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A NEW SEQUENCE SPACE ISOMORPHIC TO THE SPACE $\ell(p)$ AND COMPACT OPERATORS

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Abstract. As a major issue in this work, we present the paranormed sequence space $\ell(u,v,p;\widetilde{B})$ consisting of all sequences whose $\widetilde{R}$-transforms are in the linear space $\ell(p)$ introduced by Maddox [Quart. J. Math. Oxford 18 (1967), 345-355], where $\widetilde{B}=B(\widetilde{r},\widetilde{s})$ denotes double sequential band matrix provided that $(r_n)_{n=0}^{\infty}$ and $(s_n)_{n=0}^{\infty}$ are given convergent sequences of positive real numbers. For this purpose, we have used the generalized weighted mean G and double sequential band matrix $\widetilde{B}$. Meanwhile, we have also presented the basis of this space and computed its α-, β- and γ-duals. Then, we have characterized the classes of matrix mappings from $\ell(u,v,p;\widetilde{B})$ to ℓ_∞, c and c_0. In conclusion, in order to characterize some classes of compact operators given by matrices on the space $\ell_p(u,v,\widetilde{B})$ $(1\leq p<\infty)$, we have applied the Hausdorff measure of noncompactness.

Keywords: paranormed sequence space; $\widetilde{B}=B(r_n,s_n)$ double sequential band matrix; weighted mean; α-, β- and γ-duals; Matrix mappings; Hausdorff measure of noncompactness; compact operators.

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1. Preliminaries-background

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Most of the concepts dealt with in this paper will be presented via infinite sequences or series and their certain properties. We know that the sequences of real numbers appearing nearly everywhere in summability theory play an important role. Because of these reasons, here is briefly given the basic properties of them. There are many ways to introduce a sequence, but here we have preferred to give the definition as: Let X be a set. A sequence in X is simply a function from $\mathbb{N}$ to X, i.e., $\varphi : \mathbb{N} \to X$ is a sequence, we write also $(x_n)_{n\in\mathbb{N}}$ for φ, where $x_n := \varphi(n)$ is the n^{th} term of the sequence $\varphi = (x_0, x_1, x_2, \ldots)$. Sequences in $\mathbb{K}$ are called number sequences, and the $\mathbb{K}$-vector space $\mathbb{K}^{\mathbb{N}}$ of all number sequences is denoted by ω or $\omega(\mathbb{K})$. More precisely, one says (x_n) is a real or complex sequence if $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, where $\mathbb{K}$ denotes either of fields $\mathbb{R}$ and $\mathbb{C}$. For $m \in \mathbb{N}^{\times}$, a function $\varphi : m + \mathbb{N} \to X$ is also called a sequence in X. That is, $(x_j)_{j\geq m} = (x_m, x_{m+1}, x_{m+2}, \ldots)$ is a sequence in X even though the indexing does not start with 0. This convention is justified, since after 're-indexing' using the function $\mathbb{N} \to m + \mathbb{N}$, $n \longmapsto m + n$, the shifted sequence $(x_j)_{j\geq m}$ can be identified with the unusual sequence $(x_{m+k})_{k\in\mathbb{N}} \in X^{\mathbb{N}}$. A sequence (x_n) converges to limit a if each neighborhood of a contain almost all terms of the sequence. In this case we say that (x_n) converges to a as n goes to ∞. We denote by c, the set of all convergent sequences in $\mathbb{K}$. A sequence (x_n) in $\mathbb{K}$ is called a null sequence if it converges to zero. The set of all null sequences in $\mathbb{K}$ we denote by c_0. A sequence is bounded if the set of its terms have an upper bound and a lower bound. The set of all bounded sequences is denoted by ℓ_∞. Any vector subspace of $\omega = \omega(\mathbb{K}) = \mathbb{K}^{\mathbb{N}}$ is known as a sequence space. It is clear that the sets c, c_0 and ℓ_∞ are the subspaces of the ω. Therefore, c, c_0 and ℓ_∞, equipped with a vector space structure, forms a sequence space. Also by bs, cs, ℓ_1 and ℓ_p we denote the spaces of all bounded, convergent, absolutely and p-absolutely convergent series, respectively.

In this paragraph, we shall introduce the notion of a matrix transformation from X to Y. Let X, Y be any two sequence spaces. Given any infinite matrix $A = (a_{nk})$ of real numbers a_{nk}, where $n,\ k \in \mathbb{N}$, any sequence x, we write $Ax = \big((Ax)_n\big)$, the A-transform of x, if $(Ax)_n = \sum_k a_{nk}x_k$ converges for each $n \in \mathbb{N}$. For simplicity in notation, here and in what follows, the summation without limits runs from 0 to ∞. If $x \in X$ implies that $Ax \in Y$ then we say that A defines a matrix mapping from X into Y and denote it by $A : X \to Y$. By $(X : Y)$, we mean the class of all infinite matrices such that $A : X \to Y$.

The X_A is said that matrix domain of an infinite matrix A for any subspace X of the all real-valued sequence space $w(\mathbb{R})$ is described as

$$X_A = \{x = (x_k) \in w : Ax \in X\}. \tag{1.1}$$

The new sequence space X_A generated by the limitation matrix A from the space X either includes the space X or is included by the space X, in general, i.e., the space X_A is the expansion or the contraction of the original space X.

From now on, let's assume that (p_k) be a bounded sequence of strictly positive real numbers with $\sup p_k = H$ and $M = \max\ \{1,\ H\}$ and $1/p_k + 1/p_k^{'} = 1$ provided $1 < inf\, p_k \leq H < \infty$. Then, the linear spaces $\ell_\infty(p)$, $c(p)$, $c_0(p)$ and $\ell(p)$ were defined by Maddox in [1] and [2] (see also Simons [3] and Nakano [4]) as follows:

$$\ell(p) = \left\{x = (x_k) \in w : \sum_k |x_k|^{p_k} < \infty\right\}$$

and

$$\ell_\infty(p) = \left\{x = (x_k) \in w : \sup_{k\in\mathbb{N}} |x_k|^{p_k} < \infty\right\},$$

which are the complete spaces paranormed by

$$g_1(x) = \left(\sum_k |x_k|^{p_k}\right)^{1/M} \quad \text{and} \quad g_2(x) = \sup_{k\in\mathbb{N}} |x_k|^{p_k/M} \ \text{ iff } \ \inf p_k > 0,$$

respectively.

For the sake of simplicity, here and in what follows, it will be assumed that the summation without limits runs from 0 to ∞.

In recent years, the approach to construct a new sequence space by means of the matrix domain of a particular triangle (An infinite matrix $A = (a_{nk})$ is called a triangle $a_{nk} = 0$ for $k > n$ and $a_{nn} = 0$ for all $n \in \mathbb{N}$.) has been used by some of the writers in many research articles. They defined and examined the sequence spaces $X_p = (\ell_p)_{C_1}$ in [5], $r^t(p) = (\ell(p))_{R_t}$ in [6], $e^r_p = (\ell_p)_{E^r}$ and $e^r(p) = (\ell(p))_{E^r}$ in [7-9]. $Z(u,v,\ell_p) = (\ell_p)_{G(u,v)}$ and $\ell(u,v,p) = (\ell(p))_{G(u,v)}$ in [10-11], $a^r(p) = (\ell_p)_{A^r}$ and $a^r(u,p) = (\ell(p))_{A^r_u}$ in [12-13], $bv_p = (\ell_p)_\Delta$ and $bv(u,p) = (\ell(p))_{A_u}$

in [14-16], $\overline{\ell(p)} = (\ell(p))_S$ in [17], $\ell_p^\lambda = (\ell_p)_\Lambda$ in [18], $\lambda_{B(r,s)}$ in [19] $\lambda_{B(\tilde{r},\tilde{s})}$ in [20], $f_0(B)$ and $f(B)$ in [21], $f_0(\widetilde{B})$ and $f(\widetilde{B})$ in [22] and etc., where $C_1 = \{c_{nk}\}$, $R^t = \{r_{nk}^t\}$, $E^r = \{e_{nk}^r\}$, $S = \{s_{nk}\}$, $\Delta = \{\delta_{nk}\}$, $G(u,v) = \{g_{nk}\}$, $\Delta^{(m)} = \{\Delta_{nk}^{(m)}\}$, $A^r = \{a_{nk}^r\}$, $A_u^r = \{a_{nk}(r)\}$, $A^u = \{a_{nk}^u\}$, $B(r,s) = \{b_{nk}(r,s)\}$, $B(\tilde{r},\tilde{s}) = \{b_{nk}(\tilde{r},\tilde{s})\}$, $\Lambda = \{\lambda_{nk}\}_{n,k=0}^\infty$ and $A(\lambda) = \{a_{nk}(\lambda)\}$ denote the Cesàro, Riesz, Euler, generalized weighted means or factorable matrix, summation matrix, difference matrix, generalized difference matrix and sequential band matrix, respectively. To write in a more clear way,

$$c_{nk} := \begin{cases} \frac{1}{n+1} & , \quad (0 \le k \le n), \\ 0 & , \quad (k > n). \end{cases} \qquad r_{nk}^t := \begin{cases} \frac{t_k}{T_n} & , \quad (0 \le k \le n), \\ 0 & , \quad (k > n). \end{cases}$$

$$e_{nk}^r := \begin{cases} \binom{n}{k}(1-r)^{n-k}r^k & , \quad (0 \le k \le n), \\ 0 & , \quad (k > n). \end{cases}$$

$$s_{nk} := \begin{cases} 1 & , \quad (0 \le k \le n), \\ 0 & , \quad (k > n). \end{cases} \qquad \delta_{nk} := \begin{cases} (-1)^{n-k} & , \quad (n-1 \le k \le n), \\ 0 & , \quad (0 \le k < n-1 \text{ or } k > n). \end{cases}$$

$$g_{nk} := \begin{cases} u_n v_k & , \quad (0 \le k \le n), \\ 0 & , \quad (k > n). \end{cases} \qquad \Delta_{nk}^{(m)} := \begin{cases} (-1)^{n-k}\binom{m}{n-k} & , \quad (\max\{0, n-m\} \le k \le n), \\ 0 & , \quad (0 \le k < \max\{0, n-m\} \text{ or } k > n). \end{cases}$$

$$a_{nk}^r := \begin{cases} \frac{1+r^k}{n+1} & , \quad (0 \le k \le n), \\ 0 & , \quad (k > n). \end{cases}$$

$$a_{nk}(r) := \begin{cases} \frac{1+r^k}{n+1}u_k & , \quad (0 \le k \le n), \\ 0 & , \quad (k > n). \end{cases} \quad \text{and} \quad a_{nk}^u := \begin{cases} (-1)^{n-k}u_k & , \quad (n-1 \le k \le n), \\ 0 & , \quad (0 \le k < n-1 \;\text{ or }\; k > n). \end{cases}$$

$$b_{nk}(r,s) := \begin{cases} r & , \quad (k = n), \\ s & , \quad (k = n-1), \\ 0 & , \quad (0 \le k < n-1 \;\text{ or }\; k > n). \end{cases} \qquad b_{nk}(\widetilde{r},\widetilde{s}) := \begin{cases} r_n & , \quad (k = n), \\ s_n & , \quad (k = n-1), \\ 0 & , \quad (0 \le k < n-1 \;\text{ or }\; k > n). \end{cases}$$

$$\lambda_{nk} := \begin{cases} \frac{\lambda_k - \lambda_{k-1}}{\lambda_n} & , \quad (0 \le k \le n), \\ 0 & , \quad (k > n). \end{cases} \qquad a_{nk}(\lambda) := \begin{cases} \frac{\lambda_k - 2\lambda_{k-1} + \lambda_{k-2}}{\lambda_n \lambda_{n-1}} & , \quad (0 \le k \le n), \\ 0 & , \quad (\, k > n). \end{cases}$$

for all $k, m, n \in \mathbb{N}$, $r, s \in \mathbb{R} \setminus \{0\}$, where u_n depends only on n and v_k only on k, $(r_n)_{n=0}^\infty$ and $(s_n)_{n=0}^\infty$ are given convergent sequences of positive real numbers. The approach to construct a new sequence space by means of the matrix domain of a particular limitation method has recently been employed. For instance, see [23-28]. For more detail on the domains of some triangle matrices in certain sequence spaces, the reader may refer to Başar (see [29, p. 50]). Moreover, the ones who are more interested in the subject are advised to read [30-36].

We now recall some definitions and facts which will be frequently used. The function g on X satisfies the properties of a paranorm

i) $g(\theta)=0$,

ii) $g(x)=g(-x)$,

iii) $g(x+y)=g(x)+g(y)$,

iv) $|\alpha_n-\alpha|\to 0$ and $g(x_n-x)\to 0$ imply $g(\alpha_n x_n-\alpha x)\to 0$,

for all $\alpha\in\mathbb{R}$ and all $x\in X$, where θ is the zero vector in the linear space X. Recall that a linear topological space X over the real field $\mathbb{R}$ with a paranorm obeying these rules (i-iv) is called a paranormed space. In addition to this, by notation $\mathscr{F}_r$ $(r\in\mathbb{N})$, we denote the subcollection of $\mathscr{F}$ consisting of all nonempty and finite subsets of $\mathbb{N}$ with elements that are greater than r, namely

$$\mathscr{F}_r=\{N\in\mathscr{F} : n>r \text{ for all } n\in N\}\ (r\in\mathbb{N}).$$

As well known, we call a sequence space X with a linear topology a K-space iff each of the maps $p_n : X\to\mathbb{R}$ defined by $p_n(x)=x_n$ is continuous for all $n\in\mathbb{N}$. A K-space X is called an FK-space iff X is a complete linear metric space. On the other words; we can say that an FK-space is a complete total paranormed space. An FK-space whose topology is normable is called a BK-space, so a BK-space is a normed FK-space. The space ℓ_p $(1\le p<\infty)$ is a BK-space with $\|x\|_p=(\sum_k|x_k|^p)^{\frac{1}{p}}$ and c_0, c and ℓ_∞ are BK-space with $\|x\|_\infty=sup_k|x_k|$. An FK-space X is said to have AK property, if $\phi\subset X$ and $\{e^{(k)}\}$ is a basis for X, where e^k is a sequence whose only non-zero term is a 1 in kth place for each $k\in\mathbb{N}$ and $\phi=span\{e^k\}$, the set of all finitely non-zero sequences. If ϕ is dense in X, then X is called an AD-space, thus AK implies AD. We know that the spaces c_0, cs and ℓ_p are AK-spaces, where $1\le p<\infty$.

The brief structure of this paper is as follows:

This text consists of five main section. Many of the concepts in the first chapter contains essential fundamental definitions, results and terminological materials of which we shall make frequent use later. At the end of the Section 1; it is given some notations and basic definitions including FK-space, paranormed space and others. The sequence spaces $\ell(u,v,p;\widetilde{B})$ and $\ell_p(u,v;\widetilde{B})$ have been introduced in Section 2 and it is proved linearly isomorphic of the sequence spaces $\ell(u,v,p;\widetilde{B})$ and $\ell(p)$. In Section 3, we determine the α-, β- and γ-duals of the space $\ell(u,v,p;\widetilde{B})$ which are used to find the necessary and sufficient conditions for matrix transformations. In Section 4, we characterize the matrix classes $(\ell(u,v,p;\widetilde{B}),\ell_\infty)$, $(\ell(u,v,p;\widetilde{B}),c)$ and $(\ell(u,v,p;\widetilde{B}),c_0)$. In the final section of the paper, we present the characterizations of some classes of compact operators given by infinite matrices from $\ell_p(u,v;\widetilde{B})$ to c_0, c, ℓ_∞ and ℓ_1. In addition to this, we give the necessary and sufficient conditions for $(\ell_1(u,v;\widetilde{B}),\ell_p)$ to be compact, where $1\leq p<\infty$.

2. The Paranormed Sequence Space $\ell(u,v,p\,;\widetilde{B})$

In this section, we are interested in the new sequence space $\ell(u,v,p;\widetilde{B})$ obtained by generalized weighted mean $G(u,v)$ and sequential band matrix $B(\widetilde{r},\widetilde{s})$.

The set of all sequences u such that $u_k\neq 0$ for all $k\in\mathbb{N}$ will be denoted by $\mathscr{U}$. For $u\in\mathscr{U}$, let $1/u=(1/u_k)$. Let's assume that $u,v\in\mathscr{U}$ and define the matrices $G(u,v)=\{g_{nk}\}$ and $B(\tilde{r},\tilde{s})=\{b_{nk}(\tilde{r},\tilde{s})\}$. These matrices are as clearly defined in the first chapter above.

From now on, let's assume that $u=(u_k),\ v=(v_k)\in\mathscr{U}$ and $(r_n)_{n=0}^{\infty}$ and $(s_n)_{n=0}^{\infty}$ are given convergent sequences of positive real numbers. For the sake of simplicity, we shall write

$$\widetilde{R}=R(u,\ v,\ \widetilde{B})=G(u,\ v).\,B(\widetilde{r},\ \widetilde{s})$$

and

$$\widetilde{\Delta}(j,k)=\frac{1}{r_j}\left[\frac{1}{v_k}\prod_{i=k}^{j-1}(-\frac{s_i}{r_i})-\frac{1}{v_{k+1}}\prod_{i=k+1}^{j-1}(-\frac{s_i}{r_i})\right], \tag{2.1}$$

where; $j,k\in\mathbb{N}$.

Let's define the sequence space $\ell(u,v,p\,;\widetilde{B})$ by

$$\ell(u,v,p\,;\widetilde{B})=\{x=(x_k)\in w: y=(y_k)\in\ell(p)\},$$

where the frequently used sequence $y=(y_k)$ by the $\widetilde{R}=R(u,v,\tilde{B})$-transform of any given sequence $x=(x_k)$, that is,

$$y_0=r_0u_0v_0 \text{ and } y_k=u_k\left[\sum_{j=0}^{k-1}(r_jv_j+s_jv_{j+1})x_j+r_kv_kx_k\right] \text{ for } k\geq 1. \tag{2.2}$$

If $p_k=p$ $(1\leq p<\infty)$ for every $k\in\mathbb{N}$, then we substitute $\ell_p(u,v\,;\widetilde{B})$ in place of $\ell(u,v,p\,;\widetilde{B})$. With the notation of (1.1), we can redefine the spaces $\ell(u,v,p\,;\widetilde{B})$ and $\ell_p(u,v\,;\widetilde{B})$ by

$$\ell(u,v,p\,;\widetilde{B})=(\ell(p))_{\widetilde{R}} \text{ and } \ell_p(u,v\,;\widetilde{B})=(\ell_p)_{\widetilde{R}}.$$

We should state here that the matrix $B(\widetilde{r},\widetilde{s})$ can be reduced to the generalized difference matrix $B(r,s)$ in case $r_n=r$ and $s_n=s$ for all $n\in\mathbb{N}$. So, the results related to the space $\ell(u,v,p\,;\widetilde{B})$ are more general and more inclusive than the corresponding consequences of the space $\ell(u,v,p\,;B)$ more recently defined by Başarır and Kara [37].

It is remarkable that the sequences $x=(x_k)$ and $y=(y_k)$ are connected by the relation (2,2) everywhere in the paper.

Theorem 2.1. *The set $\ell(u,v,p\,;\widetilde{B})$ is linear space.*

Proof. This well-known result can easily be shown with an elementary way.

Theorem 2.2. *(i) The space stated above $\ell(u,v,p\,;\widetilde{B})$ is the complete linear metric space paranormed by h, described with the following equality*

$$h(x)=\left(\sum_k\left|\sum_{j=0}^{k-1}u_k(r_jv_j+s_jv_{j+1})x_j+r_ku_kv_kx_k\right|^{p_k}\right)^{1/M}.$$

(ii) $\ell_p(u,v\,;\widetilde{B})$ is the BK-space with the following norm

$$\|x\|_{l_p(u,v;\widetilde{B})}=\|y\|_{l_p}$$

when $1\leq p<\infty$.

Proof. Since they are similar to each other, we will only prove part(i) and leave part(ii) to reader. We use a standard type procedure in proof of the first part of Theorem 2.2. The following inequalities satisfied for $z,\ x\in\ell(u,v,p\,;\widetilde{B})$ show linearity of $\ell(u,v,p\,;\widetilde{B})$ with respect to the coordinatewise addition and scalar multiplication (see [38, p. 30])

$$\left(\sum_k\left|\sum_{j=0}^{k-1}u_k(r_jv_j+s_jv_{j+1})(z_j+x_j)+r_ku_kv_k(z_k+x_k)\right|^{p_k}\right)^{1/M}$$
$$\leq\left(\sum_k\left|\sum_{j=0}^{k-1}u_k(r_jv_j+s_jv_{j+1})z_j+r_ku_kv_kx_k\right|\right)^{1/M}+\left(\sum_k\left|\sum_{j=0}^{k-1}u_k(r_jv_j+s_jv_{j+1})x_j+r_ku_kv_kx_k\right|\right)^{1/M} \tag{2.3}$$

for any $\alpha\in\mathbb{R}$ (see [1])

$$\left|\alpha\right|^{p_k}\leq\max\{1,\ |\alpha|^M\}. \tag{2.4}$$

The reader can obviously see that $h(\theta)=0$ and $h(x)=h(-x)$ for all $x\in\ell(u,v,p\,;\widetilde{B})$. One more time, the inequalities (2.3) and (2.4) result in the subadditivity of h and

$$h(\alpha x)\leq\max\{1,\ |\alpha|\}h(x).$$

Let's suppose that $\{x^n\}$ be any sequence of the points lying in $\ell(u,v,p\,;\widetilde{B})$ such that $h(x^n-x)\to 0$ and (α_k) also be any sequence of scalars such that $\alpha_n\to\alpha$. Thus, since the inequality

$$h(x^n)\leq h(x)+h(x^n-x)$$

holds by subadditivity of h, $\{h(x^n)\}$ is bounded and therefore we obtain

$$\begin{aligned}h(\alpha_nx^n-\alpha x)&=\left(\sum_k\left|\sum_{j=0}^{k-1}u_k(r_jv_j+s_jv_{j+1})(\alpha_nx_j^{(n)}-\alpha x_j)+r_ku_kv_k(\alpha_nx_k^{(n)}-\alpha x_k)\right|\right)^{1/M}\\&\leq|\alpha_n-\alpha|h(x^n)+|\alpha|h(x^n-x),\end{aligned}$$

which tends to be zero when $n\to\infty$. This means that the scalar multiplication is continuous. As a conclusions, h is a paranorm on the space $\ell(u,v,p\,;\widetilde{B})$.

Now, if we prove the completeness of the space $\ell(u,v,p\,;\widetilde{B})$ then the proof ends. Let's assume that $\{x^i\}$ be any Cauchy sequence in the space $\ell(u,v,p\,;\widetilde{B})$, where $x^i=\{x_0^{(i)},\ x_1^{(i)},\ x_2^{(i)},\ \ldots\}$. In that case, there exists a positive integer $n_0(\varepsilon)$

$$h(x^i-x^j)<\varepsilon \tag{2.5}$$

for all $i,\ j\geq n_0(\varepsilon)$ for any given $\varepsilon>0$. By using definition of h, for each fixed $k\in\mathbb{N}$ we get

$$\left|\left\{\widetilde{R}x^i\right\}_k-\left\{\widetilde{R}x^j\right\}_k\right|\leq\left(\sum_k\left|\left\{\widetilde{R}x^i\right\}_k-\left\{\widetilde{R}x^j\right\}_k\right|^{p_k}\right)^{1/M}<\varepsilon\ (i,\ j\geq n_0(\varepsilon)). \tag{2.6}$$

This newly obtained formula results in the fact that $\{(\widetilde{R}x^0)_k,\ (\widetilde{R}x^1)_k,\ \widetilde{R}x^2)_k,\ \ldots\}$ is a Cauchy sequence of real numbers for every fixed $k \in \mathbb{N}$. Since $\mathbb{R}$ is complete, it converges, say $(\widetilde{R}x^i)_k \to (\widetilde{R}x)_k$ as $i \to \infty$. Using these infinitely many limits $(\widetilde{R}x)_0,\ (\widetilde{R}x)_1,\ (\widetilde{R}x)_2,\ \ldots$, we define the sequence $\{(\widetilde{R}x)_0,\ (\widetilde{R}x)_1,\ (\widetilde{R}x)_2,\ \ldots\}$. We have from (2.6) for each $m \in \mathbb{N}$ and $i,\ j \geq n_0(\varepsilon)$ that

$$\sum_{k=0}^{m} \left| \left(\widetilde{R}x^i\right)_k - \left(\widetilde{R}x\right)_k \right| \leq (h(x^i - x^j))^M < \varepsilon^M. \tag{2.7}$$

Let's take any $i \geq n_0(\varepsilon)$. By passing to limit first as $j \to \infty$ and next as $m \to \infty$ in (2.7), we obtain $h(x^i - x) \leq \varepsilon$. Finally, if we take $\varepsilon = 1$ in (2.7) and $i \geq n_0(1)$ then using Minkowski's inequality for each $m \in \mathbb{N}$, we easily obtain

$$\left(\sum_{k=0}^{m} |(\widetilde{R}x)_k|^{p_k} \right)^{1/M} \leq h(x^i - x) + h(x^i) \leq 1 + h(x^i).$$

This shows that the sequence $\widetilde{R}x$ belongs to the space $\ell(p)$. Due to the fact that $h(x^i - x) \leq \varepsilon$ for all $i \geq n_0(\varepsilon)$ it results in $x^i \to x$ as $i \to \infty$. As $\{x^i\}$ was an arbitrary Cauchy sequence, it follows that the space $\ell(u,v,p;\widetilde{B})$ is complete. This, in fact, concludes the proof.

Here, let us give the definition of the isomorphism. A bijective linear transformation $\tau : X \to Y$ is called an isomorphism from X to Y. When an isomorphism from Y to X exist, we say that X to Y are isomorphic and write $X \approx Y$.

Theorem 2.3. *When $0 < p_k \leq H < \infty$, $\ell(u,v,p;\widetilde{B})$ is linearly isomorphic to $\ell(p)$.*

Proof. According to the definition of the linear isomorphism, we must ensure the existence of a linear bijection between the spaces $\ell(u,v,p;\widetilde{B})$ and $\ell(p)$ for $1 \leq p_k \leq H < \infty$. Consider the transformation τ from $\ell(u,v,p;\widetilde{B})$ to $\ell(p)$ is defined by $x \mapsto y = \tau(x) = G(u,v).B(\widetilde{r},\widetilde{s}).x$, using the notation of (2.2). Showing the linearity of τ is fairly easy. Moreover, it is clear that $x = \theta$ whenever $\tau x = \theta$, which indicates that τ is injective.

Let $y \in \ell(p)$ and define the sequence $x = (x_k)$ by

$$x_k = \sum_{j=0}^{k-1} \frac{1}{u_j} \widetilde{\Delta}(k,j) y_j + \frac{1}{r_k u_k v_k} y_k \ ;\ (k \in \mathbb{N}).$$

Then, the following can be easily obtained

$$h(x) = \left(\sum_k \left| \sum_{j=0}^{k-1} u_k (r_j v_j + s_j v_{j+1}) x_j + r_k u_k v_k x_k \right|^{p_k} \right)^{1/M} = \left(\sum_k |y_k|^{p_k} \right)^{1/M} = g_1(y) < \infty.$$

Thus, we have that $\ell(u,v,p;\widetilde{B})$ and consequently τ is surjective and is paranorm preserving. Therefore, it is concluded that τ is a linear bijection and this clearly states that the spaces $\ell(u,v,p;\widetilde{B})$ and $\ell(p)$ are linearly isomorphic. This conclusion is what was sought for.

The concept of convergence of a series can be used to define a basis as follows. Let (X,h) be a paranormed space. Then the sequence (e_k) in X is called a Schauder basis for X if for every $x \in X$ there exists a unique sequence of scalars (α_k) such that

$$\lim_{n\to\infty} h\left(x - \sum_{k=0}^{n} \alpha_k e_k\right) = 0.$$

In this case, the series $\sum_{k=1}^{\infty} \alpha_k e_k$ which has the sum x is then called the expansion of x with respect to (e_n), and we write $x = \sum_k \alpha_k e_k$.

Now, since the transformation τ defined from $\ell(u,v,p;\widetilde{B})$ to $\ell(p)$ in the proof of Theorem 2.3 is an isomorphism, the inverse image of basis of the space $\ell(p)$ is the basis for the new space $\ell(u,v,p;\widetilde{B})$. Therefore, following theorem is deduced.

Theorem 2.4. *Let $\alpha_k = \{\widetilde{R}x\}_k$ and $0 < p_k \le H < \infty$ for all $k \in \mathbb{N}$. The sequence $c^{(k)} = \{c_n^{(k)}\}_{n\in\mathbb{N}}$ is given by the following piecewise function*

$$c_n^{(k)} = \begin{cases} \frac{\widetilde{\Delta}(n,k)}{u_k} & , \quad (0 \le n \le k-1), \\ \frac{1}{r_k u_k v_k} & , \quad (n = k), \\ 0 & , \quad (n > k), \end{cases} \tag{2.8}$$

for every fixed $k \in \mathbb{N}$. In that case, the sequence $\{c^{(k)}\}_{k\in\mathbb{N}}$ is a basis for the sequence $\ell(u,v,p;\widetilde{B})$ and any x in $\ell(u,v,p;\widetilde{B})$ has a unique representation of the form

$$x = \sum_k \alpha_k c^{(k)}.$$

3. The α-, β- and γ-duals of the space $\ell(u,v,p\,;\widetilde{B})$

Before giving the alpha-, beta- and gamma-duals of the sequence space $\ell(u,v,p\,;\widetilde{B})$, we require define the concept of the multiplier space. The set $M(X,\,Y)$ defined as follows is known the multiplier space of any sequence spaces X and Y.

$$M(X,\,Y)=\{z=(z_k)\in w : xz=(x_kz_k)\in Y \text{ for all } x\in X\}. \tag{3.1}$$

It can be observed for a sequence space Z with $Y\subset Z$ and $Z\subset X$ that the inclusions $M(X,\,Y)\subset M(X,\,Z)$ and $M(X,\,Y)\subset M(Z,\,Y)$ hold, respectively. With the notation of (3.1), the alpha-, beta- and gamma-duals of a sequence space X, which are respectively denoted by X^{α}, X^{β} and X^{γ} are defined by

$$X^{\alpha}=M(X,\,\ell_1),\; X^{\beta}=M(X,\,cs) \quad \text{and} \quad X^{\gamma}=M(X,\,bs).$$

It is obvious that $X^{\alpha}\subset X^{\beta}\subset X^{\gamma}$. Also it can be seen that the inclusions $X^{\alpha}\subset Y^{\alpha}$, $X^{\beta}\subset Y^{\beta}$ and $X^{\gamma}\subset Y^{\gamma}$ hold whenever $Y\subset X$.

The results of the following lemmas will be used in the proofs of our theorems.

Lemma 3.1. [39] *(i) Let's assume that $1<p_k\le H<\infty$ for every given k. In that case, $A\in(\ell(p),\ell_1)$ iff there exists an integer $K>1$ provided that*

$$\sup_{N\in\mathscr{F}}\sum_k\left|\sum_{N\in N}a_{nk}K^{-1}\right|^{p_k'}<\infty. \tag{3.2}$$

(ii) Again, let's assume that $0<p_k\le 1$ for every given k. In that case, $A\in(\ell(p),\ell_1)$ iff the following inequalities is satisfied

$$\sup_{N\in\mathscr{F}}\sup_{k\in\mathbb{N}}\left|\sum_{n\in N}a_{nk}\right|^{p_k}<\infty. \tag{3.3}$$

Lemma 3.2. [40] *(i) $A\in(\ell(p),\ell_\infty)$ iff there exists an integer $K>1$ provided that*

$$\sup_{n\in\mathbb{N}}\sum_k\left|a_{nk}K^{-1}\right|^{p_k'}<\infty, \tag{3.4}$$

where $1<p_k\le H<\infty$ for every given k.

(ii) $A\in(\ell(p),\ell_\infty)$ iff

$$\sup_{n,k\in\mathbb{N}}|a_{nk}|^{p_k}<\infty, \tag{3.5}$$

where $0<p_k\le 1$ for every given k.

Lemma 3.3. [40] *$A\in(\ell(p),c)$ iff there exists an integer $K>1$ provided that (3.4) and (3.5) hold,*

$$\lim_{n\to\infty} a_{nk} = \alpha_k (k \in \mathbb{N}) \tag{3.6}$$

also holds, where $0 < p_k \leq H < \infty$ *for every given k.*

Theorem 3.4. *It is assume that* $N_k^* = N \cap \{n \in \mathbb{N} : n \geq k\}$ *for* $n \in \mathscr{F}$ *and the sets* $d_1(p)$ *and* $d_2(p)$ *are described with the following equations.*

$$d_1(p) = \left\{ a = (a_n) \in w : \sup_{n\in\mathscr{F}} \sup_{k\in\mathbb{N}} \left| \sum_{n\in N_k^*} d_{nk} \right|^{p_k} < \infty \right\}$$

and

$$d_2(p) = \bigcup_{K>1} \left\{ a = (a_n) \in w : \sup_{n\in\mathscr{F}} \sum_k \left| \sum_{n\in N_k^*} d_{nk} K^{-1} \right|^{p_k'} < \infty \right\},$$

here, the matrix $D = (D_{nk})$ *is given by the following piecewise function*

$$d_{nk} = \begin{cases} \frac{\widetilde{\Delta}(n,k)}{u_k} a_n & , \ (0 \leq k \leq n-1), \\ \frac{a_n}{r_n u_n v_n} & , \ (k = n), \\ 0 & , \ (\ k > n). \end{cases}$$

(i) *It is true that* $\{\ell(u,v,p\,;\widetilde{B})\}^\alpha = d_1(p)$ *in case that* $0 < p_k \leq 1$ *for all* $k \in \mathbb{N}$.

(ii) *Again, it is true that* $\{\ell(u,v,p\,;\widetilde{B})\}^\alpha = d_2(p)$ *in case that* $1 < p_k \leq H \leq \infty$ *for all* $k \in \mathbb{N}$.

Proof. Since it is routine to prove the first part of the theorem, only the second part of it will be proved here. The method for determining the α-dual of the space $\ell(u,v,p\,;\widetilde{B})$ is based on its definition. Therefore, we will need the following steps. Let's assume that $a = (a_n) \in w$. In that case, we get

$$a_n x_x = \sum_{k=0}^{n-1} \frac{\widetilde{\Delta}(n,k)}{u_k} y_k a_n + \frac{1}{r_n u_n v_n} y_n a_n = D_n(y)\ (n \in \mathbb{N}), \tag{3.7}$$

where $D = (d_{nk})$ defined by (2.2).

Thus, we can easily deduce from (3.7) that $ax = (a_n x_n) \in \ell_1$ whenever $x = (x_k) \in \ell(u,v,p\,;\widetilde{B})$ iff $Dy \in \ell_1$ whenever $y = (y_k) \in \ell(p)$. This means that the sequence $a = (a_n)$ in the α-dual of the space $\ell(u,v,p\,;\widetilde{B})$ iff $D \in (\ell(p), \ell_1)$. Therefore, we derive from Lemma 3.1(i) with D instead of A that $\{\ell(u,v,p\,;\widetilde{B})\}^\alpha = d_2(p)$. This completes the proof.

Theorem 3.5. *Let's define the sets $d_3(p), d_4(p)$ and $d_5(p)$ with the following equations*

$$d_3(p)=\left\{a=(a_n)\in w:\ \sup_{n,k\in\mathbb{N}}|\widetilde{a}_k(n)|^{p_k}<\infty \text{ and } \left(\frac{a_k}{r_k u_k v_k}\right)\in \ell_\infty(p)\right\},$$

$$d_4(p)=\left\{a=(a_n)\in w:\ \sum_{j=k+1}^{\infty}\widetilde{\Delta}(j,k)\,a_j \text{ exists for all } k\in\mathbb{N}\right\},$$

and

$$d_5(p)=\bigcup_{K>1}\left\{a=(a_n)\in w:\sup_{n\in\mathbb{N}}\sum_{k=0}^{n-1}\left|\widetilde{a}_k(n)K^{-1}\right|^{p_k'}<\infty \text{ and } \left(\frac{a_k}{r_k u_k v_k}K^{-1}\right)\in \ell_\infty(p')\right\}.$$

Here $\widetilde{a}_k(n)=\frac{1}{r_k u_k v_k}a_k+\frac{1}{u_k}\sum_{j=k+1}^{n}\widetilde{\Delta}(j,k)\,a_j\ (k<n)$.

(i) $\{\ell(u,v,p\,;\widetilde{B})\}^{\beta}=d_3(p)\cap d_4(p)$ *is held when* $0<p_k\le 1$ *for all* $k\in\mathbb{N}$.

(ii) $\{\ell(u,v,p\,;\widetilde{B})\}^{\beta}=d_4(p)\cap d_5(p)$ *is held when* $1<p_k\le H\le\infty$ *for all* $k\in\mathbb{N}$.

Proof. (i) The first part should be clear with the following rudimentary calculations

$$\begin{aligned}\sum_{k=0}^{n}a_k x_k&=\sum_{k=0}^{n}a_k\left(\sum_{j=0}^{k-1}\frac{1}{u_j}\widetilde{\Delta}(k,j)y_j+\frac{1}{r_k u_k v_k}y_k\right)\\&=\sum_{k=0}^{n-1}\left(\frac{1}{r_k u_k v_k}a_k+\frac{1}{u_k}\sum_{j=k+1}^{n}\widetilde{\Delta}(j,k)a_j\right)y_k+\frac{a_n}{r_n u_n v_n}y_n\\&=\sum_{k=0}^{n-1}\widetilde{a}_k(n)y_k+\frac{a_n}{r_n u_n v_n}y_n=E_n(y)\ (n\in\mathbb{N}),\end{aligned}\tag{3.8}$$

where the matrix $E=(e_{nk})$ is defined by

$$e_{nk}=\begin{cases}\widetilde{a}_k(n) & ,\ (k<n),\\ \frac{a_n}{r_n u_n v_n} & ,\ (k=n),\ (n,k\in\mathbb{N}).\\ 0 & ,\ (k>n).\end{cases}$$

Thus, one can easily deduce from (3.8) that $ax=(a_k x_k)\in cs$ whenever $x=(x_k)\in\ell(u,v,p;\widetilde{B})$ iff $Dy\in c$ whenever $y=(y_k)\in\ell(p)$. This means that $a=(a_k)\in\{\ell(u,v,p;\widetilde{B})\}^{\beta}$ iff $E\in(\ell(p),c)$. Finally, it is obtained that $\{\ell(u,v,p;\widetilde{B})\}^{\beta}=d_3(p)\cap d_4(p)$ using Lemma 3.3 with the help of (3.8).

Part (ii) can also be proved in a much similar way to part (i), thus it is left to the reader.

Theorem 3.6.

(i) $\{\ell(u,v,p;\widetilde{B})\}^{\beta} = d_3(p)$ *is held when* $0 < p_k \leq 1$ *for all* $k \in \mathbb{N}$.

(ii) $\{\ell(u,v,p;\widetilde{B})\}^{\beta} = d_5(p)$ *is held when* $0 < p_k \leq H < \infty$ *for all* $k \in \mathbb{N}$.

Proof. From Lemma 3.2 and (3.8), it can be obtained that $ax = (a_k x_k)$ whenever $x = (x_k) \in \{\ell(u,v,p;\widetilde{B})\}$ iff $Ey \in \ell_\infty$ whenever $y = (y_k) \in \ell(p)$. Thus, $\{\ell(u,v,p;\widetilde{B})\}^{\gamma} = d_3(p)$ for $0 < p_k \leq 1$, $\{\ell(u,v,p;\widetilde{B})\}^{\gamma} = d_5(p)$ for $p_k > 1$ are derived from (3.4) and (3.5) respectively. This ends the proof.

4. Some matrix mappings on the space $\ell(u,v,p;\widetilde{B})$

For the sake of brevity, for an infinite matrix $A = (a_{nk})$, write

$$\widetilde{a}_{nk}(m) = \frac{1}{r_k u_k v_k} a_{nk} + \frac{1}{u_k} \sum_{j=k+1}^{m} \widetilde{\Delta}(j,k) a_{nj} \ (k < m) \tag{4.1}$$

and

$$\widetilde{a}_{nk} = \frac{1}{r_k u_k v_k} a_{nk} + \frac{1}{u_k} \sum_{j=k+1}^{\infty} \widetilde{\Delta}(j,k) a_{nj} \tag{4.2}$$

for all $n,k,m \in \mathbb{N}$ provided the convergence of the series.

Now, let's give the characterization of the classes $\Big(\ell(u,v,p;\widetilde{B}),\ell_\infty\Big)$, $\Big(\ell(u,v,p;\widetilde{B}),c\Big)$ and $\Big(\ell(u,v,p;\widetilde{B}),c_0\Big)$.

Theorem 4.1. *(i)* $A \in \Big(\ell(u,v,p;\widetilde{B}),\ell_\infty\Big)$ *iff there exists an integer* $K > 1$ *provided that*

$$C(K) = \sup_{n\in\mathbb{N}} \sum_k \left|\widetilde{a}_{nk} K^{-1}\right|^{p_k'} < \infty, \tag{4.3}$$

and

$$\{a_{nk}\}_{k\in\mathbb{N}} \in d_3(p) \cap d_4(p) \ (n \in \mathbb{N}). \tag{4.4}$$

Where $1 < p_k \leq H < \infty$ *for every given* $k \in \mathbb{N}$.

(ii) $A \in (\ell(u,v,p;\widetilde{B}),\ell_\infty)$ *iff*

$$\sup_{n,k\in\mathbb{N}} \sum_k |\widetilde{a}_{nk}|^{p_k} < \infty, \tag{4.5}$$

and

$$\{a_{nk}\}_{k\in\mathbb{N}} \in d_4(p) \cap d_5(p), \tag{4.6}$$

where $0 < p_k \leq 1$ *for every given* $k \in \mathbb{N}$.

Proof. We consider only the case $1 < p_k \leq H < \infty$ and leave the case $0 < p_k \leq 1$ to the reader since its proof is similar to this one.

Let us suppose that the conditions (4.3) and (4.4) are met and take an arbitrary $x = (x_k) \in \ell(u, v, p; \widetilde{B})$. In this case, using Theorem 3.5(ii) we obtain $\{a_{nk}\}_{k\in\mathbb{N}} \in \ell(u, v, p; \widetilde{B})$ for every fixed $n \in \mathbb{N}$. This shows the existence of the A-transform of x, that is, Ax exists. Now, let's take the following equality resulted from the relation (2.2) from the mth partial sum of the series $\sum_k a_{nk}x_k$:

$$\sum_{k=0}^{m} a_{nk}x_k = \sum_{k=0}^{m-1} \widetilde{a}_{nk}(m)y_k + \frac{1}{r_m u_m v_m} a_{nm}y_m \ (n, m \in \mathbb{N}). \tag{4.7}$$

If we consider the hypothesis, then we can derive from (4.7) as $m \to \infty$ that

$$\sum_k a_{nk}x_k = \sum_k \widetilde{a}_{nk}(m)y_k \ (n \in \mathbb{N}). \tag{4.8}$$

Now, by joining (4.8) and inequality holding for an arbitrary $K > 0$ and complex numbers a, b

$$|ab| \leq K\left\{\left|aK^{-1}\right|^{p'} + |b|^p\right\},$$

where $p > 1$ and $1/p + 1/p' = 1$ (refer to [40]). Thus one can easily obtain

$$\begin{aligned} \sup_{n\in\mathbb{N}} \left|\sum_k a_{nk}x_k\right| &\leq \sup_{n\in\mathbb{N}} \sum_k |\widetilde{a}_{nk}||y_k| \\ &\leq K\left[C(K) + g_1^K(y)\right] \\ &< \infty. \end{aligned}$$

In a converse way, let's suppose that $A \in (\ell(u, v, p; \widetilde{B}), \ell_\infty)$ and $1 < p_k \leq H < \infty$ for every $k \in \mathbb{N}$. In this condition, Ax exists for every $x \in \ell(u, v, p; \widetilde{B})$ and this shows that $\{a_{nk}\}_{k\in\mathbb{N}} \in \{\ell(u, v, p; \widetilde{B})\}^\beta$ for each $n \in \mathbb{N}$. Using Theorem 3.5(ii) immediately results in the necessity of (4.4). Additionally, equality (4.8) indicates that $\widetilde{A} = (\widetilde{a}_{nk})$ is in class $(\ell(p), \ell_\infty)$. Therefore $\widetilde{A}$ satisfies the condition (3.4) which is also equivalent to (4.3). This ends the proof.

Theorem 4.2. *$A \in (\ell(u, v, p; \widetilde{B}), c)$ is held iff (4.3)-(4.6) are true for $0 < p_k \leq H < \infty$ for every given $k \in \mathbb{N}$. Moreover, there exists a sequence $(\widetilde{\alpha}_k)$ of the scalars such that*

$$\lim_{n\to\infty} \widetilde{a}_{nk} = \widetilde{\alpha}_k \ (k \in \mathbb{N}). \tag{4.9}$$

Proof. Let $A \in (\ell(u,v,p;\widetilde{B}),c)$ and $0 < p_k \leq H < \infty$ for every $k \in \mathbb{N}$. Then, since the inclusion $c \subset \ell_\infty$, the necessities (4.3)-(4.6) are trivial by Theorem 4.1(i) and (ii).

In proving the necessity of (4.9), let's consider the sequence $c^{(k)}$ which is described by (2.8) lying is the space $\ell(u,v,p;\widetilde{B})$ for every fixed $k \in \mathbb{N}$. Since the A-transform of every $x \in \ell(u,v,p;\widetilde{B})$ exists and moreover lies in c according to the hypothesis, $Ac^{(k)} = \{\widetilde{a}_{nk}\}_{n\in\mathbb{N}}$ also lies in c for every fixed $k \in \mathbb{N}$. This clearly indicates the necessity (4.9).

In a converse way, let's suppose that the conditions (4.3)-(4.6) and (4.9) are met and take any $x \in \ell(u,v,p;\widetilde{B})$. In this condition, using Theorem 3.5 we obtain that $\{a_{nk}\}_{k\in\mathbb{N}} \in \{\ell(u,v,p;\widetilde{B})\}^\beta$ for every fixed $n \in \mathbb{N}$. This requires the existence of the A-transform of x, that is, Ax exists. Moreover, it is obviously seen that the associated sequence lies in the space ℓ_p. Moreover, if we combine Lemma 3.3 together with the conditions (4.3) and (4.9), we see that the matrix $\widetilde{A}$ is in the class $(\ell(p),c)$. Therefore, it is seen that $y \in \ell(p)$ and $\widetilde{A}y \in c$. In conclusion, using (4.9) results in the fact that $Ax \in c$ whenever $x \in \ell(u,v,p;\widetilde{B})$. In fact, this is exactly what we want to prove.

Replacing c by c_0 in Theorem 4.2 results in.

Corollary 4.3. *Let's assume that $0 < p_k \leq H < \infty$ for every given $k \in \mathbb{N}$. In that case, $A \in (\ell(u,v,p;\widetilde{B}),c_0)$ iff (4.3)-(4.6) hold and (4.9) also holds with $\widetilde{\alpha}_k = 0$ for all $k \in \mathbb{N}$.*

5. The Hausdorff measure of noncompactness and compact operators on the space $\ell_p(u,v;\widetilde{B})(1 \leq p < \infty)$

Here the classes of compact operators given by infinite matrices from $\ell_p(u,v;\widetilde{B})$ to c_0, c, ℓ_∞ and ℓ_1 are characterized. Meanwhile, the necessary and sufficient conditions for $A \in (\ell_1(u,v;\widetilde{B}),\ell_p)$ to be compact are given, where $1 \leq p < \infty$.

Goldenštein, Gohberg and Markus in 1957 first defined the Hausdorff measure of noncompactness and then Goldenštein and Markus studied it. A matrix transformation between BK-spaces is continuous. Thus it is expected to find conditions for a matrix map between BK-spaces to define a compact operator. Applying the Hausdorff measure of noncompactness achieves this. Some authors have set up classes of compact operators in terms of infinite matrices on some sequence spaces via this method. For example see [41-50]. In recent years, Malkowsky and Rakočević [51] , Djolović and Malkowsky [52] and Mursaleen and Noman [53] have set up some identities or estimates for the operator norms and Haudsdorff measures of noncompactness of linear operators. These operators are represented by infinite matrices mapping an arbitrary BK-space or the matrix domains of triangles in any BK-spaces.

Let's assume that X is a normed space. Then S_x is written for the unit sphere in X, that means $S_x = \{x \in X : x = 1\}$. If X and Y are Banach spaces then $B(X,Y)$ consists of all continuous linear operators $L : X \to Y$; $B(X,Y)$ is a Banach space with the operator norm defined by $\|L\| = sup\{\|L(x)\| : \|x\| \leq 1\}$ for all $L \in B(X,Y)$.

We have the right to write $\|a\|_X^* = \sup_{x \in S_x} \left| \sum_{k=0}^{\infty} a_k x_k \right|$ for $a \in w$ provided the expression on the right-hand side exists and is finite which is the case whenever X is a BK-space and $a \in X^\beta$ iff $(X, \|.\|)$ is a normed sequence space [54].

From now on, let $1 \leq p < \infty$ and q denote the conjugate of p, that is, $q = p/p-1$ for $1 < p < \infty$ or $q = \infty$ for $p = 1$. Again, all finite sequences terminating in zeros are written in a set, namely ϕ.

For our study, the following fundamental results will be used.

Lemma 5.1. [54] *Let's assume that X and Y be BK-spaces. In that case, we get $(X,Y) \subset B(X,Y)$, namely, every $A \in (X,Y)$ defines a linear operator $L_A \in B(X,Y)$, where $L_A(x) = Ax$ for all $x \in X$.*

Lemma 5.2. [42] *Let's assume that $X \supset \phi$ be BK-space and $Y \in \{c_0, c, \ell_\infty\}$. Then*

$$\|L_A\| = \|A\|_{(X,\ell_\infty)} = \sup_n \|A_n\|_X^* < \infty$$

when $A \in (X,Y)$.

Lemma 5.3. [55] *It is true that $\ell_p^\beta = \ell_q$ and $\|a\|_{\ell_p}^* = \|a\|_{\ell_q}$ for all $a = (a_k) \in \ell_q$, when $1 \leq p < \infty$.*

Lemma 5.4. [46] *We assume that $1 \leq p < \infty$ and q be the conjugate of p. If $A \in (\ell_p, c)$, then the following equalities hold.*

$$\alpha_k = \lim_{n\to\infty} a_{nk} \text{ exists for every } k \in \mathbb{N},$$
$$\alpha = (\alpha_k) \in \ell_q,$$

$\underset{n}{sup}\|A_n-\alpha\|_{\ell_q}<\infty,$

$\underset{n\to\infty}{lim}A_n(x)=\sum_{k=0}^{\infty}\alpha_k x_k$ *for all* $x=(x_k)\in\ell_p$.

Theorem 5.5. *Let* $a=(a_k)\in\{\ell_p(u,v;\widetilde{B})\}^{\beta}$. *In that case* $\widetilde{a}=(\widetilde{a}_k)\in\ell_q$ *and the following*

$$\sum_{k=0}^{\infty}a_k x_k=\sum_{k=0}^{\infty}\widetilde{a}_k y_k \tag{5.1}$$

is valid for every given $x=(x_k)\in\ell_p(u,v;\widetilde{B})$, *where*

$$\widetilde{a}_k=\frac{a_k}{r_k u_k v_k}+\frac{1}{u_k}\sum_{j=k+1}^{\infty}\widetilde{\Delta}(j,k)a_j \ (k\in\mathbb{N}).$$

Proof. This can easily be seen from [[51] Theorem 3.2].

If $1\le p<\infty$, then the inclusion $\ell_p(u,v;\widetilde{B})\supset\phi$ holds iff $u=(u_k)\in\ell_p$. Thus, we shall suppose that $u=(u_k)\in\ell_p$ whenever we study the space $\ell_p(u,v;\widetilde{B})$.

Theorem 5.6. *If* $\widetilde{a}=(\widetilde{a}_k)$ *is the newly introduced one above, then we get*

$$\|a\|^*_{\ell_p(u,v;\widetilde{B})}=\|\widetilde{a}\|_{\ell_q}=\begin{cases}\left(\sum_{k=0}^{\infty}|\widetilde{a}_k|^q\right)^{1/q}, (1<p<\infty),\\ \underset{k}{sup}|\widetilde{a}_k|\ ,\ (p=1)\end{cases}$$

for all $a=(a_k)\in\{\ell_p(u,v;\widetilde{B})\}^{\beta}$ *when* $1\le p<\infty$.

Proof. We assume that $a=(a_k)\in\{\ell_p(u,v;\widetilde{B})\}^{\beta}$. Obviously $\widetilde{a}=(\widetilde{a}_k)\in\ell_q$ and the equality (5.1) holds from Theorem 5.5 for all $x=(x_k)\in\ell_p(u,v;\widetilde{B})$ and $y=(y_k)\in\ell_p$ which are connected by the relation (2.2). It is immediate by Theorem 2.2(ii) that $x\in S_{\ell_p(u,v;\widetilde{B})}$ iff $y\in S_{\ell_p}$. Thus, we can write from (5.1) that

$$\|a\|^*_{\ell_p(u,v;\widetilde{B})}=\underset{x\in S_{\ell_p(u,v;\widetilde{B})}}{sup}\left|\sum_{k=0}^{\infty}a_k x_k\right|=\underset{y\in S_{\ell_p}}{sup}\left|\sum_{k=0}^{\infty}\widetilde{a}_k y_k\right|=\|\widetilde{a}\|^*_{\ell_p}. \tag{5.2}$$

Moreover, since $\widetilde{a}\in\ell_q$, we obtain from Lemma 5.3 and (5.2) that

$$\|a\|^*_{\ell_p(u,v;\widetilde{B})}=\|\widetilde{a}\|^*_{\ell_p}=\|\widetilde{a}\|_{\ell_q}<\infty,$$

which terminates the proof.

Theorem 5.7. *Let's assume that* $A=(a_{nk})$ *is an arbitrary infinite matrix, X is any given sequence space when* $1\le p<\infty$. *In case* $A\in(\ell_p(u,v;\widetilde{B}),X)$, *we have* $\widetilde{A}\in(\ell_p,X)$ *such that*

$Ax=\widetilde{A}y$ for every $x\in\ell_p(u,v;\widetilde{B})$ and $y\in\ell_p$, in which the sequences x and y are connected via the relation given by (2.2) and $\widetilde{A}=(\widetilde{a}_{nk})$ is described as (4.2) .

Proof. Let's take any $x\in\ell_p(u,v;\widetilde{B})$ and $A\in(\ell_p(u,v;\widetilde{B}),X)$. Then, $A_n\in\{\ell_p(u,v;\widetilde{B})\}^{\beta}$ for all $n\in\mathbb{N}$, where A_n is the sequence in the nth row of the matrix A. Hence, we have by Theorem 5.5 that $\widetilde{A}_n\in\ell_p^{\beta}=\ell_q$ for all $n\in\mathbb{N}$ and the equality $Ax=\widetilde{A}y$ holds for all sequences $x=(x_k)$ and $y=(y_k)$ which are connected by the relations (2.2). Thus $\widetilde{A}y\in X$. Since every $y\in\ell_p$ is the associated sequence of $x\in\ell_p(u,v;\widetilde{B})$. Therefore, we deduce that $\widetilde{A}\in(\ell_p,X)$, which concludes the proof.

Before going to Haussdorff measure of noncompactness, let us have one more definitions first. We remember the idea of compact operator. A linear operator L from a Banach space X to another Banach space Y is said to be compact if its domain is all of X and for each bounded sequence (x_n) in X, the sequence $(L(x_n))$ contains convergent subsequence in Y. $K(X,Y)$ is the space of compact operators from X to Y.

Let X is a metric space. In this case, we write M_X for the class of all bounded subsets of X. In a metric space (X,d), the open ball of radius $r\in\mathbb{R}$ centered at $x\in X$, is the set $B_r(x)=\{y\in X: d(x,y)<r\}$. $\chi(Q)$ is defined by the following set is known the Hausdorff measure of noncompactness of the set $Q\in M_X$.

$$\chi(Q)=inf\left\{\varepsilon>0:Q\subset\bigcup_{i=0}^{n}B(x_i,r_i),\ x_i\in X,\ r_i<\varepsilon\ (i=0,1,\cdots),\ n\in\mathbb{N}\right\}.$$

The function $\chi = M_X\to[\,0,\,\infty)$ is called the Hausdorf measure of noncompactness in the literature.

The certain fundamental properties related to Hausdorff measure of noncompactness can be reached in [55], for instance if Q, Q_1 and Q_2 are bounded subsets of a metric space (X,d), then

$$\chi(Q)=0 \text{ iff } Q \text{ is totally bounded } Q_1\subset Q_2 \text{ implies } \chi(Q_1)\le\chi(Q_2).$$

Moreover, suppose X is a normed space. In this case the function χ has some additional properties connected with the linear structure, e.g.

$$\chi(Q_1+Q_2)\le\chi(Q_1)+\chi(Q_2),\chi(\alpha Q)=|\alpha|\chi(Q) \text{ for all } \alpha\in\mathbb{C},$$

where $\mathbb{C}$ is the complex field.

Lemma 5.8. [46] *We assume that X is a Banach space having a Schauder basis $(d_k)_{k=0}^{\infty}$, $Q \in M_X$ and $P_n : X \to X$ $(n \in \mathbb{N})$ is the projector onto the linear span given by $\{d_0, d_1, \cdots, d_n\}$. Thus, we obtain the following inequalities*

$$\frac{1}{a} \limsup_{n\to\infty} \left(\sup_{x\in Q} \|(I-P_n)(x)\| \right) \leq \chi(Q) \leq \limsup_{n\to\infty} \left(\sup_{x\in Q} \|(I-P_n)(x)\| \right),$$

here $a = \limsup_{n\to\infty} \|I - P_n\|$.

Particularly, the following result indicates how to compute the Hausdorff measure of noncompactness in the spaces ℓ_p $(1 \leq p < \infty)$ and c_0.

Lemma 5.9. [42] *We assume that Q is a bounded subset of the normed space X in which X is ℓ_p for ℓ_p $(1 \leq p < \infty)$ or c_0. In case $P_n : X \to X$ is the operator described by the equality $P_n(x) = x^{[n]} = (x_0, x_1, x_2, \cdots, x_n, 0, 0, \cdots)$ for every $x = (x_k) \in X$, we obtain*

$$\chi(Q) = \lim_{n\to\infty} \left(\sup_{x\in Q} \|(I-P_n)(x)\| \right).$$

The following lemma is related to the Hausdorff measure of noncompactness of a bounded linear operator.

Lemma 5.10. [55] *Let's assume that X and Y are Banach spaces and $L \in B(X,Y)$. Thus we obtain*

$$\|L\|_\chi = \chi(L(S_X)) \tag{5.3}$$

and

$$L \in K(X,Y) \text{ iff } \|L\|_\chi = 0. \tag{5.4}$$

Lemma 5.11. [53] *Assuming that $X \supset \phi$ is a BK-space, we obtain*

(a) In case $A \in (X, c_0)$, then

$$\|L_A\|_\chi = \limsup_{n\to\infty} \|A_n\|_X^*$$

and

$$L_A \text{ is compact iff } \lim_{n\to\infty} \|A_n\|_X^* = 0.$$

(b) In case $A \in (X, \ell_\infty)$, then

$$0 \leq \|L_A\|_\chi \leq \limsup_{n\to\infty} \|A_n\|_\chi^*$$

and

$$L_A \text{ is compact iff } \lim_{n\to\infty} \|A_n\|_\chi^* = 0.$$

(c) In case $A \in (X, \ell_1)$, then

$$\lim_{r\to\infty}\left(\sup_{N\in\mathscr{F}_r}\left\|\sum_{n\in N} A_n\right\|_X^*\right) \leq \|L_A\|_\chi \leq 4.\lim_{r\to\infty}\left(\sup_{N\in\mathscr{F}_r}\left\|\sum_{n\in N} A_n\right\|_X^*\right)$$

$$L_A \text{ is compact iff } \lim_{r\to\infty}\left(\sup_{N\in\mathscr{F}_r}\left\|\sum_{n\in N} A_n\right\|_X^*\right) = 0.$$

The necessary and sufficient conditions for a matrix transformation from a BK-space X to c_0, ℓ_1 and ℓ_∞ to be compact (only sufficient condition for ℓ_∞) are given by this Lemma. As a result, we obtain

Theorem 5.12. *Assuming $1 < p < \infty$ and $q = p/(p-1)$, we obtain*

(a) In case $A \in (\ell_p(u,v;\widetilde{B}), c_0)$, it is true that

$$\|L_A\|_\chi = \limsup_{n\to\infty}\left(\sum_{k=0}^{\infty} |\widetilde{a}_{nk}|^q\right)^{1/q} \tag{5.5}$$

and

$$L_A \text{ is compact iff } \lim_{n\to\infty}\left(\sum_{k=0}^{\infty} |\widetilde{a}_{nk}|^q\right)^{1/q} = 0. \tag{5.6}$$

(b) In case $A \in (\ell_p(u,v;\widetilde{B}), \ell_\infty)$, it is true that

$$0 \leq \|L_A\|_\chi = \limsup_{n\to\infty}\left(\sum_{k=0}^{\infty} |\widetilde{a}_{nk}|^q\right)^{1/q} \tag{5.7}$$

and

$$L_A \text{ is compact iff } \lim_{n\to\infty}\left(\sum_{k=0}^{\infty} |\widetilde{a}_{nk}|^q\right)^{1/q} = 0. \tag{5.8}$$

Proof. Let us assume that $A \in (\ell_p(u,v;\widetilde{B}), c_0)$. Since $A_n \in \{\ell_p(u,v;\widetilde{B})\}^\beta$ for all $n \in \mathbb{N}$, using Theorem 5.6 we easily get

$$\|A_n\|_{\ell_p(u,v;\widetilde{B})}^* = \|\widetilde{A}_n\|_{\ell_q} = \left(\sum_{k=0}^{\infty} |\widetilde{a}_{nk}|^q\right)^{1/q} \tag{5.9}$$

for all $n \in \mathbb{N}$. Thus we have (5.5) and (5.6) (5.9) and Lemma 5.11(a).

Because the second part of the theorem has exactly same idea as in the first part of it, the proof of the second part can be obtained similarly.

Theorem 5.13. *Assuming $1 \le p < \infty$ and $A \in (\ell_p(u,v;\widetilde{B}),c)$, then*

$$\frac{1}{2}.\lim_{r\to\infty}\left(\sup_{n\ge r}\|\widetilde{A}_n - \widetilde{\alpha}\|_{\ell_q}\right) \le \|L_A\|_\chi \le \lim_{r\to\infty}\left(\sup_{n\ge r}\|\widetilde{A}_n - \widetilde{\alpha}\|_{\ell_q}\right), \tag{5.10}$$

where $\widetilde{\alpha} = (\widetilde{\alpha}_k)$ is described as in (4.9).

Proof. Let's suppose that $A \in (\ell_p(u,v;\widetilde{B}),c)$. Then, using Theorem 5.7, we will have $\widetilde{A} \in (\ell_p,c)$. Therefore, it is seen that the sequence $(sup_{n\ge r}\|\widetilde{A}_n - \widetilde{\alpha}\|_{\ell_q})_{r=0}^{\infty}$ of non-negative reals is nonincreasing and bounded according to Lemma 5.4. Consequently, it is obvious that the limit given in (5.10) exists.

Now, we can shortly write $S = S_{\ell_p(u,v;\widetilde{B})}$. Thus, using (5.3) and Lemma 5.1, we easily get

$$\|L_A\|_\chi = \chi(AS) \tag{5.11}$$

and $AS \in M_c$, here M_c denotes the class of all bounded subsets of c. Moreover, it is known that every $z = (z_n) \in c$ has a distinct representation as $z = \bar{z}e + \sum_{n=0}^{\infty}(z_n - \bar{z})e^{(n)}$, here $\bar{z} = lim_{n\to\infty} z_n$. Therefore the projectors $P_r : c \to c$ $(r \in \mathbb{N})$ by $P_0(z) = \bar{z}e$ and $P_r(z) = \bar{z}e + \sum_{n=0}^{r-1}(z_n - \bar{z})e^{(n)}$ can be defined for $r \ge 1$. Consequently, we will have for every $r \in \mathbb{N}$ that $(1 - P_r)(z) = \sum_{n=r}^{\infty}(z_n - \bar{z})e^{(n)}$ and thus

$$\|(I - P_r)(z)\|_{\ell_\infty} = \sup_{n\ge r}|z_n - \bar{z}| \tag{5.12}$$

for all $z \in c$ and every $r \in \mathbb{N}$. In fact, it is obvious seen that $\|I - P_r\| = 2$ for all $r \in \mathbb{N}$. Thus, from (5.11) and Lemma 5.8 we obtain

$$\frac{1}{2}.\mu(A) \le \|L_A\|_\chi \le \mu(A), \tag{5.13}$$

where

$$\mu(A) = \limsup_{r\to\infty}\left(\sup_{x\in S}\|(I - P_r)(Ax)\|_{\ell_\infty}\right).$$

Let's suppose that $y \in \ell_p$ be associated with sequence defined by (2.2) for every given $x \in \ell_p(u,v;\widetilde{B})$, we will have $\widetilde{A} \in (\ell_p,c)$ and $Ax = \widetilde{A}y$ by Theorem 5.7 because of the fact that $A \in$

$(\ell_p(u,v;\widetilde{B}),c)$. Moreover, the limits $\alpha_k = lim_{n\to\infty} a_{nk}$ exist for all k, $\widetilde{\alpha} = (\widetilde{\alpha})_k \in \ell_q$ and

$$\lim_{n\to\infty} \widetilde{A}_n(y) = \sum_{k=0}^{\infty} \widetilde{\alpha}_k x_k.$$

using Lemma 5.4.

For all $r \in \mathbb{N}$, the equalities

$$\|(I-P_r)(Ax)\|_{\ell_\infty} = \left\|(I-P_r)(\widetilde{A}y)\right\|_{\ell_\infty} = \sup_{n\geq r}\left|\widetilde{A}_n(y) - \sum_{k=0}^{\infty}\widetilde{\alpha}_k y_k\right| = \sup_{n\geq r}\left|\sum_{k=0}^{\infty}(\widetilde{a}_{nk} - \widetilde{\alpha}_k)y_k\right|$$

can be derived from (5.12). Furthermore, for all $r \in \mathbb{N}$ we obtain

$$\sup_{x\in S}\|(I-P_r)(Ax)\|_{\ell_\infty} = \sup_{n\geq r}\left(\sup_{y\in S_{\ell_p}}\left|\sum_{k=0}^{\infty}(\widetilde{a}_{nk} - \widetilde{\alpha}_k)y_k\right|\right) = \sup_{n\geq r}\|\widetilde{A}_n - \widetilde{\alpha}\|^*_{\ell_p} = \sup_{n\geq r}\|\widetilde{A}_n - \widetilde{\alpha}\|_{\ell_q}$$

using the definition of the norm $\|.\|^*_X$ and Lemma 5.3, because of the fact that $x \in S = S_{\ell_p(u,v;\widetilde{B})}$ iff $y \in \ell_p$. In conclusion, we obtain (5.10) from (5.13), due to the fact that the limit in (5.10) exists. This marks the end of the proof.

Now remember that the upper limit (limit superior) of a bounded real sequence $x = (x_n)$ can be defined by

$$\limsup_{n\to\infty} x_n = \lim_{n\to\infty}\left(\sup_{n\geq r} x_n\right). \tag{5.14}$$

Moreover, if $x_n \geq 0$ for all n, then

$$\limsup_{n\to\infty} x_n = 0 \text{ iff } \lim_{n\to\infty} x_n = 0.$$

Using the newly obtained notation results in:

Corollary 5.14. *When it is assumed that* $1 < p < \infty$, $q = p/(p-1)$ *and* $A \in (\ell_p(u,v;\widetilde{B}),c)$, *then*

$$\frac{1}{2}.\limsup_{n\to\infty}\left(\sum_{k=0}^{\infty}|\widetilde{a}_{nk} - \widetilde{\alpha}_k|^q\right)^{1/q} \leq \|L_A\|_\chi \leq \limsup_{n\to\infty}\left(\sum_{k=0}^{\infty}|\widetilde{a}_{nk} - \widetilde{\alpha}_k|^q\right)^{1/q}$$

and

$$L_A \text{ is compact iff } \lim_{n\to\infty}\left(\sum_{k=0}^{\infty}|\widetilde{a}_{nk} - \widetilde{\alpha}_k|^q\right)^{1/q} = 0,$$

here $\widetilde{\alpha} = (\widetilde{\alpha}_k)$ *is described as in (4.6).*

Proof. This result can directly be proved from Theorem 5.13 by using (5.14) and (5.4).

Theorem 5.15. *Assuming* $1 \leq p < \infty$ *and* $A \in (\ell_1(u,v;\widetilde{B}), \ell_p)$*, then*

$$\|L_A\|_\chi = \lim_{r\to\infty}\left(\sup_k \left(\sum_{n=r}^{\infty} |\widetilde{a}_{nk}|^p\right)^{1/p}\right). \tag{5.15}$$

Proof. Let's suppose that $S = S_{\ell_1(u,v;\widetilde{B})}$. Then, using Lemma 5.1; we have $L_A(S) = AS \in \ell_p$. Therefore, using (5.3) and Lemma 5.9, it can be written as

$$\|L_A\|_\chi = \chi(AS) = \lim_{r\to\infty}\left(\sup_{x\in S} \|(I-P_r)(Ax)\|_{\ell_p}\right). \tag{5.16}$$

here $P_r : \ell_p \to \ell_p$ $(r \in \mathbb{N})$ is the operator defined by $P_r(x) = (x_0, x_1, \cdots, x_r, 0, 0, \cdots)$ for all $x = (x_k) \in \ell_p$.

Secondly, let's suppose that $x = (x_k) \in \ell_1(u,v;\widetilde{B})$. Because of the fact that $A \in (\ell_1(u,v;\widetilde{B}), \ell_p)$ from Theorem 5.7, it is obtained that $\widetilde{A} \in (\ell_1, \ell_p)$ and $Ax = \widetilde{A}y$, where $y = (y_k) \in \ell_1$ is the associated sequence defined by (2.2). Thus, we easily obtain the following inequality

$$\begin{aligned}
\|(I-P_r)(Ax)\|_{\ell_p} &= \left\|(I-P_r)(\widetilde{A}y)\right\|_{\ell_p} \\
&= \left(\sum_{n=r+1}^{\infty} |\widetilde{A}_n(y)|^p\right)^{1/p} \\
&= \left(\sum_{n=r+1}^{\infty} \left|\sum_{k=0}^{\infty} \widetilde{a}_{nk} y_k\right|^p\right)^{1/p} \\
&\leq \sum_{k=0}^{\infty}\left(\sum_{n=r+1}^{\infty} |\widetilde{a}_{nk} y_k|^p\right)^{1/p} \\
&\leq \|y\|_{\ell_1}\left(\sup_k \left(\sum_{n=r+1}^{\infty} |\widetilde{a}_{nk}|^p\right)^{1/p}\right) \\
&= \|x\|_{r_1^q(B^m)}\left(\sup_k \left(\sum_{n=r+1}^{\infty} |\widetilde{a}_{nk}|^p\right)^{1/p}\right)
\end{aligned}$$

holding for every $n \in \mathbb{N}$. This results in

$$\sup_{x\in S} \|(I-P_r)(Ax)\|_{\ell_p} \leq \sup_k \left(\sum_{n=r+1}^{\infty} |\widetilde{a}_{nk}|^p\right)^{1/p}$$

holding for every $n \in \mathbb{N}$. Hence, using (5.16) we obtain

$$\|L_A\|_\chi \leq \lim_{r\to\infty}\left(\sup_k\left(\sum_{n=r+1}^{\infty}|\widetilde{a}_{nk}|^p\right)^{1/p}\right). \tag{5.17}$$

In a converse way, let's suppose that $c^k \in \ell_1(u,v;\widetilde{B})$ such that $R(c^{(k)}) = e^{(k)}$ $(k \in \mathbb{N})$, here $c^{(k)} = \{c_n^{(k)}\}_{n\in\mathbb{N}}$ is described using (2.8) for each $k \in \mathbb{N}$. Therefore, using Theorem 5.7 we get $Ac^{(k)} = \widetilde{A}e^{(k)} = (\widetilde{a}_{nk})_{n=0}^{\infty}$ for every $k \in \mathbb{N}$. Now, let's suppose that $E = \{c^{(k)} : k \in \mathbb{N}\}$. In this condition, $E \subset S$ and therefore $AE \subset AS$ and this indicates that

$$\chi(AE) \leq \chi(AS) = \|L_A\|_\chi. \tag{5.18}$$

Additionally, using Lemma 5.8 and (5.18), it can be written as

$$\chi(AE) = \lim_{r\to\infty}\left(\sup_k\left(\sum_{n=r+1}^{\infty}|\widetilde{a}_{nk}|^p\right)^{1/p}\right) \leq \|L_A\|_\chi.$$

In conclusion, (5.15) can be easily obtained from (5.17) and (5.18).

Corollary 5.16. *When it is assumed that* $1 \leq p < \infty$ *and* $A \in (\ell_1(u,v;\widetilde{B}),\ell_p)$ *then*

$$L_A \text{ is compact iff } \lim_{r\to\infty}\left(\sup_k\left(\sum_{n=r+1}^{\infty}|\widetilde{a}_{nk}|^p\right)^{1/p}\right) = 0.$$

Proof. This immediately follows from Theorem 5.15 and (5.4).

Theorem 5.17. *When it is assumed that* $1 < p < \infty$, $q = p/(p-1)$ *and* $A \in (\ell_p(u,v;\widetilde{B}),\ell_1)$, *then*

$$\lim_{r\to\infty}\left(\sup_{N\in\mathscr{F}_r}\left(\sum_{k=0}^{\infty}\left|\sum_{n\in N}\widetilde{a}_{nk}\right|^q\right)^{1/q}\right) \leq \|L_A\|_\chi \leq 4.\lim_{r\to\infty}\left(\sup_{N\in\mathscr{F}_r}\left(\sum_{k=0}^{\infty}\left|\sum_{n\in N}\widetilde{a}_{nk}\right|^q\right)^{1/q}\right) \tag{5.19}$$

$$L_A \text{ is compact iff } \lim_{r\to\infty}\left(\sup_{N\in\mathscr{F}_r}\left(\sum_{k=0}^{\infty}\left|\sum_{n\in N}\widetilde{a}_{nk}\right|^q\right)^{1/q}\right) = 0. \tag{5.20}$$

Proof. Let's suppose that $A \in (\ell_p(u,v;\widetilde{B}),\ell_1)$. Since $A_n \in [\ell_p(u,v;\widetilde{B})]^\beta$ for all $n \in \mathbb{N}$, using Theorem 5.6 we have that

$$\left\|\sum_{n\in N}A_n\right\|^*_{r_p^q(B)} = \left\|\sum_{n\in N}\widetilde{A}_n\right\|_{\ell_q}. \tag{5.21}$$

Therefore, (5.19) and (5.20) can be obtained from Lemma 5.11 (c) and (5.21).

Remark 5.18. The obtained conclusions from Theorem 5.12, Corollary 5.14 and Theorem 5.17 are still valid for $\ell_1(u,v;\widetilde{B})$ in place of $\ell_p(u,v;\widetilde{B})$ having $q=1$ and by substituting the summation over k by the supremum over k.

Conflict of Interests

The author declares that there is no conflict of interests.

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