Available online at http://scik.org
J. Math. Comput. Sci. 2 (2012), No. 5, 1475-1497
ISSN: 1927-5307

VARIATIONAL ITERATION METHOD FOR SOLVING OPTIMIZATION PROBLEMS

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Abstract. In this paper, we use variational iteration method (VIM) to solve some optimization problems. Exact solutions for many of the problems discussed in this paper are found. The main idea is to use both Euler's equations together with Lagrange multiplier in solving correction functionals for the problems. We use He's VIM to handle many kinds of the variational problems, such as problems with fixed and moving boundaries, also, we solve some variational problems with extremals having corner points. Moreover, we found the solution of the variational problems involving conditional extremum such as, isoperimetric problems, via the variational iteration method. In addition, we introduced the relation between the conditional problems and the eigenvalue problems.

Keywords: Optimization problems, Variational iteration methoth.

2000 AMS Subject Classification: 49J05; 35A15

1. Introduction

In a large number of problems arising in analysis, mechanics, geometry, etc., it is necessary to determine the maximal and minimal of a certain functional [1]. Problems in which it is required to investigate a function for maximum or minimum are called optimization problem. Calculus of variations (C.V) began to develop in 1696, and became an independent mathematical branch with its own methods of investigation, after the fundamental works of Euler's (1707-1783), whom we may justifiably consider the founder of the

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Received February 12, 2012

calculus of variations [8]. Minimization problems that can be analyzed by the calculus of variation serve to characterize the equilibrium configurations of almost all continuous physical systems [6, 7]. Finding the solution of these problems needs to solve the corresponding ordinary (partial) differential equation that are generally nonlinear and difficult to find exact solutions. In recent years, He's variational iteration method VIM, proposed by He [2], have received much attention of the researcher's for solving nonlinear problems [10, 11]. It has been applied to a wide class of deterministic and stochastic problems, linear and nonlinear, in physics, biology and chemical reactions etc [15, 16]. Also, it is used for solving some problems in calculus of variations [4, 5] and Computation of eigenvalues for Sturm-Liouville problems [17]. The method gives rapidly convergent successive approximations of the exact solution, if a solution exists.

The main concepts in the VIM are the general Lagrange multiplier, restricted variation, correction functional [3]. In this method, general Lagrange multipliers are introduced to construct correction functional for the variational problems. The multipliers in functionals can identified optimally via the variational theory. The initial approximations can be freely chosen with possible unknown constant which can be determined by imposing the boundary or initial conditions [4]. In this paper, the (VIM) will be used to solve the optimization problems and try to find an exact solution for this kind of problems. Some examples will be presented to test the efficiency of the proposed technique. For more details see [12].

The arrangement of this paper is as follows: In section two, we introduce the basic concepts of calculus of variation and VIM. In section three, we handle some optimization problems with moving boundaries, also, we solve some constructed problems in which their solutions have corner points in section four. Finally, in section five, we solve some examples of the conditional problems, such as the isoperimetric problems.

2. Calculus of variations with Fixed Boundaries

The (VIM) was proposed by He [2] initially with the aims to solve frontier physical problem. It has been applied to a wide class of deterministic and stochastic problems, linear and nonlinear, in physics, biology and chemical reactions etc., [13, 14, 15]. Also, it is used for solving some problems in calculus of variations [5].

2.1. Functions of Single Derivatives

The main idea of VIM is to construct a correction functional form using general Lagrange multipliers. These multipliers should be chosen such that its correction solution is superior to its initial approximation, called trial function. It is the best within the flexibility of trial functions. Accordingly, Lagrange multipliers can be identified by the variational theory [5]. The initial approximation can be freely chosen with possible unknowns, which can be determined by imposing boundary/initial conditions, and end up with finding the approximate solution [9]. In this paper, He's VIM will be employed for solving some problems in calculus of variation. The examples ranging through different applications in physics and it will be presented to show the efficiency of the proposed technique. In the VIM, we will consider the general problem as

$$Ly + Ny = g\left(s\right),$$

where L is a linear operator, N is a nonlinear operator, and $g(s)$ is the non-homogeneous term. Using variational iteration method, the following correct functional is considered

$$y_{n+1} = y_n + \int_{x_0}^{x} \lambda \left(Ly_n\left(s\right) + N\tilde{y}_n\left(s\right) - g\left(s\right)\right) ds, \tag{1}$$

where λ is Lagrange multiplier [5], which can be identified optimally via the variational theory, the subscript n denotes the nth approximation, and $\tilde{y}_n$ is considered as a restricted variation, i.e. $\delta\tilde{y}_n = 0$. The main idea of this paper is to replace λ in equation (1) by the result obtained from Euler's equation.

Recall that the boundary-value problem

$$F_y - \frac{d}{dx}F_{y'} = 0, \qquad y\left(x_0\right) = y_0, \qquad y\left(x_1\right) = y_{1},$$

does not always have a solution, and if the solution exists, it may not be unique. Note that in many variational problems the existence of a solution is obvious from the physical or geometrical meaning of the problem, and if the solution of Euler's equation satisfying the boundary conditions is unique, then this unique extremal will be the solution of the given variational problem. Consider,

$$v[y_1, y_2, ..., y_n] = \int_{x_0}^{x_1} F(x, y_1, y_2, ..., y_n, y_1', y_{2y}', ..., y_n')dx$$

with given boundary conditions

$$y_1\left(x_0\right) = y_{10}, \qquad y_2\left(x_0\right) = y_{20}, ... \ , y_n\left(x_0\right) = y_{n0}$$

$$y_1\left(x_1\right) = y_{11}, \qquad y_2\left(x_1\right) = y_{21}, ... \ ,y_n\left(x_1\right) = y_{n1},$$

for the maximum or the minimum. In order to do that, we need to obtain the necessary conditions, $\delta v = 0$.

Therefore, we can vary only one function $y_j\left(x\right)$, $\left(j = 1, 2, ..., n\right)$, and holding the other functions unchanged. Then the functional will reduce to a functional dependent only on a single function $y_j\left(x\right)$.

$$v[y_1, y_2, ..., y_n] = \tilde{v}[y_i],$$

so, the extremizing function must satisfy Euler's equation

$$F_{y_i} - \frac{d}{dx} F_{y_i'} = 0.$$

Since, we can do that for any function $y_i(x)$, $i = 1, 2, ..., n$, we get a system of second-order differential equations

$$F_{y_i} - \frac{d}{dx} F_{y_i'} = 0 \qquad i = 1, 2, ..., n.$$

In order to handle problems arising in the nature, specially in optics, we will display the Fermat's principle which discuss the least time that the ray of light taken to propagated from point A to another B, with given speed. Now to solve the optics problem, we will convert it to form in calculus of variation. Just to keep the reader with us, we illustrate it by the following example.

Example 1. *Find the lines of propagation of light between $A(0,0,0)$, and $B(1,1,1)$, such that they investigate the least time, where the velocity is given by $v(x,y,z) = c$, where c is constant.*

Since we want the least time t of the propagation, we can say

$$dt = \frac{ds}{v},$$

where ds is the distance between two close points, as we know $ds^2 = dx^2 + dy^2 + dz^2$.

Simplify, we can get

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2 + \left(\frac{dz}{dx}\right)^2}\, dx.$$

Hence,

$$t = \int_{x_0}^{x_1} \frac{\sqrt{1 + y'^2 + z'^2}}{c} dx$$

Now, we can solve this functional, by finding the curves that satisfy the system of Euler's equations

$$F_y - \frac{d}{dx} F_{y'} = z_n'' z_n' y_n' - y_n'' z_n'^2 - y_n'' = 0,$$

$$F_z - \frac{d}{dx} F_{z'} = y_n'' y_n' z_n' - z_n'' y_n'^2 - z_n'' = 0.$$

Which are the differential equations of the lines of propagation of light between A, and B. Using He's variational iteration method, we have the following correctional functionals:

$$y_{n+1}(x) = y_n(x) + \int_0^x \lambda_1(t) \left(z_n''(t)\, z_n'(t)\, y_n'(t) - y_n''(t)\, z_n'^2(t) - y_n''(t)\right) dt, \tag{2}$$

$$z_{n+1}(x) = z_n(x) + \int_0^x \lambda_2(t) \left(y_n''(t)\, y_n'(t)\, z_n'(t) - z_n''(t)\, y_n'^2(t) - z_n''(t)\right) dt. \tag{3}$$

Taking the variation of equations (2) *and* (3) *with respect to* y_n , z_n, *respectively, noting that* $\delta y_{n+1} = \delta z_{n+1} = 0$, *we get*

$$\delta y_{n+1}(x) = \delta y_n(x) + \delta \int_0^x \lambda_1(t)\left(z_n''(t)\, z_n'(t)\, y_n'(t) - y_n''(t)\, z_n'^2(t) - y_n''(t)\right) dt,$$

$$\delta z_{n+1}(x) = \delta z_n(x) + \delta \int_0^x \lambda_2(t)\left(y_n''(t)\, y_n'(t)\, z_n'(t) - z_n''(t)\, y_n'^2(t) - z_n''(t)\right) dt.$$

Then we can get the following stationary conditions

$$1 + \lambda_1'(t)\mid_{t=x} = 0, \qquad \lambda_1(t)\mid_{t=x} = 0, \qquad \lambda_1''(t) = 0,$$

$$1 + \lambda_2'(t)\mid_{t=x} = 0, \qquad \lambda_2(t)\mid_{t=x} = 0, \qquad \lambda_2''(t) = 0.$$

these yield to

$$\lambda_1(t) = \lambda_2(t) = x - t,$$

Therefore, we have

$$y_{n+1}(x) = y_n(x) + \int_0^x (x-t)\left(z_n''(t)\, z_n'(t)\, y_n'(t) - y_n''(t)\, z_n'^2(t) - y_n''(t)\right) dt,$$

$$z_{n+1}(x) = z_n(x) + \int_0^x (x-t)\left(y_n''(t)\, y_n'(t)\, z_n'(t) - z_n''(t)\, y_n'^2(t) - z_n''(t)\right) dt.$$

Set,

$$y_0(x) = ax + b, \qquad z_0(x) = cx + d,$$

y_1 *and* z_1 *are easy to find. By imposing the boundary conditions on the obtained* y_1 *and* z_1, *we have:*

$$a = 1, \quad b = 0, \quad c = 1, \quad d = 0,$$

thus,

$$y_1(x) = z_1(x) = x,$$

Where these are the lines of propagation of light between $A(0,0,0)$, *and* $B(1,1,1)$.

2.2. Functionals Depends on Higher-Order Derivatives

In order to obtain the necessary conditions for the extremum of the functional

$$v[y(x)] = \int_{x_0}^{x_1} F\left(x, y(x), y'(x), ..., y^{(n)}(x)\right) dx,$$

with the boundary conditions

$$y(x_0) = y_0, \quad y'(x_0) = y_0', ..., y^{(n-1)}(x_0) = y_0^{(n-1)};$$

$$y(x_1) = y_1, \quad y'(x_1) = y_1', ..., y^{(n-1)}(x_1) = y_1^{(n-1)}.$$

we need first to consider the function F which has $(n+2)$ derivatives with respect to all arguments, and the extremal curve $y(x)$ which has $2n$ derivatives.

Now, we will reduce the functional $v[y(x)]$ to a function of one-parameter α, by considering the value of the functional $v[y(x)]$ only on curves of the family with one parameter α

$$y(x,\alpha) = y(x) + \alpha\delta y,$$

where $\delta y = \bar{y}(x) - y(x)$, with $\bar{y}(x)$ belongs to the admissible curves, which is also $2n$ times differentiable. Note that, $v[y(x)]$ has an extremum value at $\alpha = 0$, hence $\delta v = \frac{d}{d\alpha} v[y(x,\alpha)]\,|_{\alpha=0} = 0$. This derivative is called the variation of the functional, denoted by δv, and given by

$$\delta v = \frac{d}{d\alpha} v[y(x,\alpha)]\,|_{\alpha=0} = \int_{x_0}^{x_1} \left(F_y \delta y + F_{y'}\delta y' + F_{y''}\delta y'' + F_{y^{(n)}}\delta y^{(n)} \right) dx.$$

Integrating by parts the second summand once, the third summand twice, and so forth; the last summand n times, implies that:

$$\int_{x_0}^{x_1} F_{y^{(n)}} \delta y^{(n)} dx = \left[F_{y^{(n)}} \delta y^{(n-1)} \right]_{x_0}^{x_1} - \left[\frac{d}{dx} F_{y^{(n)}} \delta y^{(n-1)} \right]_{x_0}^{x_1} + \ldots + (-1)^n \int_{x_0}^{x_1} \frac{d^n}{dx^n} F_{y^{(n)}} \delta y dx.$$

Now, taking the boundary conditions, for $x = x_0$ and for $x = x_1$, the variations $\delta y = \delta y' = \delta y'' = \ldots = \delta y^{(n-1)} = 0$, we finally get

$$\delta v = \int_{x_0}^{x_1} \left(F_y - \frac{d}{dx} F_{y'} + \frac{d^2}{dx^2} F_{y''} + \ldots + (-1)^n \frac{d^n}{dx^n} F_{y^{(n)}} \right) \delta y dx,$$

on the extremizing curve we have

$$\delta v = \int_{x_0}^{x_1} \left(F_y - \frac{d}{dx} F_{y'} + \frac{d^2}{dx^2} F_{y''} + \ldots + (-1)^n \frac{d^n}{dx^n} F_{y^{(n)}} \right) \delta y dx = 0.$$

because δy is arbitrary, and the first factor is continuous function of x on the same curve $y = y(x)$, then by the fundamental Lemma, we have:

$$F_y - \frac{d}{dx} F_{y'} + \frac{d^2}{dx^2} F_{y''} + \ldots + (-1)^n \frac{d^n}{dx^n} F_{y^{(n)}} = 0.$$

Thus, the function $y = y(x)$, which extremizes the functional

$$v[y(x)] = \int_{x_0}^{x_1} F\left(x, y(x), y'(x), \ldots, y^{(n)}(x) \right) dx,$$

must be a solution of the equation

$$F_y - \frac{d}{dx} F_{y'} + \frac{d^2}{dx^2} F_{y''} + \ldots + (-1)^n \frac{d^n}{dx^n} F_{y^{(n)}} = 0,$$

which is called the Euler-poisson equation, and it is of order $2n$. The general solution of this equation contains $2n$ arbitrary constants that may be determined by the $2n$ boundary conditions:

$$y\left(x_0\right) = y_0, y'\left(x_0\right) = y_0', ..., y^{(n-1)}\left(x_0\right) = y_0^{(n-1)};$$

$$y\left(x_1\right) = y_1, y'\left(x_1\right) = y_1', ..., y^{(n-1)}\left(x_1\right) = y_1^{(n-1)}.$$

2.3. Functionals of Several Independent Variables

In this section, we consider functions with two variables, as in the following form,

$$v\left[z\left(x,y\right)\right]=\iint\limits_{D}F\left(x,y,z\left(x,y\right),\frac{\partial z}{\partial x},\frac{\partial z}{\partial y}\right)dx,dy;$$

where we determine the value of the function z on the boundary C of the domain D. We consider a function F that is three times differentiable and the extremizing surface $z=z\left(x,y\right)$ twice differentiable. Now, again as we did before, we reduce the functional v to function of one parameter α, by considering the family of surfaces $z\left(x,y,\alpha\right)=z\left(x,y\right)+\alpha\delta z$, where $\delta z=\bar{z}\ \left(x,y\right)-z\left(x,y\right)$. Note that, the functional $v\left[z\left(x,y,\alpha\right)\right]$ extremized at $\alpha=0$. So $\frac{\partial}{\partial\alpha}v\left[z\left(x,y,\alpha\right)\right]_{\alpha=0}=0$, we call the derivative of $v\left[z\left(x,y,\alpha\right)\right]$ with respect to α, for $\alpha=0$, the variation of the functional $\left(\delta v\right)$, just to simplify, put $\frac{\partial z}{\partial x}=p$, $\frac{\partial z}{\partial y}=q$, we have,

$$\delta v=[\frac{\partial}{\partial\alpha}\iint\limits_{D}F\left(x,y,z\left(x,y,\alpha\right),p\left(x,y,\alpha\right),q\left(x,y,\alpha\right)\right)dx,dy]_{\alpha=0}=\iint\limits_{D}[F_z\delta z+F_p\delta p+F_q\delta q]dxdy,$$

where,

$$p\left(x,y,\alpha\right)=\frac{\partial z\left(x,y,\alpha\right)}{\partial x}=p\left(x,y\right)+\alpha\delta p,\quad q\left(x,y,\alpha\right)=\frac{\partial z\left(x,y,\alpha\right)}{\partial y}=q\left(x,y\right)+\alpha\delta q.$$

Using,

$$\frac{\partial}{\partial x}\left[F_p\delta z\right]=\frac{\partial}{\partial x}\left[F_p\right]\delta z+F_p\delta p,\quad \frac{\partial}{\partial y}\left[F_q\delta z\right]=\frac{\partial}{\partial y}\left[F_q\right]\delta z+F_q\delta q,$$

It follows that,

$$\iint\limits_{D}[F_p\delta p+F_q\delta q]dxdy=\iint\limits_{D}[\frac{\partial}{\partial x}\left[F_p\delta z\right]+\frac{\partial}{\partial y}\left[F_q\delta z\right]]dxdy-\iint\limits_{D}[\frac{\partial}{\partial x}\left[F_p\right]+\frac{\partial}{\partial y}\left[F_q\right]]\delta zdxdy,$$

where,

$$\frac{\partial}{\partial x}[F_p]=F_{px}+F_{pz}\frac{\partial z}{\partial x}+F_{pp}\frac{\partial p}{\partial x}+F_{pq}\frac{\partial q}{\partial x}$$
$$\frac{\partial}{\partial x}[F_p]=F_{qy}+F_{qz}\frac{\partial z}{\partial y}+F_{qp}\frac{\partial p}{\partial y}+F_{qq}\frac{\partial q}{\partial y}.$$

Using Green's theorem, we get,

$$\iint\limits_{D}[\frac{\partial}{\partial x}\left[F_p\delta z\right]+\frac{\partial}{\partial y}\left[F_q\delta z\right]]dxdy=\int\limits_{C}\left(F_pdy-F_qdx\right)\delta z=0.$$

Since on the contour C, $\delta z=0$ because all permissible surfaces pass through the same path $\tilde{C}$. So,

$$\iint\limits_{D}[F_p\delta p+F_q\delta q]dxdy=-\iint\limits_{D}[\frac{\partial}{\partial x}\left[F_p\right]+\frac{\partial}{\partial y}\left[F_q\right]]\delta zdxdy,$$

now the necessary condition $\delta v=0$,

$$\iint\limits_{D}[F_z\delta z+F_p\delta p+F_q\delta q]dxdy=\iint\limits_{D}[F_z-\frac{\partial}{\partial x}\left[F_p\right]+\frac{\partial}{\partial y}\left[F_q\right]]\delta zdxdy=0.$$

because δz is an arbitrary, continuous function and the first factor is continuous, it follows from the fundamental lemma of calculus for variations that on the extremizing the curve $z = z(x, y)$, we have

$$F_z - \frac{\partial}{\partial x} F_p - \frac{\partial}{\partial y} F_q = 0. \tag{4}$$

Which is called Ostrogradsky equation.

3. Problems with Moving-Boundaries

In this section, we investigate the extreme value of a functional with assumption that one or both of the boundary points can move [8].

3.1. Moving-Boundary Problem of Single Function

we investigate the extreme value of the functional

$$v[y(x)] = \int_{x_0}^{x_1} F(x, y, y')\, dx,$$

with assumption that one or both of the boundary points can move, so we can note that, the class of permissible curves is extended. Therefore, if on a curve $y(x)$ an extremum is all the more attained relative to a narrower class of curves having common boundary points with the curve $y = y(x)$ and, hence, the basic condition for achieving an extremum in a problem with fixed boundaries must be a solution of Euler's equation. For the purpose of simplification, we shall assume that one of the boundary points, say (x_0, y_0) is fixed, and the other (x_1, y_1) can moved and passes to point $(x_1 + \delta x_1, y_1 + \delta y_1)$. We will call the permissible curves $y = y(x)$ and $\tilde{y} = y(x) + \delta y$ neighboring if the absolute values of δy and $\delta y'$ are small, and the absolute values of δx_1 and δy_1 are also small. Let us compute the variation of the functional.

Now,

$$\begin{aligned} \delta v &= \int_{x_0}^{x_1+\delta x_1} F(x, y + \delta y, y' + \delta y')\, dx - \int_{x_0}^{x_1} F(x, y, y')\, dx \\ &= \int_{x_1}^{x_1+\delta x_1} F(x, y + \delta y, y' + \delta y')\, dx + \int_{x_0}^{x_1} [F(x, y + \delta y, y' + \delta y') - F(x, y, y')] dx. \end{aligned} \tag{5}$$

Using the Mean value theorem for integral, the first integral in the above equation can be reduced to:

$$\int_{x_1}^{x_1+\delta x_1} F(x, y + \delta y, y' + \delta y')\, dx = F\,|_{x = x_1 + \Theta \delta x_1}\, \delta x_1, \qquad 0 \prec \Theta \prec 1.$$

by the continuity of F, we have

$$F\,|_{x = x_1 + \Theta \delta x_1}\, \delta x_1 = F(x, y, y')\,|_{x = x_1} + \epsilon_1,$$

where $\epsilon_1 \to 0$ as $\delta x_1 \to 0$, and $\delta y_1 \to 0$. Thus,

$$\int_{x_1}^{x_1+\delta x_1} F\left(x, y+\delta y, y'+\delta y'\right) dx = F\left(x, y, y'\right)|_{x=x_1} \delta x_1 + \epsilon_1 \delta x_1.$$

Using the Taylor's expansion of the integrand on the second term of equation (5), we get:

$$\int_{x_0}^{x_1} [F\left(x, y+\delta y, y'+\delta y'\right) - F\left(x, y, y'\right)] dx = \int_{x_0}^{x_1} [F_y\left(x, y, y'\right)\delta y + F_{y'}\left(x, y, y'\right)\delta y'] dx + R_1,$$

where R_1 is an infinitesimal of higher order than δy or $\delta y'$, so the linear part is

$$\int_{x_0}^{x_1} [F_y \delta y + F_{y'} \delta y'] dx,$$

integrating by parts the second summand of the integrand

$$\int_{x_0}^{x_1} [F_y \delta y + F_{y'} \delta y'] dx = [F_{y'} \delta y]_{x_0}^{x_1} + \int_{x_0}^{x_1} \left(F_y - \frac{d}{dx} F y'\right) \delta y dx$$

but $F_y - \frac{d}{dx} F y' = 0$, since y is extremal and satisfy Euler's equation, and the boundary point (x_0, y_0) is fixed, it follows that $\delta y \,|_{x=x_0}$. So,

$$\int_{x_0}^{x_1} [F_y \delta y + F_{y'} \delta y'] dx = [F_{y'} \delta y]_{x=x_1}$$

note that, δy_1: is the increment of y_1 when the boundary point is displaced to $(x_1, +\delta x_1, y_1 + \delta y_1)$, and $\delta y_1 \,|_{x=x_1}$is the increment of the ordinate at the point x_1 when going from extremal passing through the points (x_0, y_0) and (x_1, y_1) to the extremal passing through the points (x_0, y_0) and $(x_1, +\delta x_1, y_1 + \delta y_1)$. Now, to determine the value of $\delta y_1 \,|_{x=x_1}$, and note that,

$$\delta y_1 \,|_{x=x_1} = bc, \qquad \delta y_1 = fd,$$

$$ed \approx y'\left(x_1\right) \delta x_1, \qquad bc = fd - ed.$$

Therefore,

$$\delta y_1 \,|_{x=x_1} \approx \delta y_1 - y'\left(x_1\right) \delta x_1.$$

hence, from equation (5) the basic necessary condition is

$$\delta v = \left(F - y' F_{y'}\right)|_{x=x_1} \delta x_1 + F_{y'} \,|_{x=x_1} \delta y_1 = 0. \tag{6}$$

In order to know how we can determine the extremal function $y\left(x\right)$ using this condition and to handle one of the basic problems in the calculus of variations via VIM, we construct the following example.

Example 2. *(Brachistochrone problem) Find the line connecting two points A $(0, 0)$ and $B\left(x_1, y_1\right)$, such that the point B can move along the curve $h\left(x\right) = x - 1$, that satisfying the shortest time and the velocity $v\left(x, y\right) = \sqrt{1 - y\left(x\right)}$.*

Since we want to satisfy the least time t of the propagation, it is easy to verify that

$$t = \int_0^{x_1} \frac{\sqrt{1+y'^2}}{v(x,y)} dx = \int_0^{x_1} \sqrt{\frac{1+y'^2}{1-y}} dx.$$

Now, for this functional to satisfy Euler's equation, we have

$$y'' - yy'' - \frac{y'^2}{2} - \frac{1}{2} = 0.$$

which is nonlinear differential equation, therefore, we use the VIM. By construct the following correct functional

$$y_{n+1}(x) = y_n(x) + \int_0^x \lambda(s) \left(y_n''(s) - \tilde{y}_n(s)\tilde{y}_n''(s) - \frac{(\tilde{y}_n'(s))^2}{2} - \frac{1}{2} \right) ds,$$

taking variation with respect to the independent variable y_n, we have

$$\delta y_{n+1}(x) = \delta y_n(x) + \delta \int_0^x \lambda(s) \left(y_n''(s) - \tilde{y}_n(s)\tilde{y}_n''(s) - \frac{(\tilde{y}_n'(s))^2}{2} - \frac{1}{2} \right) ds = \delta y_n(x) + \delta \int_0^x \lambda(s) y_n''(s).$$

So, we will get the stationary conditions

$$1 - \lambda'(s)\,|_{s=x} = 0, \quad \lambda(s)\,|_{s=x} = 0, \quad \lambda''(s) = 0.$$

These yield to

$$\lambda(s) = s - x.$$

Therefore, our iteration formula becomes

$$y_{n+1}(x) = y_n(x) + \int_0^x (s-x) \left(y_n''(s) - \tilde{y}_n(s)\tilde{y}_n''(s) - \frac{\tilde{y}_n'^2(s)}{2} - \frac{1}{2} \right) ds.$$

If we set

$$y_0 = ax + b,$$

and find y_1, y_2 we arrive at

$$y_2(x) = b + ax + 0.25x^2 + 0.25a^2x^2 + (1. + a^2)x^2(0.25b + x(0.166667a + 0.0208333x + 0.0208333a^2x))$$

Now, to obtain the constants a and b, we need to satisfy the necessary condition in equation (6), since δx_1 and δy_1 are dependent, (x_1, y_1) move along the curve $h(x) = x - 1$, then $\delta y_1 = h'(x_1)\,\delta x_1$, and so

$$\begin{aligned}
\delta v &= (F - y'F_{y'})\mid_{x=x_1} \delta x_1 + F_{y'}\mid_{x=x_1} \delta y_1 \\
&= (F - y'F_{y'})\mid_{x=x_1} \delta x_1 + F_{y'}\mid_{x=x_1} h'(x)\,\delta x_1 \\
&= (F - y'F_{y'})\mid_{x=x_1} \delta x_1 + F_{y'}\mid_{x=x_1} \delta x_1 \\
&= (F + (1 - y')\,F_{y'})\mid_{x=x_1} \delta x_1 \\
&= 0,
\end{aligned}$$

since δx_1 is arbitrary, therefore

$$(F + (1 - y')\,F_{y'})\mid_{x=x_1} = 0,$$

can be reduced to

$$\begin{aligned}
(F + (1 - y')\,F_{y'})\mid_{x=x_1} &= \left(\sqrt{\frac{1 + y'^2}{1 - y}} + (1 - y')\,\frac{y'}{\sqrt{(1 + y'^2)(1 - y)}}\right)\mid_{x=x_1} \\
&= \left(\frac{(1 + y')}{\sqrt{(1 + y'^2)(1 - y)}}\right)\mid_{x=x_1} = 0.
\end{aligned}$$

So, this condition yield that $y'(x_1) = -1$. Now, we have three conditions

$$y(0) = 0, \qquad y(x_1) = x_1 - 1, \qquad \text{and } y'(x_1) = -1,$$

which are sufficient to determine the constants a, b and x_1, by imposing these conditions on y_2, we get

$$a = -1.30955, \qquad b = 0, \qquad x_1 = 0.475206.$$

So,

$$y_2(x) = -1.30955x + 0.678733x^2 - 0.592559x^3 + 0.15356x^4.$$

Where the extremals of the Brachistochrone problem are cycloids.

3.2. Moving-Boundary Problems of Several Functions

In order to investigate for extremum for the functional

$$v[y(x), z(x)] = \int_{x_0}^{x_1} F(x, y, z, y', z')\,dx,$$

one of the boundary points, say $B(x_1, y_1, z_1)$ is moved, and the other point $A(x_0, y_0, z_0)$ is fixed (or both boundary points are movable), then it is obvious that an extremum may be achieved only on the integral curves of the system of Euler's equations

$$F_y - \frac{d}{dx}F_{y'} = 0; \qquad F_z - \frac{d}{dx}F_{z'} = 0. \tag{7}$$

In order to find the solution of the system of Euler's equations (7) , we need to satisfy the condition $\delta v = 0$; therefore, similar to the previous case, we get

$$\delta v = [F - y' F_{y'} - z' F_{z'}]_{x=x_1} \delta x_1 + F_{y'} \mid_{x=x_1} \delta y_1 + F_{z'} \mid_{x=x_1} \delta z_1 = 0.$$

If we consider the functional

$$v = \int_{x_0}^{x_1} F\left(x, y_1, y_2, ..., y_n, y_1', y_2', ..., y_n'\right) dx,$$

in the same way, if we consider the moving point $B\left(x_{11}, y_{11}, y_{21}, ..., y_{n1}\right)$, we have the following condition

$$\left(F - \sum_{i=1}^{n} y_i' F_{y_i'}\right) \mid_{x=x_1} \delta x_1 + \sum_{i=1}^{n} F_{y_i'} \mid_{x=x_1} \delta y_{i1} = 0.$$

Example 3. *Minimize* $v = \int_0^{x_1} \left(y'^2 + z'^2 + 2yz\right) dx$, *with* $y(0) = 0$; $z(0) = 0$, *and the point* (x_1, y_1, z_1) *can move over the plane* $x = x_1$.

The necessary condition for the solution of this problem is to satisfy the system of Euler's equations

$$y'' - z = 0; \qquad z'' - y = 0.$$

Using He's variational iteration method, we have the following correctional functionals:

$$y_{n+1}(x) = y_n(x) + \int_0^x \lambda_1(t) \left(y_n''(t) - z_n(t)\right) dt, \tag{8}$$

and,

$$z_{n+1}(x) = z_n(x) + \int_0^x \lambda_2(t) \left(z_n''(t) - y_n(t)\right) dt. \tag{9}$$

Taking the variation of the both sides of the equations (8) *and* (9) *with respect to* y_n, z_n, *respectively, we obtain*

$$\delta y_{n+1}(x) = \delta y_n(x) + \delta \int_0^x \lambda_1(t) \left(y_n''(t) - z_n(t)\right) dt,$$

and,

$$\delta z_{n+1}(x) = \delta z_n(x) + \delta \int_0^x \lambda_2(t) \left(z_n''(t) - y_n(t)\right) dt,$$

using the integration by parts, we get

$$\delta y_{n+1}(x) = \delta y_n(x) + \lambda_1(t) \, \delta y_n'(x) \mid_{t=x} - \lambda_1'(t) \, \delta y_n(x) \mid_{t=x} + \delta \int_0^x \lambda_1''(t) \, \delta y_n(x) dt,$$

and,

$$\delta z_{n+1}(x) = \delta z_n(x) + \lambda_1(t) \, \delta z_n'(x) \mid_{t=x} - \lambda_1'(t) \, \delta z_n(x) \mid_{t=x} + \delta \int_0^x \lambda_1''(t) \, \delta z_n(x) dt,$$

since δy_n, δz_n are arbitrary, $\delta y_{n+1} = 0$, $\delta z_{n+1} = 0$, and by the fundamental Lemma of calculus of variation, we obtain the following stationary conditions:

$$1 - \lambda_1'(t)\mid_{t=x} = 0, \qquad \lambda_1(t)\mid_{t=x} = 0, \qquad \lambda_1''(t) = 0.$$

and,

$$1 - \lambda_2'(t)\mid_{t=x} = 0, \qquad \lambda_2(t)\mid_{t=x} = 0, \qquad \lambda_2''(t) = 0.$$

Then, we obtain

$$\lambda_1(t) = t - x, \qquad \lambda_2(t) = t - x.$$

Therefore, we have

$$y_{n+1}(x) = y_n(x) + \int_0^x (t-x)\left(y_n''(t) - z_n(t)\right) dt,$$

and,

$$z_{n+1}(x) = z_n(x) + \int_0^x (t-x)\left(z_n''(t) - y_n(t)\right) dt.$$

Set,

$$y_0(x) = a\sin(x) + b\cos(x), \qquad z_0(x) = c\sin(x) + d\cos(x),$$

we get,

$$y_2(x) = \frac{1}{6}\left(x^2\left(3b + 3d + (a+c)x\right) + 6b\cos x + 6a\sin x\right),$$
$$z_2(x) = \frac{1}{6}\left(x^2\left(3b + 3d + (a+c)x\right) + 6d\cos x + 6c\sin x\right).$$

Using the conditions $y(0) = 0$, $z(0) = 0$, noting that $b = d = 0$, and the condition at the moving boundary point

$$[F - y'F_{y'} - z'F_{z'}]_{x=x_1}\delta x_1 + F_{y'}\mid_{x=x_1}\delta y_1 + F_{z'}\mid_{x=x_1}\delta z_1 = 0$$

since $\delta x_1 = 0$ and δy_1, δz_1 are arbitrary, then the conditions yield to

$$F_{y'}\mid_{x=x_1} = 0 \qquad \text{and} \qquad F_{z'}\mid_{x=x_1} = 0,$$

hence,

$$y'(x_1) = 0 \qquad \text{and} \qquad z'(x_1) = 0.$$

Imposing these conditions on y_2 and z_2, we obtain

$$y_2'(x) = \frac{x^2}{2}(a+c) + 6a\cos x, \quad z_2'(x) = \frac{x^2}{2}(a+c) + 6c\cos x,$$

if $\cos x_1 \neq 0$, then $y_2(x) = z_2(x) = 0$, which are trivial solutions, but if $\cos x_1 = 0$, then $a = -c$, i.e.,

$$y_2(x) = c\sin x, \quad z_2(x) = -c\sin x,$$

where c is an arbitrary constant, which is also the exact solution.

4. Extremals with Corners

There are many problems in nature involving corner points on their extremals, i.e., maybe the desired function $y = y(x)$ does not have a continuous derivative. For instance problems involving the reflection and refraction of light. Therefore, if we want to find the curve that extremizes the functional $v = \int_{x_0}^{x_2} F(x, y, y')\, dx$ and passes through the given points $a(x_0, y_0)$ and $b(x_2, y_2)$; such that, the curve must arrive at b only after reflected from a given line $y = \varphi(x)$, so at this point of reflection $c(x_1, y_1)$, there exist a corner point of the desired extremal y, therefore, at this point y' is discontinuous.

In order to obtain the extremal, represent the functional v in the form

$$v[y(x)] = \int_{x_0}^{x_1} F(x, y, y')\, dx + \int_{x_1}^{x_2} F(x, y, y')\, dx,$$

here, the derivative y' is assumed to be continuous on the intervals $[x_0, x_1)$ and $(x_1, x_2]$.

The basic necessary condition for an extremum, $\delta v = 0$. Since the point (x_1, y_1) can move along the curve $y = \varphi(x)$, so we are involved in the conditions of the problem with a boundary point moving along a given curve, as in the previous section, It is obvious that the curves ac and cb are extremals. Indeed, on these segments $y = y(x)$ is a solution of Euler's equation [8]. In order to keep the reader in the mode, we construct the following example.

Example 4. *Find the curves that satisfy the shortest distance between $a(0, 1)$ and $b\left(\frac{1}{2}, \frac{3}{2}\right)$, such that the curves must arrive at $b\left(\frac{1}{2}, \frac{3}{2}\right)$ after reflected from $\varphi(x) = x$, at the point (x_1, y_1).*

To get the shortest distance, we need to extremize the functional

$$v[y(x)] = \int_{0}^{x_1} \sqrt{1 + (y'(x))^2}dx + \int_{x_1}^{\frac{1}{2}} \sqrt{1 + (y'(x))^2}dx,$$

Here, we divide the desired extremum curve $y(x)$ into two segments, $k(x)$ and $m(x)$ which define at $[0, x_1]$ and $[x_1, \frac{1}{2}]$, respectively. In order to find these segments, we must solve Euler's equation

$$y''(x) = 0.$$

Using He's variation iteration method, we have

$$k_{n+1}(x) = k_n(x) + \int_{0}^{x} \lambda(s)\, k_n''(s)\, ds,$$

taking the variation with respect to the independent variable k_n, we have

$$\delta k_{n+1}(x) = \delta k_n(x) + \delta \int_{0}^{x} \lambda(s)\, k_n''(s)\, ds.$$

So, we get the stationary conditions

$$1-\lambda'(s)\mid_{s=x}=0, \qquad \lambda(s)\mid_{s=x}=0,$$

$$\lambda''(s)=0.$$

These yield to

$$\lambda(s)=s-x.$$

Therefore, our iteration formula becomes

$$k_{n+1}(x)=k_n(x)+\int_0^x (s-x)\,k_n''(s)\,ds.$$

If we set

$$k_0(x)=ax+b,$$

then, we get the exact solution

$$k(x)=ax+b.$$

Similarly, we can get

$$m(x)=cx+d.$$

Now, to determine these unknown constants, a, b, c, and d, we need to satisfy the necessary conditions, in equation (6), in the previous section

$$\begin{aligned}\delta v &= \left((F-k'F_{k'})\mid_{x=x_1^-}\delta x_1+F_{k'}\mid_{x=x_1^-}\delta y_1\right)\\ &\quad-\left((F-m'F_{m'})\mid_{x=x_1^+}\delta x_1+F_{m'}\mid_{x=x_1^+}\delta y_1\right)\\ &= 0,\end{aligned}$$

Since δx_1 and δy_1 are dependent, (x_1, y_1) move along the curve $\varphi(x)=x$, then $\delta y_1=\delta x_1$. Therefore, we have

$$[F+(1-k')F_{k'}]_{x=x_1^-}\,\delta x_1-[F+(1-m')F_{m'}]_{x=x_1^+}\,\delta x_1=0,$$

since δx_1 is arbitrary, we get

$$[F+(1-k')F_{k'}]_{x=x_1^-}=[F+(1-m')F_{m'}]_{x_1^+},$$

then, we obtain

$$\left[\sqrt{1+k'^2}+(1-k')\frac{k'}{\sqrt{1+k'^2}}\right]_{x=x_1^-}=\left[\sqrt{1+m'^2}+(1-m')\frac{m'}{\sqrt{1+m'^2}}\right]_{x=x_1^+},$$

or,

$$\frac{1+k'}{\sqrt{1+k'^2}}\mid_{x=x_1^-}=\frac{1+m'}{\sqrt{1+m'^2}}\mid_{x=x_1^+}. \tag{10}$$

To simplify equation (10), *we can set the following*

$$k'(x_1) = \tan(\gamma_1), \qquad m'(x_1) = \tan(\gamma_2),$$

then, we can write equation (10) *as*

$$\frac{1+\tan(\gamma_1)}{\sec(\gamma_1)} = \frac{1+\tan(\gamma_2)}{\sec(\gamma_2)},$$

or,

$$\cos(\gamma_1) + \sin(\gamma_1) = \cos(\gamma_2) + \sin(\gamma_2).$$

Since the extremal curve $y(x)$ *is continuous, we obtain*

$$k(x_1) = m(x_1) = \varphi(x_1),$$

together with

$$k(0) = 1, \qquad m\left(\frac{1}{2}\right) = \frac{3}{2}.$$

These conditions are enough to determine the unknown constants

$$a = -.3333, \qquad b = 1, \qquad c = -3, \qquad d = 3, \qquad x_1 = .75.$$

Therefore,

$$k(x) = -.3333x + 1,$$

and

$$m(x) = -3x + 3.$$

5. Conditional Problems

In this section, we discuss variational problems involving conditional extremum, such that the Isoperimetric problems. Then we introduce the relation between the eigenvalue problems and the isoperimetric problems

5.1. Isoperimetric Problems

In this section, we will talk about specific type of variational problems, which involve a conditional extremum. For example, it is required to investigate for an extremum for the functional

$$v[y_1, y_2, ..., y_n] = \int_{x_0}^{x_1} F(x, y_1, y_2, ..., y_n, y_1', y_2', ..., y_n')\, dx,$$

and so-called isoperimetric conditions

$$\int_{x_0}^{x_1} K_i(x, , y_1, y_2, ..., y_n, y_1', y_2', ..., y_n')\, dx = l_i, \qquad (i = 1, 2, ..., m),$$

where l_i are constants, and $m \geq 1$.

Now, to obtain the basic necessary condition in an isoperimetric problem, it is necessary to form the auxiliary functional

$$v^* = \int_{x_0}^{x_1} F^* dx, \quad \text{where} \quad F^* = F + \sum_{i=1}^{m} \lambda_i K_i,$$

where λ_i are constants [8], and then find Euler's equation for it, for more details, we refer the reader to [8].

Example 5. *Find the extremal of the following functional*

$$v[y(x)] = \int_0^1 (y'(x))^2 \, dx,$$

with boundary conditions

$$y(0) = 0, \qquad y(1) = 2.$$

and the isoperimetric condition

$$\int_0^1 y(x) \, dx = 6.$$

The auxiliary functional

$$v^* = \int_0^1 \left((y'(x))^2 + \lambda y(x) \right) dx,$$

then Euler's equation for v^* is

$$\lambda - 2y''(x) = 0.$$

Using He's variational iteration method we have the following correct functional

$$y_{n+1}(x) = y_n(x) + \int_0^x \gamma(s) \left(\lambda - 2y_n''(s) \right) ds,$$

taking variation with respect to the independent variable y_n, we have

$$\delta y_{n+1}(x) = \delta y_n(x) + \delta \int_0^x \gamma(s) \left(\lambda - 2y_n''(s) \right) ds.$$

So, we get the stationary conditions

$$1 + 2\gamma'(s) \mid_{s=x} = 0, \qquad \gamma(s) \mid_{s=x} = 0, \qquad \gamma''(s) = 0.$$

Therefore, we have

$$\gamma(s) = \frac{1}{2}(x - s).$$

Then the iteration formula becomes

$$y_{n+1}(x) = y_n(x) + \int_0^x \frac{1}{2}(x - s) \left(\lambda - 2y_n''(s) \right) ds.$$

If we set

$$y_0 = ax + b,$$

then y_1 *is easy to find. By imposing the boundary conditions on the obtained* y_1*, and using the isoperimetric condition, we have*

$$a = -1, \qquad b = 0, \qquad \lambda = 12.$$

$$y_1(x) = -x + 3x^2.$$

Where the exact solution of this problem is also the same.

Example 6. *Find the extremal in the isoperimetric problem of the extremization of the functional*

$$v[y(x), z(x)] = \int_0^1 \left(y'^2 + z'^2 - 4xz' - 4z\right) dx,$$

given that

$$\begin{aligned} \int_0^1 (y'^2 - z'^2 - xy')dx &= 2, \qquad y(0) = 0, \qquad y(1) = 1, \\ z(0) &= 0, \qquad z(1) = 1. \end{aligned}$$

The auxiliary functional

$$v^* = \int_0^1 \left((1+\lambda) y'^2 + (1-\lambda) z'^2 - 4xz' - 4z - \lambda xy'\right) dx,$$

Now, the system of Euler's differential equations of v^* *is as follows*

$$\begin{aligned} F_y^* - \frac{d}{dx}\left(F_{y'}^*\right) &= -2y''(1+\lambda) + \lambda = 0, \\ F_z^* - \frac{d}{dx}\left(F_{z'}^*\right) &= z'' = 0. \end{aligned}$$

Using He's variational iteration method we have the following correctional functionals:

$$y_{n+1}(x) = y_n(x) + \int_0^x \gamma_1(t)\left(-2y_n''(t)(1+\lambda) + \lambda\right) dt,$$

$$z_{n+1}(x) = z_n(x) + \int_0^x \gamma_2(t)\, z_n''(t)\, dt.$$

Making these correction functionals, we can obtain the following stationary conditions:

$$1 + 2(1+\lambda)\ \gamma_1'(t)\,|_{t=x} = 0, \qquad \gamma_1(t)\,|_{t=x} = 0, \qquad \gamma_1''(t) = 0.$$

$$1 - \gamma_2'(s)\,|_{s=x} = 0, \qquad \gamma_2(s)\,|_{s=x} = 0, \qquad \gamma_2''(s) = 0.$$

These conditions yield

$$\gamma_1(t) = \frac{1}{2(1+\lambda)}(x-t), \qquad \gamma_2(t) = t - x.$$

Therefore, we have

$$y_{n+1}(x) = y_n(x) + \int_0^x \frac{1}{2(1+\lambda)}(x-t)\left(-2y_n''(t)(1+\lambda)+\lambda\right)dt,$$

$$z_{n+1}(x) = z_n(x) + \int_0^x (t-x)\left(z_n''(t)\right)dt.$$

Set

$$y_0(x) = a\sin(x) + b\cos(x), \qquad z_0(x) = c\sin(x) + d\cos(x),$$

y_1 *and* z_1 *are easy to find. Imposing the boundary conditions on* y_1 *and* z_1, *we obtain:*

$$a = 3.5, \quad b = 0, \quad c = 1, \quad d = 0, \quad \lambda = -1.1.$$

Thus

$$y_1(x) = (3.5 - 2.5x)x,$$

$$z_1(x) = x.$$

Which is also the exact solution of this problem.

5.2. Eigenvalues of Isoperimetric

In order to solve the Isoperimetric problem, the idea is in finding differential equation with some conditions, that comes from Euler's equation, these together form an eigenvalue problem, for the eigenvalues λ_i and the solution of it are the eigenfunctions. We should mention that the solutions of the isoperimetric problem exists only for some values of the parameter λ_i. In order to get the point, we construct the following example.

Example 7. *Find the extremals of the following isoperimetric problem*

$$v[y(x)] = \int_0^2 \left(y'(x)\right)^2 dx,$$

subject to

$$y(0) = 0, \quad y(2) = 0, \int_0^2 y^2 dx = 6.$$

The auxiliary functional is given by

$$v^* = \int_0^2 \left(\left(y'(x)\right)^2 + \lambda y^2(x)\right)dx,$$

then Euler's equation for v^* *is*

$$\lambda y(x) - y''(x) = 0. \tag{11}$$

Note that, equation (11) *appears as an eigenvalue problem. Using He's variational iteration method, we have the following correct functional*

$$y_{n+1}(x) = y_n(x) + \int\limits_0^x \gamma(s)\left(\lambda y_n(x) - y_n''(x)\right) ds.$$

Taking variation with respect to the independent variable y_n, we have

$$\delta y_{n+1}(x) = \delta y_n(x) + \delta \int\limits_0^x \gamma(s)\left(\lambda y_n(x) - y_n''(x)\right) ds.$$

So, we will get the stationary conditions

$$1 + \gamma'(s)\,|_{s=x} = 0, \qquad \gamma(s)\,|_{s=x} = 0, \qquad \lambda\gamma(s) - \gamma''(s) = 0.$$

Now, to find $\gamma(s)$, we need to take three cases for λ.

Case 1. $\lambda < 0$, *by solving the above differential equation, we have*

$$\gamma(s) = \frac{-1}{\sqrt{-\lambda}} \sin\left(\sqrt{-\lambda}\,(s-x)\right).$$

Then the iteration formula

$$y_{n+1}(x) = y_n(x) + \int\limits_0^x -\frac{1}{\sqrt{-\lambda}} \sin\left(\sqrt{-\lambda}\,(s-x)\right)\left(\lambda y_n(s) - y_n''(s)\right) ds.$$

If we set

$$y_0 = ax + b,$$

we find,

$$y_1(x) = b\cos\left(\sqrt{-\lambda}x\right) + \frac{a\sin\left(\sqrt{-\lambda}x\right)}{\sqrt{-\lambda}},$$

By imposing the boundary conditions on the obtained y_1, we have

$$b = 0, \qquad \lambda = -\left(\frac{n\pi}{2}\right)^2, \qquad \textit{where } n = 1, 2, ...,$$

therefore,

$$y_1(x) = \frac{2a\sin\left(\frac{n\pi}{2}x\right)}{n\pi}.$$

Now imposing the Isoperimetric condition, we get $a = \pm\pi\sqrt{6}$,

so

$$y_1(x) = \frac{\pm\sqrt{6}2\sin\left(\frac{n\pi}{2}x\right)}{n}.$$

Case 2. $\lambda > 0$, *then we have*

$$\gamma(s) = \frac{-e^{\sqrt{\lambda}(s-x)} + e^{\sqrt{\lambda}(x-s)}}{2\sqrt{m}}.$$

Then the iteration formula

$$y_{n+1}(x) = y_n(x) + \int_0^x \left(\frac{-e^{\sqrt{\lambda}(s-x)} + e^{\sqrt{\lambda}(x-s)}}{2\sqrt{\lambda}} \right) \left(\lambda y_n(s) - y_n''(s) \right) ds.$$

If we set $y_0 = ax + b$, *then we find*

$$y_1(x) = b\cosh\left(\sqrt{\lambda}x\right) + \frac{a\sinh\left(\sqrt{\lambda}x\right)}{\sqrt{\lambda}},$$

By imposing the boundary conditions on y_1, *then we have the trivial solution* $y_1(x) = 0$, *but that does not satisfy the Isoperimetric condition, i.e., when* $\lambda > 0$, *we do not have solution.*

Case 3. $\lambda = 0$, *we have*

$$\delta y_{n+1}(x) = \delta y_n(x) + \delta \int_0^x \gamma(s) \left(-y_n''(x) \right) ds.$$

So, we will get the stationary conditions

$$1 + \gamma'(s)\,|_{s=x} = 0, \qquad \gamma(s)\,|_{s=x} = 0, \qquad \gamma''(s) = 0.$$

$$\gamma(s) = \frac{1}{2}(x - s).$$

Then the iteration formula becomes

$$y_{n+1}(x) = y_n(x) + \int_0^x \left(\frac{1}{2}(x - s) \right) \left(-y_n''(s) \right) ds.$$

If, we set $y_0 = ax + b$, *then we find*

$$y_1(x) = ax + b,$$

By imposing the boundary conditions on y_1, *then we also have the trivial solution* $y_1(x) = 0$, *therefore, for* $\lambda = 0$, *we do not have solution.*

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