x), & x \in (\theta_m, \theta_{m+1}]. \end{cases} \tag{5}$$

Let us consider the initial value problem

$$-u'' + q(x)u = \lambda u, \; x \in [\theta_0, \theta_1], \tag{6}$$

$$u(\theta_0) = \beta_2, \quad u'(\theta_0) = -\beta_1, \tag{7}$$

has a unique solution $u = \phi_{1\lambda}(x)$ which is an entire function of $\lambda \in \mathbb{C}$ for each fixed $x \in [\theta_0, \theta_1]$. Similarly, employing the same method as in proof of Theorem 1.5 in $[15]$, the problem

$$-u'' + q(x)u = \lambda u, \; x \in [\theta_m, \theta_{m+1}], \tag{8}$$

$$u(\theta_{m+1}) = \lambda\alpha_2' + \alpha_2, \quad u'(\theta_{m+1}) = \lambda\alpha_1' + \alpha_1, \tag{9}$$

has a unique solution $u = \chi_{(m+1)\lambda}(x)$ which is an entire function of $\lambda \in \mathbb{C}$ for each fixed $x \in [\theta_m, \theta_{m+1}]$.

Now the functions $\phi_{(k+1)\lambda}(x)$ and $\chi_{k\lambda}(x)$ are defined in terms of $\phi_{k\lambda}(x)$ and $\chi_{(k+1)\lambda}(x)$ $\left(k = \overline{1,m}\right)$ respectively, as follows: $\phi_{(k+1)\lambda}(x)$ is a solution of the equation (1) on $[\theta_k, \theta_{k+1}]$ by the transmission condition

$$\begin{pmatrix} u(\theta_k) \\ u'(\theta_k) \end{pmatrix} = D_k \begin{pmatrix} \phi_k(\theta_k - 0) \\ \phi_k'(\theta_k - 0) \end{pmatrix}, k = \overline{1,m} \tag{10}$$

and $\chi_{k\lambda}(x)$ is a solution of the equation (1) on $[\theta_{k-1}, \theta_k]$ by the transmission condition

$$\begin{pmatrix} u(\theta_k) \\ u'(\theta_k) \end{pmatrix} = D_k^{-1} \begin{pmatrix} \chi_{k+1}(\theta_k + 0) \\ \chi_{k+1}'(\theta_k + 0) \end{pmatrix}, k = \overline{1,m}. \tag{11}$$

Hence $\phi_{\lambda}(x)$ satisfies the equation (1) on $[\theta_0, \theta_{m+1}]$, the boundary condition (2) and the transmission condition (4), $\chi_{k\lambda}(x)$ satisfies the equation (1) on $[\theta_0, \theta_{m+1}]$, the boundary condition (3) and the transmission condition (4). Since the Wronskians $W(\phi_{k\lambda}, \chi_{k\lambda}; x)$, $\left(k = \overline{1, m+1}\right)$ are independent of variable x, then the functions $\omega_k(\lambda) := W(\phi_{k\lambda}, \chi_{k\lambda}; x)$, $\left(k = \overline{1, m+1}\right)$ are the entire functions of parameter λ. Let $\omega(\lambda) := \omega_1(\lambda)$ and then we obtain

$$\omega_{k+1}(\lambda) = \prod_{l=1}^{k} |D_l| \, \omega(\lambda), \; k = \overline{1, m}. \tag{12}$$

We defined the Hilbert space $H := \left(\bigoplus_{k=1}^{m+1} L_2(\theta_{k-1}, \theta_k) \right) \oplus \mathbb{C}$ with an inner product

$$\langle F, G \rangle_H := \sum_{k=1}^{m+1} \frac{1}{\prod_{l=1}^{k} |D_{l-1}|} \int_{\theta_{k-1}}^{\theta_k} f(x) \overline{g(x)} dx + \frac{1}{\rho |D_1 D_2 ... D_m|} f_1 \overline{g_1}, \tag{13}$$

where $F = \begin{pmatrix} f(x) \\ f_1 \end{pmatrix}, G = \begin{pmatrix} g(x) \\ g_1 \end{pmatrix} \in H$ and defined a symmetric operator A in this Hilbert space such a way that the problem (1)-(4) could be considered as the eigenvalue problem of this operator. Also we gave in [6] the eigenvalues of the problem (1)-(4) are real, bounded below, coincide with the zeros of $\omega(\lambda)$ and two eigenfunctions corresponding to different eigenvalues are orthogonal.

2. Green's Function

To study the completeness of the eigenelements of A, and hence the completeness of the eigenfunctions of the problem (1)-(4), we construct the resolvent of A as well as Green's function of the problem (1)-(4). We assume without any loss of generality that $\lambda = 0$ is not an eigenvalue of A. Otherwise, from discreteness of eigenvalues, we can find a real number μ such that $\mu \neq \lambda_n$ for all n and replace the eigenparameter λ by $\lambda - \mu$. Now let $\lambda \in \mathbb{C}$ not be an eigenvalue of A and consider the inhomogenous problem for $F = \begin{pmatrix} f(x) \\ f_1 \end{pmatrix} \in H, U = \begin{pmatrix} u(x) \\ R'(u) \end{pmatrix} \in D(A)$

$$(\lambda \mathrm{I} - A) U = F, \tag{14}$$

where $R(u) = \alpha_1 u(\theta_{m+1}) - \alpha_2 u'(\theta_{m+1})$, $R'(u) = \alpha_1' u(\theta_{m+1}) - \alpha_2' u'(\theta_{m+1})$ and I is the identity operator. Since

$$(\lambda \mathrm{I}-A)U=\lambda\begin{pmatrix} u(x) \\ R'(u) \end{pmatrix}-\begin{pmatrix} \tau(u) \\ -R(u) \end{pmatrix}=\begin{pmatrix} f(x) \\ f_1 \end{pmatrix} \tag{15}$$

then we have

$$(\lambda \mathrm{I}-\tau)\,u(x)=f(x),\ x\in I, \tag{16}$$

$$\lambda R'(u)+R(u)=f_1. \tag{17}$$

Now, we can represent the general solution of (16) in the following form:

$$u(x,\lambda)=\begin{cases} a_1\phi_{1\lambda}(x)+b_1\chi_{1\lambda}(x), & x\in[\theta_0,\theta_1) \\ a_2\phi_{2\lambda}(x)+b_2\chi_{2\lambda}(x), & x\in(\theta_1,\theta_2) \\ \vdots & \\ a_{m+1}\phi_{(m+1)\lambda}(x)+b_{m+1}\chi_{(m+1)\lambda}(x), & x\in(\theta_m,\theta_{m+1}]. \end{cases} \tag{18}$$

By applying the method of variation of the constants to (18), thus, the functions $a_i(x,\lambda)$, $b_i(x,\lambda)$ $\left(i=\overline{1,m+1}\right)$ satisfy the linear system of equation

$$\begin{gathered} \begin{cases} a_1'(x,\lambda)\,\phi_{1\lambda}(x)+b_1'(x,\lambda)\,\chi_{1\lambda}(x)=0 \\ a_1'(x,\lambda)\,\phi_{1\lambda}'(x)+b_1'(x,\lambda)\,\chi_{1\lambda}'(x)=f(x) \end{cases},\ x\in[\theta_0,\theta_1) \\ \begin{cases} a_2'(x,\lambda)\,\phi_{2\lambda}(x)+b_2'(x,\lambda)\,\chi_{2\lambda}(x)=0 \\ a_2'(x,\lambda)\,\phi_{2\lambda}'(x)+b_2'(x,\lambda)\,\chi_{2\lambda}'(x)=f(x) \end{cases},\ x\in(\theta_1,\theta_2) \\ \vdots \\ \begin{cases} a_{m+1}'(x,\lambda)\,\phi_{(m+1)\lambda}(x)+b_{m+1}'(x,\lambda)\,\chi_{(m+1)\lambda}(x)=0 \\ a_{m+1}'(x,\lambda)\,\phi_{(m+1)\lambda}'(x)+b_{m+1}'(x,\lambda)\,\chi_{(m+1)\lambda}(x)=f(x) \end{cases},\ x\in(\theta_m,\theta_{m+1}]. \end{gathered} \tag{19}$$

Since λ is not an eigenvalue and $\omega_{k+1}(\lambda) \neq 0$ $(k = \overline{1,m})$, each of the linear systems in (19) has a unique solution which leads

$$\begin{cases} a_1(x,\lambda) = \frac{1}{\omega_1(\lambda)} \int\limits_x^{\theta_1} \chi_{1\lambda}(y) f(y)\,dy + a_1(\lambda) \\ b_1(x,\lambda) = \frac{1}{\omega_1(\lambda)} \int\limits_{\theta_0}^{x} \phi_{1\lambda}(y) f(y)\,dy + b_1(\lambda) \end{cases}, \quad x \in [\theta_0, \theta_1)$$
$$\begin{cases} a_2(x,\lambda) = \frac{1}{\omega_2(\lambda)} \int\limits_x^{\theta_2} \chi_{2\lambda}(y) f(y)\,dy + a_2(\lambda) \\ b_2(x,\lambda) = \frac{1}{\omega_2(\lambda)} \int\limits_{\theta_1}^{x} \phi_{2\lambda}(y) f(y)\,dy + b_2(\lambda) \end{cases}, \quad x \in (\theta_1, \theta_2) \tag{20}$$
$$\vdots$$
$$\begin{cases} a_{m+1}(x,\lambda) = \frac{1}{\omega_{m+1}(\lambda)} \int\limits_x^{\theta_{m+1}} \chi_{(m+1)\lambda}(y) f(y)\,dy + a_{m+1}(\lambda) \\ b_{m+1}(x,\lambda) = \frac{1}{\omega_{m+1}(\lambda)} \int\limits_{\theta_m}^{x} \phi_{(m+1)\lambda}(y) f(y)\,dy + b_{m+1}(\lambda) \end{cases}, \quad x \in (\theta_m, \theta_{m+1}]$$

where $a_i(\lambda), b_i(\lambda)$ $\left(i = \overline{1,m+1}\right)$ are arbitrary constants. Substituting (20) into (18), then from (17) and the transmission conditions (4), we obtain

$$\begin{aligned}
a_1(\lambda) &= \frac{1}{\omega_2(\lambda)} \int\limits_{\theta_1}^{\theta_2} \chi_{2\lambda}(y) f(y)\,dy + \frac{1}{\omega_3(\lambda)} \int\limits_{\theta_2}^{\theta_3} \chi_{3\lambda}(y) f(y)\,dy + \ldots + \\
&\quad \frac{1}{\omega_{m+1}(\lambda)} \int\limits_{\theta_m}^{\theta_{m+1}} \chi_{(m+1)\lambda}(y) f(y)\,dy + \frac{f_1}{\omega_{m+1}(\lambda)} \\
a_2(\lambda) &= \frac{1}{\omega_3(\lambda)} \int\limits_{\theta_2}^{\theta_3} \chi_{3\lambda}(y) f(y)\,dy + \frac{1}{\omega_4(\lambda)} \int\limits_{\theta_3}^{\theta_4} \chi_{4\lambda}(y) f(y)\,dy + \ldots + \\
&\quad \frac{1}{\omega_{m+1}(\lambda)} \int\limits_{\theta_m}^{\theta_{m+1}} \chi_{(m+1)\lambda}(y) f(y)\,dy + \frac{f_1}{\omega_{m+1}(\lambda)} \\
&\vdots \\
a_m(\lambda) &= \frac{1}{\omega_{m+1}(\lambda)} \int\limits_{\theta_m}^{\theta_{m+1}} \chi_{(m+1)\lambda}(y) f(y)\,dy + \frac{f_1}{\omega_{m+1}(\lambda)} \\
a_{m+1}(\lambda) &= \frac{f_1}{\omega_{m+1}(\lambda)}
\end{aligned}$$

$$b_1(\lambda) = 0$$

$$b_2(\lambda) = \frac{1}{\omega_1(\lambda)} \int_{\theta_0}^{\theta_1} \phi_{1\lambda}(y) f(y)\, dy$$

$$\vdots$$

$$b_m(\lambda) = \frac{1}{\omega_{m-1}(\lambda)} \int_{\theta_{m-2}}^{\theta_{m-1}} \phi_{(m-1)\lambda}(y) f(y)\, dy + \frac{1}{\omega_{m-2}(\lambda)} \int_{\theta_{m-3}}^{\theta_{m-2}} \phi_{(m-2)\lambda}(y) f(y)\, dy$$
$$+ \dots + \frac{1}{\omega_1(\lambda)} \int_{\theta_0}^{\theta_1} \phi_{1\lambda}(y) f(y)\, dy$$

$$b_{m+1}(\lambda) = \frac{1}{\omega_m(\lambda)} \int_{\theta_{m-1}}^{\theta_m} \phi_{m\lambda}(y) f(y)\, dy + \frac{1}{\omega_{m-1}(\lambda)} \int_{\theta_{m-2}}^{\theta_{m-1}} \phi_{(m-1)\lambda}(y) f(y)\, dy$$
$$+ \dots + \frac{1}{\omega_1(\lambda)} \int_{\theta_0}^{\theta_1} \phi_{1\lambda}(y) f(y)\, dy.$$

Then we obtain the solution

(21)

$$u(x,\lambda)=\begin{cases}\frac{\phi_{1\lambda}(x)}{\omega_1(\lambda)}\int\limits_{x}^{\theta_1}\chi_{1\lambda}(y)f(y)dy+\frac{\chi_{1\lambda}(x)}{\omega_1(\lambda)}\int\limits_{\theta_0}^{x}\phi_{1\lambda}(y)f(y)dy+\\ \frac{\phi_{1\lambda}(x)}{\omega_2(\lambda)}\int\limits_{\theta_1}^{\theta_2}\chi_{2\lambda}(y)f(y)dy+\frac{\phi_{1\lambda}(x)}{\omega_3(\lambda)}\int\limits_{\theta_2}^{\theta_3}\chi_{3\lambda}(y)f(y)dy+...+\\ \frac{\phi_{1\lambda}(x)}{\omega_{m+1}(\lambda)}\int\limits_{\theta_m}^{\theta_{m+1}}\chi_{(m+1)\lambda}(y)f(y)dy+\frac{f_1\phi_{1\lambda}(x)}{\omega_{m+1}(\lambda)}, & x\in[\theta_0,\theta_1)\\ \\ \frac{\phi_{2\lambda}(x)}{\omega_2(\lambda)}\int\limits_{x}^{\theta_2}\chi_{2\lambda}(y)f(y)dy+\frac{\chi_{2\lambda}(x)}{\omega_2(\lambda)}\int\limits_{\theta_1}^{x}\phi_{2\lambda}(y)f(y)dy+\frac{\phi_{2\lambda}(x)}{\omega_3(\lambda)}\int\limits_{\theta_2}^{\theta_3}\chi_{3\lambda}(y)f(y)dy+\\ \frac{\phi_{2\lambda}(x)}{\omega_4(\lambda)}\int\limits_{\theta_3}^{\theta_4}\chi_{4\lambda}(y)f(y)dy+...+\frac{\phi_{2\lambda}(x)}{\omega_{m+1}(\lambda)}\int\limits_{\theta_m}^{\theta_{m+1}}\chi_{(m+1)\lambda}(y)f(y)dy+\\ \frac{\chi_{2\lambda}(x)}{\omega_1(\lambda)}\int\limits_{\theta_0}^{\theta_1}\phi_{1\lambda}(y)f(y)dy+\frac{f_1\phi_{2\lambda}(x)}{\omega_{m+1}(\lambda)}, & x\in(\theta_1,\theta_2)\\ \quad\vdots\\ \frac{\phi_{(m+1)\lambda}(x)}{\omega_{m+1}(\lambda)}\int\limits_{x}^{\theta_{m+1}}\chi_{(m+1)\lambda}(y)f(y)dy+\frac{\chi_{(m+1)\lambda}(x)}{\omega_{m+1}(\lambda)}\int\limits_{\theta_m}^{x}\phi_{(m+1)\lambda}(y)f(y)dy+\\ \frac{\chi_{(m+1)\lambda}(x)}{\omega_m(\lambda)}\int\limits_{\theta_{m-1}}^{\theta_m}\phi_{m\lambda}(y)f(y)dy+\frac{\chi_{(m+1)\lambda}(x)}{\omega_{m-1}(\lambda)}\int\limits_{\theta_{m-2}}^{\theta_{m-1}}\phi_{(m-1)\lambda}(y)f(y)dy+...+\\ \frac{\chi_{(m+1)\lambda}(x)}{\omega_1(\lambda)}\int\limits_{\theta_0}^{\theta_1}\phi_{1\lambda}(y)f(y)dy+\frac{f_1\phi_{(m+1)\lambda}(x)}{\omega_{m+1}(\lambda)}, & x\in(\theta_m,\theta_{m+1}].\end{cases}$$

Substituting (12) into (21) and from (13), then (21) can be written in the form

$$u(x,\lambda)=\sum_{k=1}^{m+1}\frac{1}{\prod\limits_{l=1}^{k}|D_{l-1}|}\int\limits_{\theta_{k-1}}^{\theta_k}G(x,y;\lambda)f(y)dy+\frac{f_1\phi_\lambda(x)}{|D_1D_2...D_m|\,\omega(\lambda)},\tag{22}$$

where

$$G(x,y;\lambda)=\begin{cases}\dfrac{\phi_\lambda(y)\chi_\lambda(x)}{\omega(\lambda)}, & \theta_0\le y\le x\le\theta_{m+1},\ x,y\neq\theta_k,\ k=\overline{1,m}\\ \dfrac{\phi_\lambda(x)\chi_\lambda(y)}{\omega(\lambda)}, & \theta_0\le x\le y\le\theta_{m+1},\ x,y\neq\theta_k,\ k=\overline{1,m}\end{cases}\tag{23}$$

is Green's function of the problem (1)-(4).

Hence, we have

$$U=(\lambda \mathrm{I}-A)^{-1}F= \tag{24}$$
$$\begin{pmatrix} \sum\limits_{k=1}^{m+1}\frac{1}{\prod\limits_{l=1}^{k}|D_{l-1}|}\int\limits_{\theta_{k-1}}^{\theta_k}G(x,y;\lambda)f(y)\,dy+\frac{f_1\phi_\lambda(x)}{|D_1D_2...D_m|\omega(\lambda)} \\ R'(u) \end{pmatrix}$$

the resolvent of the problem (1)-(4).

Denoting $\mathbf{G}_{x,\lambda}=\begin{pmatrix} G(x,\cdot;\lambda) \\ R'(G(x,\cdot;\lambda)) \end{pmatrix}, \overline{F}=\begin{pmatrix} \overline{f(x)} \\ \overline{f_1} \end{pmatrix}$ and from (13) and (24) the resolvent of the operator can be represented in the form

$$U=\begin{pmatrix} \left\langle \mathbf{G}_{x,\lambda},\overline{F}\right\rangle \\ R'\left(\left\langle \mathbf{G}_{x,\lambda},\overline{F}\right\rangle\right) \end{pmatrix}. \tag{25}$$

3. Eigenfunction Expansion for Green's Function and the Modified Parseval Equality

In this section, we derive the eigenfunction expansion for Green's function of the problem (1)-(4), and establish the modified Parseval equality in the associated Hilbert space H. Without loss of generality we assume that $\lambda=0$ is not an eigenvalue. Let $\mathbf{G}_{x,0}=G(x,y;0)$.

From (25), the solution of the equation $A\Phi=\lambda\Phi$ can be written in the form

$$\Phi=\begin{pmatrix} \left\langle \mathbf{G}_{x,0},-\overline{\lambda\Phi}\right\rangle \\ R'\left(\left\langle G_{x,0},-\overline{\lambda\Phi}\right\rangle\right) \end{pmatrix} \tag{26}$$

and the first component of this solution can be written as

$$\phi(x)=-\lambda\left\langle \mathbf{G}_{x,0},\Phi\right\rangle. \tag{27}$$

Theorem 3.1. *Let λ_n be the eigenvalues of the problem (1)-(4) and $\Psi_n(x)$ be the corresponding normalized eigenelements. Then*

$$\mathbf{G}_{x,0}=-\sum_{n=1}^{\infty}\frac{\Psi_n(x)\Psi_n(y)}{\lambda_n}. \tag{28}$$

Proof. Let $P(x,y) = \mathbf{G}_{x,0} + \sum_{n=1}^{\infty} \frac{\Psi_n(x)\Psi_n(y)}{\lambda_n}$. Then $P(x,y)$ is continuous and symmetric. We assume $P(x,y) \neq 0$. Then by the Fredholm integral equation, there is a number λ_0 and a function $\Phi_0(x) \neq 0$ in H such that $\Phi_0 = \begin{pmatrix} \phi_0(x) \\ R'(\phi_0) \end{pmatrix}$ and satisfy

$$\phi_0(x) = \lambda_0 \langle P(x,y), \Phi_0 \rangle . \tag{29}$$

Since each $\Psi_n(x)$ is an eigenelement, we obtain from (27) that

$$\psi_n(x) + \lambda_n \left\langle \mathbf{G}_{x,0}, \Psi_n \right\rangle = 0. \tag{30}$$

Then, substituting from (30), we obtain

$$\begin{aligned} \langle P(x,y), \Psi_m \rangle &= \left\langle \mathbf{G}_{x,0} + \sum_{n=1}^{\infty} \frac{\Psi_n(x)\Psi_n(y)}{\lambda_n}, \Psi_m \right\rangle \\ &= \left\langle \mathbf{G}_{x,0}, \Psi_m \right\rangle + \sum_{n=1}^{\infty} \frac{\Psi_n(x)}{\lambda_n} \langle \Psi_n, \Psi_m \rangle \\ &= -\frac{1}{\lambda_m}\Psi_m + \frac{1}{\lambda_m}\Psi_m \\ &= 0. \end{aligned} \tag{31}$$

From (29), we get

$$\Phi_0 = \lambda_0 \begin{pmatrix} \langle P(x,y), \Phi_0 \rangle \\ R'(\langle P(x,y), \Phi_0 \rangle) \end{pmatrix}. \tag{32}$$

By the definition of the inner product (13), we get

$$
\begin{aligned}
\langle \Phi_0, \Psi_n \rangle &= \sum_{k=1}^{m+1} \frac{1}{\prod_{l=1}^{k} |D_{l-1}|} \int_{\theta_{k-1}}^{\theta_k} \lambda_0 \langle P(x,y), \Phi_0 \rangle \psi_n(x)\, dx + \\
&\frac{1}{\rho |D_1 D_2 ... D_m|} \lambda_0 R' (\langle P(x,y), \Phi_0 \rangle) R'(\psi_n) \\
&= \lambda_0 \left\{ \sum_{k=1}^{m+1} \frac{1}{\prod_{l=1}^{k} |D_{l-1}|} \int_{\theta_{k-1}}^{\theta_k} \left\{ \sum_{k=1}^{m+1} \frac{1}{\prod_{l=1}^{k} |D_{l-1}|} \int_{\theta_{k-1}}^{\theta_k} P(x,y)\, \phi_0(y)\, dy + \right.\right. \\
&\left. \frac{1}{\rho |D_1 D_2 ... D_m|} R'(P(x,y)) R'(\phi_0) \right\} \psi_n(x)\, dx + \\
&\frac{1}{\rho |D_1 D_2 ... D_m|} \left\{ \sum_{k=1}^{m+1} \frac{1}{\prod_{l=1}^{k} |D_{l-1}|} \int_{\theta_{k-1}}^{\theta_k} R'(P(x,y))\, \phi_0(y)\, dy + \right. \\
&\left.\left. \frac{1}{\rho |D_1 D_2 ... D_m|} R'(R'(P(x,y))) R'(\phi_0) \right\} R'(\psi_n) \right\} \\
&= \lambda_0 \left\{ \sum_{k=1}^{m+1} \frac{1}{\prod_{l=1}^{k} |D_{l-1}|} \int_{\theta_{k-1}}^{\theta_k} \langle P(x,y), \Psi_n \rangle \phi_0(y)\, dy + \right. \\
&\left. \frac{1}{\rho |D_1 D_2 ... D_m|} R'(\langle P(x,y), \Psi_n \rangle) R'(\phi_0) \right\}.
\end{aligned} \tag{33}
$$

Substituting (31) into (33), we obtain

$$\langle \Phi_0, \Psi_n \rangle = 0. \tag{34}$$

Then from (29) and (34), we see that

$$
\begin{aligned}
\phi_0(x) = \lambda_0 \langle P(x,y), \Phi_0 \rangle &= \lambda_0 \left\langle \mathbf{G}_{x,0} + \sum_{n=1}^{\infty} \frac{\Psi_n(x)\Psi_n(y)}{\lambda_n}, \Phi_0 \right\rangle \\
&= \lambda_0 \left\langle \mathbf{G}_{x,0}, \Phi_0 \right\rangle + \lambda_0 \sum_{n=1}^{\infty} \frac{\Psi_n(x)}{\lambda_n} \langle \Psi_n, \Phi_0 \rangle \\
&= \lambda_0 \left\langle \mathbf{G}_{x,0}, \Phi_0 \right\rangle.
\end{aligned} \tag{35}
$$

This implies Φ_0 is the eigenelement of the problem (1)-(4). So from (34) and the completeness of the eigenfunctions, we know $\Phi_0 = 0$. Thus we get a contradiction. Consequently $P(x,y) = 0$. The proof is completed.

Theorem 3.2. *Let $f \in L_2(\theta_0, \theta_{m+1})$. Then $f(x)$ can be expanded into an absolutely and uniformly convergent series of eigenfunctions, that is*

$$f(x) = \sum_{n=1}^{\infty} \left\{ \sum_{k=1}^{m+1} \frac{1}{\prod_{l=1}^{k} |D_{l-1}|} \int_{\theta_{k-1}}^{\theta_k} f(y)\, \psi_n(y)\, dy + \frac{1}{\rho\, |D_1 D_2 ... D_m|} R'(f)\, R'(\psi_n) \right\} \psi_n(x). \tag{36}$$

Proof. Let $f \in L_2(\theta_0, \theta_{m+1})$. Then $F = \begin{pmatrix} f(x) \\ R'(f) \end{pmatrix} \in D(A)$. It follows that the function $f(x)$ can be written as

$$f(x) = \left\langle \mathbf{G}_{x,0}, -\overline{AF} \right\rangle. \tag{37}$$

Substituting (28) into (37) we get

$$f(x) = \sum_{n=1}^{\infty} \frac{\Psi_n(x)}{\lambda_n} \left\langle \Psi_n, \overline{AF} \right\rangle. \tag{38}$$

Since the operator A is symmetric (see [6]), we have

$$\left\langle \Psi_n, \overline{AF} \right\rangle = \langle AF, \Psi_n \rangle = \langle F, A\Psi_n \rangle = \langle F, \lambda \Psi_n \rangle = \lambda_n \langle F, \Psi_n \rangle. \tag{39}$$

From (38) and (39), we obtain

$$F = \sum_{n=1}^{\infty} \langle F, \Psi_n \rangle \Psi_n. \tag{40}$$

If the equality (40) is written as open and the first component being equal the proof is completed.

Theorem 3.3. *Let $f \in L_2(\theta_0, \theta_{m+1})$. Then the modified Parseval equality holds, that is*

$$\|f\|^2_{L_2(\theta_0, \theta_{m+1})} = \sum_{n=1}^{\infty} |c_n|^2 \tag{41}$$

where $\|f\|^2_{L_2(\theta_0, \theta_{m+1})} = \langle f, f \rangle$ and c_n is

$$c_n = \sum_{k=1}^{m+1} \frac{1}{\prod_{l=1}^{k} |D_{l-1}|} \int_{\theta_{k-1}}^{\theta_k} f(x)\, \psi_n(x)\, dx + \frac{1}{\rho\, |D_1 D_2 ... D_m|} R'(f)\, R'(\psi_n). \tag{42}$$

Proof. For any $f \in L_2(\theta_0, \theta_{m+1})$, there exists a sequence $\{f_k\}_{k\in\mathbb{N}} \subset \widetilde{C}_0^\infty$ converging to f in $L_2(\theta_0, \theta_{m+1})$. Since $\widetilde{C}_0^\infty$ is dense in $L_2(\theta_0, \theta_{m+1})$ (see [3]), we find from Theorem 3.2 becomes

$$f_k(x) = \sum_{n=1}^{\infty} c_n^{(k)} \psi_n(x), \tag{43}$$

where $c_n^{(k)}$ is

$$c_n^{(k)} = \sum_{k=1}^{m+1} \frac{1}{\prod\limits_{l=1}^{k} |D_{l-1}|} \int\limits_{\theta_{k-1}}^{\theta_k} f_k(x)\,\psi_n(x)\,dx + \frac{1}{\rho\,|D_1 D_2 ... D_m|} R'(f_k)\,R'(\psi_n) \tag{44}$$

and we get

$$\langle f_k - f_l, \Psi_n \rangle_{L_2(\theta_0,\theta_{m+1})} = c_n^{(k)} - c_n^{(l)}.$$

It follows that

$$\|f_k - f_l\|^2_{L_2(\theta_0,\theta_{m+1})} = \sum_{n=1}^{\infty} \left| c_n^{(k)} - c_n^{(l)} \right|^2. \tag{45}$$

By the Cauchy-Schwartz inequality, we have

$$\left| c_n - c_n^{(k)} \right| = \left| \langle f - f_k, \Psi_n \rangle_{L_2(\theta_0,\theta_{m+1})} \right| = \|f_k - f_l\|_{L_2(\theta_0,\theta_{m+1})}.$$

Hence $\lim\limits_{k\to\infty} c_n^{(k)} = c_n$. Since (45), we can write

$$\sum_{n=1}^{N} \left| c_n^{(k)} - c_n^{(l)} \right|^2 \le \|f_k - f_l\|^2_{L_2(\theta_0,\theta_{m+1})}. \tag{46}$$

Letting $k \to \infty$, we find the inequality (46) becomes

$$\sum_{n=1}^{N} \left| c_n - c_n^{(l)} \right|^2 \le \|f - f_l\|^2_{L_2(\theta_0,\theta_{m+1})}.$$

Letting $N \to \infty$, we have

$$\sum_{n=1}^{\infty} \left| c_n - c_n^{(l)} \right|^2 \le \|f - f_l\|^2_{L_2(\theta_0,\theta_{m+1})}. \tag{47}$$

Then by the Minkowski inequality

$$\sum_{n=1}^{\infty} |c_n|^2 = \sum_{n=1}^{\infty} \left| c_n - c_n^{(l)} + c_n^{(l)} \right|^2 \le \left(\left(\sum_{n=1}^{\infty} \left| c_n - c_n^{(l)} \right|^2 \right)^{1/2} + \left(\sum_{n=1}^{\infty} \left| c_n^{(l)} \right|^2 \right)^{1/2} \right)^2 < \infty$$

and by the Hölder's inequality

$$\left|\sum_{n=1}^{\infty}|c_n|^2-\sum_{n=1}^{\infty}\left|c_n^{(k)}\right|^2\right|=\left|\sum_{n=1}^{\infty}\left|c_n-c_n^{(k)}\right|\left|c_n+c_n^{(k)}\right|\right|$$
$$\leq\left(\sum_{n=1}^{\infty}\left|c_n-c_n^{(k)}\right|^2\right)^{1/2}\left(\sum_{n=1}^{\infty}\left|c_n+c_n^{(k)}\right|^2\right)^{1/2}\to 0, \text{ as } k\to\infty.$$

This means $\lim\limits_{k\to\infty}\sum_{n=1}^{\infty}\left|c_n^{(k)}\right|^2=\sum_{n=1}^{\infty}|c_n|^2$. Since $f_k\to f$ in $L_2(\theta_0,\theta_{m+1})$ as $k\to\infty$,

$$\lim_{k\to\infty}\|f_k\|_{L_2(\theta_0,\theta_{m+1})}=\|f\|_{L_2(\theta_0,\theta_{m+1})}.$$

Thus, we obtain

$$\|f\|^2_{L_2(\theta_0,\theta_{m+1})}=\lim_{k\to\infty}\|f_k\|^2_{L_2(\theta_0,\theta_{m+1})}=\lim_{k\to\infty}\sum_{n=1}^{\infty}\left|c_n^{(k)}\right|^2=\sum_{n=1}^{\infty}|c_n|^2.$$

This completes the proof.

Conflict of Interests

The authors declare that there is no conflict of interests.

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