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EVERY 3-CONNECTED CLAW-FREE $B_{i,7-i}$-FREE GRAPH IS HAMILTONIAN

KHIKMAT SABUROV[1,*], MANSOOR SABUROV[2]

[1]Department of Mathematics, University of West Bohemia, and Institute for Theoretical Computer Science (ITI), Charles University, P.O. Box 314, 306 14 Pilsen, Czech Republic
[2]Department of Computational & Theoretical Science, Faculty of Science, International Islamic University Malaysia, P.O. Box 141, 25710 Kuantan, Pahang, Malaysia

Abstract. In this paper we will proof that every 3-connected $K_{1,3}B_{2,5}$ (or $K_{1,3}B_{3,4}$)-free graph is Hamiltonian.

Keywords: Claw-free graph; forbidden graphs; Hamiltonian graphs; k-stabilizer.

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1. Introduction

We consider finite simple undirected graphs $G = (V(G), E(G))$ without loops and multiple edges. We follow the most common terminology and notation and for concepts not defined here we refer e.g. to [2].

The *circumference* of G (i.e., the length of a longest cycle in G) will be denoted by $circ(G)$. A subgraph H of a graph G is called an *induced subgraph* of G if, for any pair of vertices x and y of H, xy is an edge of H if and only if xy is an edge of G. If H is an induced subgraph of G, then we use the notation $H \overset{\text{IND}}{\subset} G$. For $M \subset V(G)$, $\langle M\rangle_G$ denotes the induced subgraph of G on the set M. Given graphs $H_1, \dots H_k$, we say that G is $\{H_1, \dots H_k\}$-free if G contains no induced subgraph isomorphic to any of the graphs $H_1, \dots, H_k$. The graphs $H_1, \dots, H_k$ will be referred to in this context as *forbidden induced subgraphs*. Specifically, the four-vertex star $X = K_{1,3}$ will be also called the *claw* and in this case we say that G is *claw-free*. For a vertex $u \in V(G)$, the set

*Corresponding author
E-mail addresses: khikmat@kma.zcu.cz (K. Saburov), msaburov@gmail.com (M. Saburov)
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$N_G(u) = \{v \in V(G) : uv \in E(G)\}$ is called the *neighborhood* of u in G and the set $N_G[u] = N_G(u) \cup \{u\}$ is called the *closed neighborhood* of u in G.

We say that x is a *eligible* vertex if the subgraph $\langle N_G(x)\rangle$ is connected non-complete. The set of all eligible vertices of G will be denoted by $V_{EL}(G)$. For an eligible vertex $x \in V_{EL}(G)$ set $B_x = \{uv : u, v \in N_G(x), uv \notin E(G)\}$ and let G^*_x be the graph with vertex set $V(G^*_x) = V(G)$ and with edge set $E(G^*_x) = E(G) \cup B_x$ (i.e., G^*_x is obtained from G by adding to $\langle N_G(x)\rangle_G$ the set B_x of all missing edges). The graph G^*_x is called the *local completion of* G *at* x. If $V_{LC}(G) = V(G)$, then we say that G is *locally connected.* A graph G is *essentially* k*-edge-connected* if the deletion of less than k edges leaves at most one component with more than one vertex.

Definition 1. *Let G be a claw-free graph. We say a graph H is a closure of G, denoted $H = cl(G)$, if*

(*i*) *there is a sequence of graphs $G_1, \ldots, G_t$ such that $G_1 = G$, $G_t = H$, $V(G_{i+1}) = V(G_i)$ and $E(G_{i+1}) = E(G_i) \cup B_{x_i}$ for some $x_i \in V_{EL}(G_i)$,*

(*ii*) *$V_{EL}(H) = \emptyset$.*

The following result summarizes basic properties of the closure.

Theorem 1 (Ryjáček [13])**.** *For every claw-free graph G,*

(*i*) *$cl(G)$ is uniquely determined,*

(*ii*) *$cl(G)$ is the line graph of a triangle-free graph,*

(*iii*) *$circ(cl(G)) = circ(G)$, and*

(*iv*) *$cl(G)$ is Hamiltonian if and only if G is Hamiltonian.*

We say that a claw-free graph G is *closed* if $G = \mathrm{cl}(G)$.

Let $\mathcal{C}$ be a subclass of the class of claw-free graphs. We say that the class $\mathcal{C}$ is *stable under the closure* (or simply *stable*) if $\mathrm{cl}(G) \in \mathcal{C}$ for every $G \in \mathcal{C}$ (equivalently, the class $\mathcal{C}$ is stable if the closure operation is internal on $\mathcal{C}$).

Suppose that Z_i, $B_{i,j}$ and $N_{i,j,k}$ are the graphs as shown in Figure 1.

Theorem 2 (Brousek et al. [3])**.**

(i) The class of $K_{1,3}P_i$-free graphs is a stable class for any $i \geq 3$.

(ii) The class of $K_{1,3}Z_i$-free graphs is a stable class for any $i \geq 1$.

(iii) The class of $K_{1,3}N_{i,j,k}$-free graphs is a stable class for any $i, j, k \geq 1$.

Also Brousek et al. [3] gave an example showing that the class of $K_{1,3}B_{i,j}$-free graphs is not stable under closure operation.

One of the first results on forbidden subgraphs and Hamiltonicity is by Goodman and Hedetniemi [8].

Theorem 3 (Goodman and Hedetniemi [8])**.** *If G is a 2-connected $\{K_{1,3}, Z_1\}$-free graph, then G is Hamiltonian.*

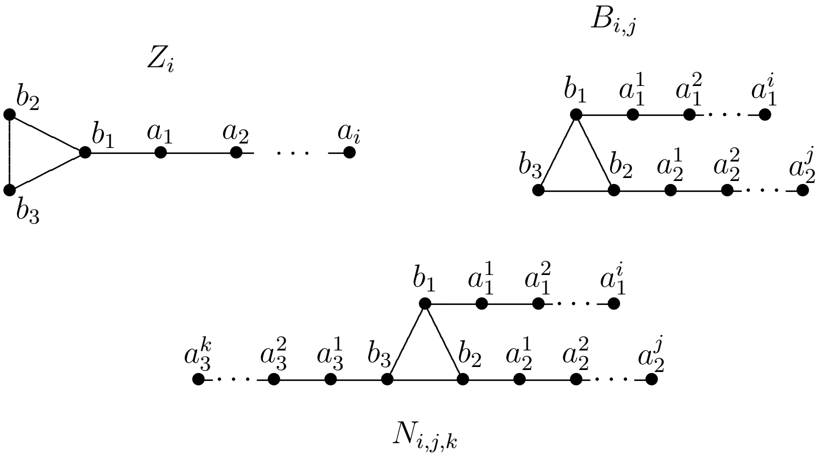

FIGURE 1. The graphs $Z_i, B_{i,j}$ and $N_{i,j,k}$

Later, in 1981 Duffus et al. [6] extended Theorem 3 to the larger class of $K_{1,3}N_{1,1,1}$-free graphs.

Theorem 4 (Duffus et al. [6]). *Every 2-connected $K_{1,3}N_{1,1,1}$-free graph is Hamiltonian.*

Bedrossian [1] and Faudree and Gould [7] characterized all pairs of connected forbidden subgraphs X, Y such that every 2-connected X, Y-free graph is Hamiltonian.

Theorem 5 (Bedrossian [1], Faudree and Gould [7]). *Let X and Y be connected graphs with $X, Y \neq P_3$, and let G be a 2-connected graph of order $n \geq 10$ that is not a cycle. Then, G being XY-free implies G is Hamiltonian if and only if (up to symmetry) $X = K_{1,3}$ and $Y = P_4, P_5, P_6, C_3, Z_1, Z_2, Z_3, B_{1,1}, N_{1,1,1}$ or $B_{1,2}$.*

A similar result for 3-connected graphs is given by Luczak and Pfender [10] in 2004.

Theorem 6 (Luczak and Pfender [10]). *Every 3-connected $K_{1,3}P_{11}$-free graph is Hamiltonian.*

Moreover, Luczak et.al. showed that there are infinitely many graphs such that they are 3-connected and $K_{1,3}P_{12}$-free, but non-Hamiltonian.

Theorem 6 give a motivation i.e. to find an upper bound for the numbers i, j such that every 3-connected $K_{1,3}B_{i,j}$-free graph is Hamiltonian.

In the next section we will present the following results:

- Every 3-connected $K_{1,3}B_{2,5}$-free graph is Hamiltonian.
- Every 3-connected $K_{1,3}B_{3,4}$-free graph is Hamiltonian.

2. Main results: 3-connected $K_{1,3}B_{i,j}$-free graphs

In the proof we need the following definitions. The following concepts were introduced by Miller et.al. [11].

A pair of pendant edges with a common vertex is called a *cherry*. If a graph G contains no cherry, we say that G is *cherry-free*. Let G be a graph, $x \in V(G)$, $d_G(x) \geq 2$ and denote all neighbours of x by $N_G(x) = \{a_1, a_2, \ldots, a_i, b_1, b_2, \ldots, b_j\}$ with $i, j \geq 1$. Let G_x^+ be the graph with

$$V(G_x^+) = (V(G) \setminus \{x\}) \cup \{x_1, x_2\}, \text{ where } x_1, x_2 \notin V(G),$$

and

$$E(G_x^+) = (E(G) \setminus \{xa_1, \ldots, xa_i, xb_1, \ldots, xb_j\}) \cup \{x_1a_1, \ldots, x_1a_i\}$$
$$\cup \{x_2b_1, \ldots, x_2b_j\} \cup \{x_1x_2\}.$$

Then we say that G_x^+ is obtained from G by *splitting of type 1 of the vertex x* (see Figure 2).

Suppose we have a graph G that contains a pendant edge e with end vertices x, u. Let us assume $x \in V(G)$, $d_G(x) \geq 3$ and denote all neighbours of x by $N_G(x) = \{u, a_1, a_2, \ldots, a_i, b_1, b_2, \ldots, b_j\}$ with $i, j \geq 1$. We will construct the graph $G_x^{+(e)}$ in the following way.

$$V(G_x^{+(e)}) = V(G),$$

and

$$E(G_x^{+(e)}) = E(G) \setminus \{xb_1, xb_2, \ldots, xb_j\} \cup \{ub_1, ub_2, \ldots, ub_j\}.$$

Then we say the graph $G_x^{+(e)}$ is obtained from G by *splitting of type 2 of x* or *rotation of e* (see Figure 2).

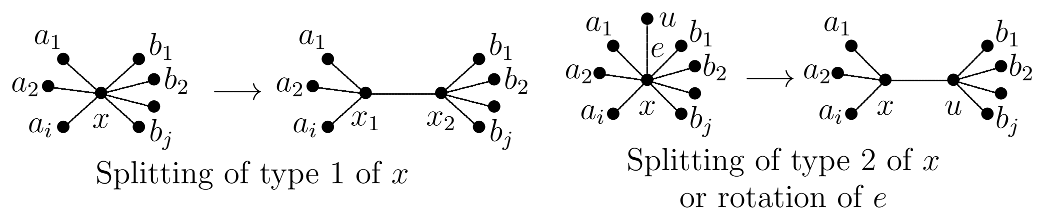

FIGURE 2. Splitting and rotation

Theorem 7 (Miller et.al. [11]). *Let G be a claw-free graph and $x \in V(G)$. If G_x^* contains an induced subgraph F such that $F = L(H)$, where H is a triangle-free and cherry-free graph, then G contains an induced subgraph F' such that $F' = L(H')$ and either $H' \simeq H$, or H' is obtained from H by splitting of type 1 of the vertex x or splitting of type 2 of x.*

Suppose that $\mathcal{Y}$ is a family of graphs. If for any $Y \in \mathcal{Y}$ and any Y' obtained from Y by splitting of type 1 or 2 of a vertex, Y' contains a subgraph $Y'' \in \mathcal{Y}$, then $\mathcal{Y}$ is called *closed under vertex splitting* (or shortly *split-closed*).

It is known that the class of $K_{1,3}B_{s,r}$-free graphs is not stable under the closure operation. Our main target is to find such a class $\mathscr{H}^{(s,r)g}$ that is split-closed and every graph in $\mathscr{H}^{(s,r)g}$ contains a subgraph isomorphic to $L^{-1}(B_{s,r})$.

The following concepts are introduced by Miller et.al. [11]. Suppose that $\mathcal{B}$ is a family of connected line graphs of triangle-free and cherry-free graphs. Let $\mathcal{G}$ be a family of connected closed $K_{1,3}$-free graph, and suppose $k \geq 2$ is an integer. Suppose that a family $\mathcal{S}$ of connected line graphs of triangle-free and cherry-free graphs satisfies the next three conditions.

(i) $\mathcal{S}$ is split-closed,
(ii) every $S \in \mathcal{S}$ contains a subgraph isomorphic to $L^{-1}(B)$ for some $B \in \mathcal{B}$,
(iii) every closed k-connected $K_{1,3}L(\mathcal{S})$-free graph is $K_{1,3}\mathcal{G}$-free.

Then, $\mathcal{S}$ is called a *k-stabilizer for $\mathcal{B}$ into $\mathcal{G}$ under closure cl*, or shortly a $(k, \mathcal{B}, \mathcal{G}, cl)$-*stabilizer*. If $\mathcal{B} = \{B\}$ and $\mathcal{G} = \{Q\}$, then we shortly say that $\mathcal{S}$ is a (k, B, Q, cl)-*stabilizer.*

Theorem 8 (Miller et.al. [11]). *Let $\mathcal{S}$ be a $(k, \mathcal{B}, \mathcal{G}, cl)$-stabilizer, $k \geq 1$, and let G be a k-connected $K_{1,3}\mathcal{B}$-free graph. Then $cl(G)$ is $K_{1,3}\mathcal{G}$-free.*

Let us assume that $\mathscr{H}$ is a class of all triangle-free and cherry-free graphs. In our proof we need a special subclass of $\mathscr{H}$. Suppose that $H \in \mathscr{H}$ and let s, r be integers. If H contains a subgraph isomorphic to $L^{-1}(B_{s,r})$ then H is called (s, r)-*good.* We denote by $\mathscr{H}^{(s,r)g}$ all (s, r)-*good* graphs i.e. $\mathscr{H}^{(s,r)g} = \{H \in \mathscr{H} | \ H$ is (s, r)-good$\}$. A pendant edge $e \in E(H)$ is called a *critical edge*, if $H \in \mathscr{H}^{(s,r)g}$ and for every subgraph of H isomorphic to $L^{-1}(B_{s,r})$, the edge e corresponds to the edge cb_0^3 (see Figure 3). For example, if H is a cycle of length at least 10 with attached pendant edge e, then H is (2,5)-good, the edge e is critical, and after rotation of e we obtain a graph $H' \in \mathscr{H} \setminus \mathscr{H}^{(2,5)g}$.

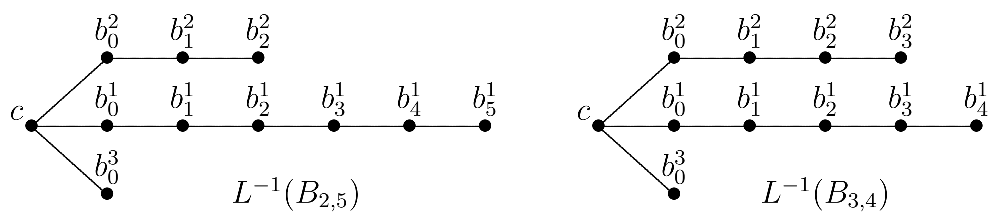

FIGURE 3. Preimages of $B_{2,5}$ and $B_{3,4}$

In the following sections we will show that $\mathscr{H}^{(2,5)g}$ is a $(3, B_{2,5}, P_{11}, cl)$-*stabilizer* and $\mathscr{H}^{(3,4)g}$ is a $(3, B_{3,4}, P_{11}, cl)$-*stabilizer* i.e. the closure of every 3-connected $K_{1,3}B_{2,5}$-free (or $K_{1,3}B_{3,4}$-free) graph is $K_{1,3}P_{11}$-free.

A graph H is called (s,r)-*good of type* A, if $H \in \mathscr{H}^{(s,r)g}$ and H has not any critical edge. The set of (s,r)-good graphs of type A is denoted by $\mathscr{H}_A^{(s,r)g}$ i.e. $\mathscr{H}_A^{(s,r)g} = \{H \in \mathscr{H}^{(s,r)g} |\ H$ is (s,r)-good of type A$\}$. For example, H is the graph which is obtained by identifying a vertex of a cycle of length at least 10 with an end vertex of a path of length at least 2, then H is $(2,5)$-good of type A.

The following proposition 1 says that the class of all (s,r)-good graphs $\mathscr{H}^{(s,r)g}$ is closed under vertex splitting.

Proposition 1. *Let $H \in \mathscr{H}^{(s,r)g}$ and let $H' \in \mathscr{H}$ be obtained from H by splitting of type 1 of a vertex or by rotation of a non-critical edge. Then $H' \in \mathscr{H}^{(s,r)g}$. Moreover, if $H \in \mathscr{H}_A^{(s,r)g}$, then also $H' \in \mathscr{H}_A^{(s,r)g}$.*

Proof is straightforward.

Let $H \in \mathscr{H}^{(s,r)g}$. We say that H is (s,r)-*good of type B* if, for every sequence $H_1, \ldots, H_j$ such that $H_1 = H$ and H_{i+1} is obtained from H_i by splitting of type 1 a vertex or by rotation of an edge, either

(*i*) $H_i \in \mathscr{H}^{(s,r)g}$, $i = 1, \ldots, j$, or

(*ii*) there is a j_0, $1 \le j_0 \le j$, such that $H_i \in \mathscr{H}^{(s,r)g}$ for $i = 1, \ldots, j_0 - 1$ and $H_{j_0} \in \mathscr{H}_A^{(s,r)g}$.

We denote $\mathscr{H}_B^{(s,r)g} = \{H \in \mathscr{H}^{(s,r)g} |\ H$ is (s,r)-good of type B$\}$.

For example, if H is obtained from a cycle C_{11} with vertices labeled $a_1 \ldots a_{11}$ by attaching a pendant edge e at a_1 and by adding the chord a_3a_{10}, then $H \in \mathscr{H}^{(3,4)g}$, e is the critical edge and after rotation of e we obtain a graph $H' \in \mathscr{H}_A^{(3,4)g}$, hence $H \in \mathscr{H}_B^{(3,4)g}$.

Theorem 7 immediately implies the following corollary, summarizing properties of the classes $\mathscr{H}$, $\mathscr{H}^{(s,r)g}$, $\mathscr{H}_A^{(s,r)g}$, $\mathscr{H}_B^{(s,r)g}$.

Corollary 1. *Let G be a claw-free graph and let G_x^* be the local completion of G at a vertex $x \in V(G)$.*

(*i*) *If G is $L(\mathscr{H})$-free, then G_x^* is $L(\mathscr{H})$-free.*

(*ii*) *If G is $L(\mathscr{H}^{(s,r)g})$-free, then G_x^* is $L(\mathscr{H}^{(s,r)g})$-free.*

(*iii*) *If G is $L(\mathscr{H}_A^{(s,r)g})$-free, then G_x^* is $L(\mathscr{H}_A^{(s,r)g})$-free.*

(*iv*) *If G is $L(\mathscr{H}_A^{(s,r)g} \cup \mathscr{H}_B^{(s,r)g})$-free, then G_x^* is $L(\mathscr{H}_A^{(s,r)g} \cup \mathscr{H}_B^{(s,r)g})$-free.*

Proof. (*i*) If $L(H) \overset{\text{IND}}{\subset} G_x^*$ for some $H \in \mathscr{H}$, then Theorem 7 implies $L(H') \overset{\text{IND}}{\subset} G$, where H' is obtained by splitting of type 1 or 2 of a vertex.

The proofs of parts (*ii*), (*iii*) and (*iv*) are similar. □

In our proof we use some special subclass of the class $\mathscr{H}$.

Let C be a cycle of length $t \ge 4$ with $V(C) = \{z_1, \ldots, z_t\}$ and let $P_1, \ldots, P_t$ be t (not necessarily nontrivial) vertex-disjoint paths with $P_j = y_0^j y_1^j \ldots y_t^j$, $j = 1, \ldots, t$. The graph, obtained by identifying $z_j = y_0^j$, $j = 1, \ldots, t$, is called an $(i_1, \ldots, i_t)$-*t-sun*, or, briefly, a *t-sun* (see Figure 4). The cycle C is called the *disc* and the paths $P_1, \ldots, P_t$ the *beams* of

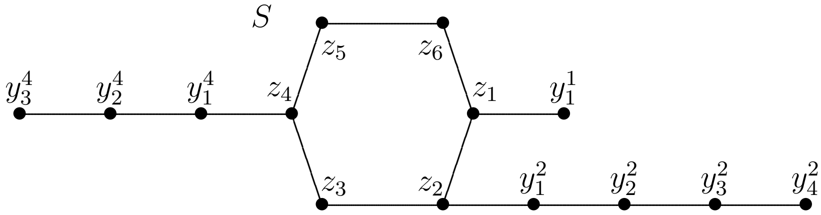

FIGURE 4. Good sun of type A

the sun, and for a beam P_j, the vertex z_j is called its *root*. The family of all suns will be denoted $\mathscr{S}$.

It is easy to verify that if $S \in \mathscr{S}_A$ and S' is obtained from S by splitting of a vertex or by rotation of an edge, then also $S' \in \mathscr{S}$. Now we define the classes of (s,r)-*good suns* and of (s,r)-*good suns of types A and B* as $\mathscr{S}^{(s,r)g} = \mathscr{S} \cap \mathscr{H}^{(s,r)g}$, $\mathscr{S}_A^{(s,r)g} = \mathscr{S} \cap \mathscr{H}_A^{(s,r)g}$ and $\mathscr{S}_B^{(s,r)g} = \mathscr{S} \cap \mathscr{H}_B^{(s,r)g}$, respectively.

For $S \in \mathscr{S}^{(s,r)g}$, the beam of S containing the center (i.e. the only vertex of degree 3) of the subgraph $Y = L^{-1}(B_{s,r})$ is called the *main beam* of S, and if an $S \in \mathscr{S}^{(s,r)g}$ contains more subgraphs that are isomorphic to Y, we will always suppose that Y is chosen such that the length of the main beam is maximum. For a sun with vertex set $\{x_1, \ldots, x_s\}$ we will use notation $S(x_1 \ldots x_s)$, where in the list of vertices we always list first the vertices of the disc, followed by lists of vertices of the beams, where the vertices of beams are ordered starting with the root and the beams are ordered in the order of the vertices of the disc; lists are separated with semicolons. If a sun is good, we always start the list of vertices of the disc with the root of the main beam (and, consequently, the main beam is also the first one in the list of beams). For example, the ((2,5)-good of type A) sun in Figure 4 will be denoted $S(z_4z_3z_2z_1z_6z_5; z_4y^4_1y^4_2y^4_3; z_2y^2_1y^2_2y^2_3y^2_4; z_1y^1_1)$.

Now we are ready to prove our main results.

2.1 $K_{1,3}B_{2,5}$-free graphs

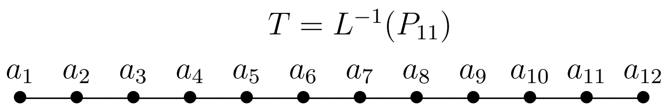

FIGURE 5. Preimage of P_{11}

Proposition 2. *Let G be a $K_{1,3}B_{2,5}$-free graph. Then* $\mathrm{cl}(G)$ *contains no induced subgraph F such that $L^{-1}(F) \in \mathscr{S}_A^{(2,5)g} \cup \mathscr{S}_B^{(2,5)g}$.*

Proof. If $F \overset{\mathrm{IND}}{\subset} \mathrm{cl}(G)$ is such that $L^{-1}(F) \in \mathscr{S}_A^{(2,5)g} \cup \mathscr{S}_B^{(2,5)g}$, then, by Corollary 1 and by induction, there is an $F' \overset{\mathrm{IND}}{\subset} G$ such that $L^{-1}(F') \in \mathscr{S}_A^{(2,5)g} \cup \mathscr{S}_B^{(2,5)g}$. Thus, $L^{-1}(F)$ contains a subgraph $L^{-1}(B_{2,5})$, contradicting the fact that G is $B_{2,5}$-free. □

Proposition 3. *Let G be a 3-connected closed claw-free graph. If G is not $K_{1,3}P_{11}$-free, then G contains an induced subgraph F such that $L^{-1}(F) \in \mathscr{S}_A^{(2,5)g} \cup \mathscr{S}_B^{(2,5)g}$.*

Proof. Let G be a 3-connected closed claw-free graph and suppose that G is not P_{11}-free. By Theorem 1, there is a graph $H = L^{-1}(G)$. Since G is not P_{11}-free, H contains a subgraph $T = L^{-1}(P_{11})$ (not necessarily induced). We will use the labeling of vertices of T as indicated in Figure 5. The graph G is 3-connected and hence H is essentially 3-edge-connected. Thus, the set $R = \{a_5a_6, a_7a_8\}$ cannot be a cut-set of H, implying there is a path $P = d_0d_1 \dots d_k$, $k \geq 1$, such that, up to symmetry, $d_0 = a_6$, $d_k \in \{a_1, a_2, a_3, a_4, a_5, a_8, a_9, a_{10}, a_{11}, a_{12}\}$ and $d_i \in V(H) \setminus V(T)$ for $1 \leq i \leq k-1$. We will show that in each of the possible cases H contains a sun $S \in \mathscr{S}_A^{(2,5)g} \cup \mathscr{S}_B^{(2,5)g}$. We list all possible cases in the following table in which the first two columns describe the case, in the third column we give minimum length of the path P which follows from the fact that H is triangle free, and in the last column we indicate the sun obtained in this case. In the following cases we obtain a sun S containing a copy of $L^{-1}(B_{2,5})$.

Case d_0	d_k	Min. k	The sun S containing a copy of $L^{-1}(B_{2,5})$	
a_6	a_1	1	$S(d_0 \dots d_ka_2a_3a_4a_5; d_0a_7 \dots a_{12}) \in \mathscr{S}_A^{(2,5)g}$	
a_6	a_2	1	$S(d_0 \dots d_ka_3a_4a_5; d_0a_7 \dots a_{12}) \in \mathscr{S}_A^{(2,5)g}$	
a_6	a_3	1	$S(d_0 \dots d_ka_4a_5; d_0a_7 \dots a_{12}; d_ka_2a_1) \in \mathscr{S}_A^{(2,5)g}$	
a_6	a_4	2	$S(d_0 \dots d_ka_5; d_0a_7 \dots a_{12}; d_ka_3a_2) \in \mathscr{S}_A^{(2,5)g}$	
a_6	a_5	3	$S(d_0 \dots d_k; d_0a_7 \dots a_{12}; d_ka_4a_3) \in \mathscr{S}_A^{(2,5)g}$	
a_6	a_8	2	$S(d_k \dots d_0a_7; d_ka_9a_{10}a_{11}; d_ka_5a_4a_3a_2a_1) \in \mathscr{S}_A^{(2,5)g}$	
a_6	a_9	1	$S(d_k \dots d_0a_8a_7; d_ka_{10}a_{11}a_{12}; d_ka_5a_4a_3a_2a_1) \in \mathscr{S}_A^{(2,5)g}$	
a_6	a_{10}	1	$S(d_k \dots d_0a_9a_8a_7; d_ka_{11}a_{12}; d_ka_5a_4a_3a_2a_1) \in \mathscr{S}_A^{(2,5)g}$	
a_6	a_{11}	1	$S(d_k \dots d_0a_{10}a_9a_8a_7; d_ka_{12}; d_ka_5a_4a_3a_2a_1) \in \mathscr{S}_B^{(2,5)g}$	
a_6	a_{12}	1	$S(d_k \dots d_0a_{11}a_{10}a_9a_8a_7; d_ka_5a_4a_3a_2a_1) \in \mathscr{S}_A^{(2,5)g}$	if $k \geq 2$

We note here that in the ninth case after rotation of its beam of length 1, we are in the tenth case.

The only remaining possibility is $d_0 = a_6, d_k = a_{12},\ k = 1$. We define $T' = (V(T'), E(T'))$ such that $V(T') = V(T),\ E(T') = E(T) \cup \{a_6a_{12}\}$. Again, since H is essentially 3-edge-connected, the set $R' = \{a_6a_7, a_8a_9\}$ cannot be a cut-set of H, implying there is a path $P' = d'_0d'_1 \dots d'_t$, $t \geq 1$, such that $d'_0 \in \{a_7, a_8\}$, $d'_t \in \{a_1, a_2, a_3, a_4, a_5, a_6, a_9, a_{10}, a_{11}, a_{12}\}$ and $d'_i \in V(H) \setminus V(T')$ for $1 \leq i \leq t-1$. Symmetrically as above, there is only one path beginning at $d'_0 = a_7$ and this path is $d'_0 = a_7, d'_t = a_1, t = 1$. There are the following possibilities.

Case d'_0	d'_t	Min. t	The sun S containing a copy of $L^{-1}(B_{2,5})$	
a_8	a_1	1	$S(a_6a_{12}a_{11}a_{10}a_9d'_0a_7; a_6a_5a_4a_3; d'_0\dots d'_ta_2) \in \mathscr{S}_A^{(2,5)g}$	
a_8	a_2	1	$S(a_6a_{12}a_{11}a_{10}a_9d'_0a_7; a_6a_5a_4a_3; d'_0\dots d'_ta_1) \in \mathscr{S}_A^{(2,5)g}$	
a_8	a_3	1	$S(d'_0a_9a_{10}a_{11}a_{12}a_6a_7; d'_0\dots d'_ta_2a_1; a_6a_5a_4) \in \mathscr{S}_A^{(2,5)g}$	
a_8	a_4	1	$S(d'_0a_9a_{10}a_{11}a_{12}a_6a_7; d'_0\dots d'_ta_3a_2a_1; a_6a_5) \in \mathscr{S}_A^{(2,5)g}$	
a_8	a_5	1	$S(d'_0a_9a_{10}a_{11}a_{12}a_6a_7; d'_0\dots d'_ta_4a_3a_2a_1) \in \mathscr{S}_A^{(2,5)g}$	if $t \geq 2$
a_8	a_6	2	$S(d'_t\dots d'_0a_7; d'_ta_5a_4a_3; d'_0a_9a_{10}a_{11}a_{12}) \in \mathscr{S}_A^{(2,5)g}$	
a_8	a_9	3	$S(d'_t\dots d'_0; d'_ta_{10}a_{11}a_{12}; d'_0a_7a_6a_5a_4a_3a_2a_1) \in \mathscr{S}_A^{(2,5)g}$	
a_8	a_{10}	2	$S(d'_0\dots d'_ta_9; d'_0a_7a_6a_5a_4a_3a_2; d'_ta_{11}a_{12}) \in \mathscr{S}_A^{(2,5)g}$	
a_8	a_{11}	1	$S(d'_0\dots d'_ta_{10}a_9; d'_0a_7a_6a_5a_4a_3a_2) \in \mathscr{S}_A^{(2,5)g}$	if $t \geq 2$
a_8	a_{12}	1	$S(d'_0\dots d'_ta_{11}a_{10}a_9; d'_0a_7a_6a_5a_4a_3a_2) \in \mathscr{S}_A^{(2,5)g}$	

The remaining possibilities are:

$$\begin{aligned} d'_0 &= a_7, \quad d'_t = a_1, \quad t = 1; \\ d'_0 &= a_8, \quad d'_t = a_{11}, \quad t = 1; \\ d'_0 &= a_8, \quad d_k = a_5, \quad t = 1. \end{aligned}$$

We consider these cases separately.

Case 1. $d'_0 = a_7,\ d'_t = a_1$ and $t = 1$. We define $T'' = (V(T''), E(T''))$ such that $V(T'') = V(T'),\quad E(T'') = E(T') \cup \{a_7a_1\}$. Since H is essentially 3-edge-connected, the set $R'' = \{a_2a_3, a_4a_5\}$ cannot be a cut-set of H, implying there is a path $P'' = d''_0d''_1\dots d''_s$, $s \geq 1$, such that $d''_0 \in \{a_3, a_4\}$, $d''_s \in \{a_1, a_2, a_5, a_6, a_7, a_8, a_9, a_{10}, a_{11}, a_{12}\}$ and $d''_i \in V(H) \setminus V(T'')$ for $1 \leq i \leq s-1$. There are the following possibilities.

Case d''_0	d''_s	Min. s	The sun S containing a copy of $L^{-1}(B_{2,5})$
a_3	a_1	2	$S(d''_0 \dots d''_s a_2; d''_0 a_4 a_5 a_6 a_{12} a_{11} a_{10}; d''_s a_7 a_8) \in \mathscr{S}_A^{(2,5)g}$
a_3	a_2	3	$S(d''_0 \dots d''_s; d''_0 a_4 a_5 a_6 a_{12} a_{11} a_{10}; d''_s a_1 a_7) \in \mathscr{S}_A^{(2,5)g}$
a_3	a_5	2	$S(d''_0 \dots d''_s a_4; d''_0 a_2 a_1 a_7; d''_s a_6 a_{12} a_{11} a_{10} a_9) \in \mathscr{S}_A^{(2,5)g}$
a_3	a_6	1	$S(d''_0 \dots d''_s a_5 a_4; d''_0 a_2 a_1 a_7; d''_s a_6 a_{12} a_{11} a_{10} a_9 a_8) \in \mathscr{S}_A^{(2,5)g}$
a_3	a_7	1	$S(d''_s \dots d''_0 a_2 a_1; d''_s a_8 a_9 a_{10} a_{11} a_{12} a_6; d''_0 a_4 a_5) \in \mathscr{S}_A^{(2,5)g}$
a_3	a_8	1	$S(d''_0 \dots d''_s a_9 a_{10} a_{11} a_{12} a_6 a_5 a_4; d''_0 a_2 a_1 a_7) \in \mathscr{S}_A^{(2,5)g}$
a_3	a_9	1	$S(d''_0 \dots d''_s a_{10} a_{11} a_{12} a_6 a_5 a_4; d''_0 a_2 a_1 a_7) \in \mathscr{S}_A^{(2,5)g}$
a_3	a_{10}	1	$S(d''_s \dots d''_0 a_4 a_5 a_6 a_7 a_8 a_9; d''_s a_{11} a_{12}; d''_0 a_2 a_1) \in \mathscr{S}_A^{(2,5)g}$
a_3	a_{11}	1	$S(d''_s \dots d''_0 a_4 a_5 a_6 a_7 a_8 a_8 a_{10}; d''_s a_{12}; d''_0 a_2 a_1) \in \mathscr{S}_B^{(2,5)g}$
a_3	a_{12}	1	$S(d''_0 \dots d''_s a_{11} a_{10} a_9 a_8 a_7 a_6 a_5 a_4; d''_0 a_2 a_1) \in \mathscr{S}_A^{(2,5)g}$
a_4	a_1	1	$S(d''_0 \dots d''_s a_2 a_3; d''_0 a_5 a_6 a_{12} a_{11} a_{10} a_9; d''_s a_7 a_8) \in \mathscr{S}_A^{(2,5)g}$
a_4	a_2	2	$S(d''_s \dots d''_0 a_3; d''_s a_1 a_7 a_8; d''_0 a_5 a_6 a_{12} a_{11} a_{10}) \in \mathscr{S}_A^{(2,5)g}$
a_4	a_5	3	$S(d''_0 \dots d''_s; d''_0 a_3 a_2 a_1 a_7 a_8 a_9; d''_s a_6 a_{12} a_{11}) \in \mathscr{S}_A^{(2,5)g}$
a_4	a_6	2	$S(d''_0 \dots d''_s a_5; d''_0 a_3 a_2 a_1 a_7 a_8 a_9; d''_s a_6 a_{12} a_{11}) \in \mathscr{S}_A^{(2,5)g}$
a_4	a_7	1	$S(d''_0 \dots d''_s a_6 a_5; d''_0 a_3 a_2 a_1; d''_s a_8 a_9 a_{10} a_{11} a_{12}) \in \mathscr{S}_A^{(2,5)g}$
a_4	a_8	1	$S(d''_0 \dots d''_s a_9 a_{10} a_{11} a_{12} a_6 a_5; d''_0 a_3 a_2 a_1) \in \mathscr{S}_A^{(2,5)g}$
a_4	a_9	1	$S(d''_0 a_3 a_2 a_1 a_7 a_6 a_5; d''_0 \dots d''_s a_{10} a_{11}; a_6 a_{12}) \in \mathscr{S}_A^{(2,5)g}$
a_4	a_{10}	1	$S(d''_0 a_3 a_2 a_1 a_7 a_6 a_5; d''_0 \dots d''_s a_9 a_8; a_6 a_{12}) \in \mathscr{S}_A^{(2,5)g}$
a_4	a_{11}	1	$S(d''_0 \dots d''_s a_{10} a_9 a_8 a_7 a_6 a_5; d''_0 a_3 a_2 a_1) \in \mathscr{S}_A^{(2,5)g}$
a_4	a_{12}	1	$S(d''_0 \dots d''_s a_{11} a_{10} a_9 a_8 a_7 a_6 a_5; d''_0 a_3 a_2 a_1) \in \mathscr{S}_A^{(2,5)g}$

Note that in the ninth case we obtain a sun S such that the rotation of its beam of length 1 results in a good sun S' with disc of length 10 and main beam of length 2.

Case 2. $d'_0 = a_8,\ d'_t = a_{11}$ and $t = 1$. We redefine $T'' = (V(T''), E(T''))$ such that $V(T'') = V(T'),\ \ E(T'') = E(T') \cup \{a_8 a_{11}\}$. Since H is essentially 3-edge-connected, the set $R'' = \{a_8 a_9, a_{10} a_{11}\}$ cannot be a cut-set of H, implying there is a path $P'' = d''_0 d''_1 \dots d''_s$, $s \geq 1$, such that $d''_0 \in \{a_9, a_{10}\}$, $d''_s \in \{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_{11}, a_{12}\}$ and $d''_i \in V(H) \setminus V(T'')$ for $1 \leq i \leq s-1$. There are the following possibilities.

Case d''_0	d''_s	Min. s	The sun S containing a copy of $L^{-1}(B_{2,5})$
a_9	a_1	1	$S(d''_0 \dots d''_s a_2a_3a_4a_5a_6a_7a_8; d''_s a_{10}a_{11}a_{12}) \in \mathscr{S}_A^{(2,5)g}$
a_9	a_2	1	$S(d''_0 \dots d''_s a_3a_4a_5a_6a_7a_8; d''_s a_{10}a_{11}a_{12}) \in \mathscr{S}_A^{(2,5)g}$
a_9	a_3	1	$S(d''_0 a_{10}a_{11}a_{12}a_6a_7a_8; d''_0 \dots d''_s a_2a_1; d''_s a_5a_4) \in \mathscr{S}_A^{(2,5)g}$
a_9	a_4	1	$S(d''_s \dots d''_0 a_8a_7a_6a_5; d''_s a_3a_2a_1; a_6a_{12}a_{11}a_{10}) \in \mathscr{S}_A^{(2,5)g}$
a_9	a_5	1	$S(a_{11}a_8d''_0 a_{10}; a_{11}a_{12}a_6a_7; d''_0 \dots d''_s a_4a_3a_2a_1) \in \mathscr{S}_A^{(2,5)g}$
a_9	a_6	1	$S(d''_0 a_8a_{11}a_{10}; d''_0 \dots d''_s a_5a_4a_3a_2a_1; a_{11}a_{12}) \in \mathscr{S}_A^{(2,5)g}$
a_9	a_7	2	$S(d''_0 \dots d''_s a_8; d''_0 a_{10}a_{11}a_{12}; d''_s a_6a_5a_4a_3a_2a_1) \in \mathscr{S}_A^{(2,5)g}$
a_9	a_8	3	$S(d''_0 \dots d''_s; d''_0 a_{10}a_{11}a_{12}; d''_s a_7a_6a_5a_4a_3a_2a_1) \in \mathscr{S}_A^{(2,5)g}$
a_9	a_{11}	2	$S(d''_0 \dots d''_s a_{10}; d''_0 a_8a_7a_6a_5a_4a_3a_2a_1; d''_s a_{12}) \in \mathscr{S}_A^{(2,5)g}$
a_9	a_{12}	1	$S(d''_s \dots d''_0 a_{10}a_{11}; d''_0 a_8a_7; d''_s a_6a_5a_4a_3a_2a_1) \in \mathscr{S}_A^{(2,5)g}$
a_{10}	a_1	1	$S(a_8a_{11}d''_0 a_9; a_8a_7a_6a_{12}; d''_0 \dots d''_s a_2a_3a_4a_5) \in \mathscr{S}_A^{(2,5)g}$
a_{10}	a_2	1	$S(a_8a_{11}d''_0 a_9; a_8a_7a_6a_{12}; d''_0 \dots d''_s a_3a_4a_5) \in \mathscr{S}_A^{(2,5)g}$
a_{10}	a_3	1	$S(a_6a_{12}a_{11}d''_0 a_9a_8a_7; a_6a_5a_4; d''_0 \dots d''_s a_2a_1) \in \mathscr{S}_A^{(2,5)g}$
a_{10}	a_4	1	$S(d''_s \dots d''_0 a_9a_8a_7a_6a_5; d''_s a_3a_2a_1; a_8a_{11}a_{12}) \in \mathscr{S}_A^{(2,5)g}$
a_{10}	a_5	1	$S(a_{11}a_8a_9d''_0; a_{11}a_{12}a_6a_7; d''_0 \dots d''_s a_4a_3a_2a_1) \in \mathscr{S}_A^{(2,5)g}$
a_{10}	a_6	1	$S(d''_0 \dots d''_s a_7a_8a_9; d''_0 a_{11}a_{12}; d''_s a_5a_4a_3a_2a_1) \in \mathscr{S}_A^{(2,5)g}$
a_{10}	a_7	1	$S(d''_s \dots d''_0 a_9a_8; d''_s a_6a_5a_4a_3a_2a_1; d''_0 a_{11}a_{12}) \in \mathscr{S}_A^{(2,5)g}$
a_{10}	a_8	2	$S(d''_s \dots d''_0 a_9; d''_s a_7a_6a_5a_4a_3a_2a_1; d''_0 a_{11}a_{12}) \in \mathscr{S}_A^{(2,5)g}$
a_{10}	a_{11}	3	$S(d''_s \dots d''_0; d''_s a_{12}a_6a_5a_4a_3a_2a_1; d''_0 a_9a_8a_7) \in \mathscr{S}_A^{(2,5)g}$
a_{10}	a_{12}	2	$S(d''_s \dots d''_0; d''_s a_6a_5a_4a_3a_2a_1; d''_0 a_9a_8) \in \mathscr{S}_A^{(2,5)g}$

Case 3. $d'_0 = a_8,\ d'_t = a_5$ and $t = 1$. We redefine $T'' = (V(T''), E(T''))$ such that $V(T'') = V(T'),\ E(T'') = E(T') \cup \{a_8a_5\}$. Since H is essentially 3-edge-connected, the set $R'' = \{a_2a_3, a_4a_5\}$ cannot be a cut-set of H, implying there is a path $P'' = d''_0 d''_1 \dots d''_s$, $s \geq 1$, such that $d''_0 \in \{a_3, a_4\}$, $d''_s \in \{a_1, a_2, a_5, a_6, a_7, a_8, a_9, a_{10}, a_{11}, a_{12}\}$ and $d''_i \in V(H) \setminus V(T'')$ for $1 \leq i \leq s-1$. There are the following possibilities.

Case d''_0	d''_s	Min. s	The sun S containing a copy of $L^{-1}(B_{2,5})$	
a_3	a_1	2	$S(a_6a_{12}a_{11}a_{10}a_9a_8a_7; a_6a_5a_4d''_0 \dots d''_sa_2) \in \mathscr{S}_A^{(2,5)g}$	
a_3	a_2	3	$S(a_6a_{12}a_{11}a_{10}a_9a_8a_7; a_6a_5a_4d''_0 \dots d''_sa_1) \in \mathscr{S}_A^{(2,5)g}$	
a_3	a_5	2	$S(d''_s \dots d''_0a_4; d''_sa_6a_7a_8a_9a_{10}a_{11}a_{12}; d''_0a_2a_1) \in \mathscr{S}_A^{(2,5)g}$	
a_3	a_6	1	$S(d''_s \dots d''_0a_4a_5; d''_sa_7a_8a_9a_{10}a_{11}a_{12}; d''_0a_2a_1) \in \mathscr{S}_A^{(2,5)g}$	
a_3	a_7	1	$S(d''_0 \dots d''_sa_6a_{12}a_{11}a_{10}a_9a_8a_5a_4; d''_0a_2a_1) \in \mathscr{S}_A^{(2,5)g}$	
a_3	a_8	1	$S(d''_sa_5a_6a_7; d''_s \dots d''_0a_2a_1; a_6a_{12}a_{11}a_{10}a_9) \in \mathscr{S}_A^{(2,5)g}$	
a_3	a_9	1	$S(d''_0 \dots d''_sa_8a_7a_6a_5a_4; d''_0a_2a_1; a_6a_{12}a_{11}a_{10}) \in \mathscr{S}_A^{(2,5)g}$	
a_3	a_{10}	1	$S(d''_0 \dots d''_sa_9a_8a_7a_6a_5a_4; d''_0a_2a_1; d''_sa_{11}a_{12}) \in \mathscr{S}_A^{(2,5)g}$	
a_3	a_{11}	1	$S(d''_s \dots d''_0a_4a_5a_6a_7a_8a_9a_{10}; d''_sa_{12}; d''_0a_2a_1) \in \mathscr{S}_B^{(2,5)g}$	
a_3	a_{12}	1	$S(d''_0 \dots d''_sa_{11}a_{10}a_9a_8a_7a_6a_5a_4; d''_0a_2a_1) \in \mathscr{S}_A^{(2,5)g}$	
a_4	a_1	1	$S(d''_0 \dots d''_sa_2a_3; d''_0a_5a_6a_7a_8a_9a_{10}) \in \mathscr{S}_A^{(2,5)g}$	if $s \geq 2$
a_4	a_2	2	$S(d''_0 \dots d''_sa_2a_3; d''_0a_5a_6a_7a_8a_9a_{10}; d''_sa_1) \in \mathscr{S}_A^{(2,5)g}$	
a_4	a_5	3	$S(d''_s \dots d''_0; d''_sa_6a_7a_8a_9a_{10}a_{11}; d''_0a_3a_2a_1) \in \mathscr{S}_A^{(2,5)g}$	
a_4	a_6	2	$S(d''_0 \dots d''_sa_5; d''_0a_3a_2a_1; d''_sa_7a_8a_9a_{10}a_{11}) \in \mathscr{S}_A^{(2,5)g}$	
a_4	a_7	1	$S(d''_0 \dots d''_sa_6a_5; d''_0a_3a_2a_1; a_5a_8a_9a_{10}a_{11}a_{12}) \in \mathscr{S}_A^{(2,5)g}$	
a_4	a_8	2	$S(d''_0 \dots d''_sa_7a_6a_5; d''_0a_3a_2a_1; d''_sa_9a_{10}a_{11}) \in \mathscr{S}_A^{(2,5)g}$	
a_4	a_9	1	$S(d''_0 \dots d''_sa_8a_5; d''_0a_3a_2a_1; a_8a_7a_6a_{12}a_{11}a_{10}) \in \mathscr{S}_A^{(2,5)g}$	
a_4	a_{10}	1	$S(d''_0 \dots d''_sa_9a_8a_5; d''_0a_3a_2a_1; a_8a_7a_6a_{12}a_{11}) \in \mathscr{S}_A^{(2,5)g}$	
a_4	a_{11}	1	$S(a_8a_5a_6a_7; a_8a_9a_{10}d''_s \dots d''_0a_3a_2a_1; a_6a_{12}) \in \mathscr{S}_A^{(2,5)g}$	
a_4	a_1	1	$S(d''_0 \dots d''_sa_{11}a_{10}a_9a_8a_7a_6a_5; d''_0a_3a_2a_1) \in \mathscr{S}_A^{(2,5)g}$	

The remaining possibility is $d''_0 = a_4, d''_s = a_1, s = 1$. We define $T^* = (V(T^*), E(T^*))$ such that $V(T^*) = V(T'')$, $E(T^*) = E(T'') \cup \{a_4a_1\}$. Since H is essentially 3-edge-connected, the set $R^* = \{a_8a_9, a_{10}a_{11}\}$ cannot be a cut-set of H, implying there is a path $P^* = d^*_0d^*_1 \dots d^*_m$, $m \geq 1$, such that $d^*_0 \in \{a_9, a_{10}\}$, $d^*_m \in \{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_{11}, a_{12}\}$ and $d^*_i \in V(H) \setminus V(T^*)$ for $1 \leq i \leq m-1$. There are the following possibilities.

Case d_0^*	Case d_m^*	Min. m	The sun S containing a copy of $L^{-1}(B_{2,5})$	
a_9	a_1	1	$S(d_m^* a_4a_3a_2; d_m^* \dots d_0^* a_{10}a_{11}; a_4a_5a_8a_7a_6a_{12}) \in \mathscr{S}_A^{(2,5)g}$	
a_9	a_2	1	$S(d_m^* a_1a_4a_3; d_m^* \dots d_0^* a_{10}a_{11}; a_4a_5a_8a_7a_6a_{12}) \in \mathscr{S}_A^{(2,5)g}$	
a_9	a_3	1	$S(d_0^* \dots d_m^* a_2a_1a_4a_5a_6a_7a_8; d_0^* a_{10}a_{11}a_{12}) \in \mathscr{S}_A^{(2,5)g}$	
a_9	a_4	1	$S(d_m^* \dots d_0^* a_8a_7a_6a_5; d_m^* a_3a_2a_1; a_6a_{12}a_{11}) \in \mathscr{S}_A^{(2,5)g}$	
a_9	a_5	2	$S(d_m^* \dots d_0^* a_8a_7a_6; d_m^* a_4a_3a_2a_1; d_0^* a_{10}a_{11}a_{12}) \in \mathscr{S}_A^{(2,5)g}$	
a_9	a_6	1	$S(d_m^* \dots d_0^* a_{10}a_{11}a_{12}; d_m^* a_7a_8a_5a_4a_3a_2a_1) \in \mathscr{S}_A^{(2,5)g}$	
a_9	a_7	2	$S(d_m^* \dots d_0^* a_8; d_0^* a_6a_5a_4a_3a_2a_1; d_0^* a_{10}a_{11}a_{12}) \in \mathscr{S}_A^{(2,5)g}$	
a_9	a_8	3	$S(d_m^* \dots d_0^*; d_0^* a_7a_6a_5a_4a_3a_2a_1; d_0^* a_{10}a_{11}a_{12}) \in \mathscr{S}_A^{(2,5)g}$	
a_9	a_{11}	2	$S(d_m^* \dots d_0^* a_{10}; d_m^* a_{12}a_6a_5a_4a_3a_2a_1; d_0^* a_8a_7) \in \mathscr{S}_A^{(2,5)g}$	
a_9	a_{12}	1	$S(d_m^* \dots d_0^* a_{10}a_{11}; d_m^* a_{12}a_6a_5a_4a_3a_2a_1; d_0^* a_8a_7) \in \mathscr{S}_A^{(2,5)g}$	
a_{10}	a_1	1	$S(d_0^* a_{11}a_{12}a_6a_7a_8a_9; d_m^* \dots d_0^* a_2a_3; a_8a_5a_4) \in \mathscr{S}_A^{(2,5)g}$	
a_{10}	a_2	1	$S(d_0^* a_{11}a_{12}a_6a_7a_8a_9; d_m^* \dots d_0^* a_2a_3; a_8a_5) \in \mathscr{S}_A^{(2,5)g}$	
a_{10}	a_3	1	$S(d_0^* a_{11}a_{12}a_6a_7a_8a_9; d_m^* \dots d_0^* a_2a_1; a_8a_5a_4) \in \mathscr{S}_A^{(2,5)g}$	
a_{10}	a_4	1	$S(d_0^* a_{11}a_{12}a_6a_7a_8a_9; d_m^* \dots d_0^* a_3a_2a_1; a_8a_5) \in \mathscr{S}_A^{(2,5)g}$	
a_{10}	a_5	1	$S(d_0^* \dots d_m^* a_6a_7a_8a_9; d_0^* a_{11}a_{12}; d_m^* a_4a_3a_2a_1) \in \mathscr{S}_A^{(2,5)g}$	if $m \geq 2$
a_{10}	a_6	1	$S(d_0^* \dots d_m^* a_7a_8a_9; d_0^* a_{11}a_{12}; d_m^* a_5a_4a_3a_2a_1) \in \mathscr{S}_A^{(2,5)g}$	
a_{10}	a_7	1	$S(d_m^* \dots d_0^* a_9a_8; d_m^* a_6a_5a_4a_3a_2a_1; d_0^* a_{11}a_{12}) \in \mathscr{S}_A^{(2,5)g}$	
a_{10}	a_8	2	$S(d_m^* \dots d_0^* a_9; d_m^* a_6a_5a_4a_3a_2a_1; d_0^* a_{11}a_{12}) \in \mathscr{S}_A^{(2,5)g}$	
a_{10}	a_{11}	3	$S(d_m^* \dots d_0^*; d_m^* a_{12}a_6a_5a_4a_3a_2a_1; d_0^* a_9a_8a_7) \in \mathscr{S}_A^{(2,5)g}$	
a_{10}	a_{12}	1	$S(d_m^* \dots d_0^* a_{11}; d_m^* a_6a_5a_4a_3a_2a_1; d_0^* a_9a_8a_7) \in \mathscr{S}_A^{(2,5)g}$	

The remaining possibility is $d_0^* = a_{10}, d_m^* = a_5, m = 1$. We define $T^{**} = (V(T^{**}), E(T^{**}))$ such that $V(T^{**}) = V(T^*)$, $E(T^{**}) = E(T^*) \cup \{a_{10}a_5\}$. Since H is essentially 3-edge-connected, the set $R^* = \{a_{10}a_{11}, a_{12}a_6\}$ cannot be a cut-set of H, implying there is a path $P^{**} = d_0^{**}d_1^{**} \dots d_n^{**}, n \geq 1$, such that $d_0^{**} \in \{a_{11}, a_{12}\}$, $d_m^{**} \in \{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}\}$ and $d_i^{**} \in V(H) \setminus V(T^{**})$ for $1 \leq i \leq n-1$. There are the following possibilities.

Case d_0^{**}	Case d_n^{**}	Min. n	The sun S containing a copy of $L^{-1}(B_{2,5})$
a_{11}	a_1	1	$S(d_0^{**}a_{12}a_6a_7a_8a_9a_{10}; d_0^{**}\dots d_n^{**}a_2a_3; a_6a_5a_4) \in \mathscr{S}_A^{(2,5)g}$
a_{11}	a_2	1	$S(d_0^{**}a_{12}a_6a_7a_8a_9a_{10}; d_0^{**}\dots d_n^{**}a_3a_4; a_6a_5) \in \mathscr{S}_A^{(2,5)g}$
a_{11}	a_3	1	$S(d_0^{**}a_{12}a_6a_7a_8a_9a_{10}; d_0^{**}\dots d_n^{**}a_2a_1; a_6a_5a_4) \in \mathscr{S}_A^{(2,5)g}$
a_{11}	a_4	1	$S(d_0^{**}a_{12}a_6a_7a_8a_9a_{10}; d_0^{**}\dots d_n^{**}a_3a_2a_1; a_6a_5) \in \mathscr{S}_A^{(2,5)g}$
a_{11}	a_5	2	$S(d_n^{**}\dots d_0^{**}a_{10}a_9a_8a_7a_6; d_n^{**}a_4a_3a_2a_1;) \in \mathscr{S}_A^{(2,5)g}$
a_{11}	a_6	2	$S(d_0^{**}\dots d_n^{**}a_{12}; d_0^{**}a_{10}a_9a_8a_7; d_0^{**}a_5a_4a_3a_2a_1) \in \mathscr{S}_A^{(2,5)g}$
a_{11}	a_7	1	$S(d_0^{**}\dots d_n^{**}a_6a_{12}; d_0^{**}a_{10}a_9a_8; d_0^{**}a_6a_5a_4a_3a_2a_1) \in \mathscr{S}_A^{(2,5)g}$
a_{11}	a_8	1	$S(d_0^{**}\dots d_n^{**}a_7a_6a_{12}; d_0^{**}a_{10}a_5a_4a_3a_2a_1) \in \mathscr{S}_A^{(2,5)g}$
a_{11}	a_9	2	$S(d_0^{**}\dots d_n^{**}a_8a_7a_6a_{12}; d_0^{**}a_{10}a_5a_4a_3a_2a_1) \in \mathscr{S}_A^{(2,5)g}$
a_{11}	a_{10}	3	$S(d_n^{**}\dots d_0^{**}a_{12}a_6a_7a_8a_9; d_n^{**}a_5a_4a_3a_2) \in \mathscr{S}_A^{(2,5)g}$
a_{12}	a_1	1	$S(d_n^{**}a_4a_3a_2; d_n^{**}\dots d_0^{**}a_{11}a_{10}a_9a_8a_7; a_4a_5a_6) \in \mathscr{S}_A^{(2,5)g}$
a_{12}	a_2	1	$S(d_n^{**}a_1a_4a_3; d_n^{**}\dots d_0^{**}a_{11}a_{10}a_9a_8a_7; a_4a_5a_6) \in \mathscr{S}_A^{(2,5)g}$
a_{12}	a_3	1	$S(d_n^{**}a_4a_1a_2; d_n^{**}\dots d_0^{**}a_{11}a_{10}a_9a_8a_7; a_4a_5a_6) \in \mathscr{S}_A^{(2,5)g}$
a_{12}	a_4	1	$S(d_n^{**}\dots d_0^{**}a_{11}a_{10}a_9a_8a_7a_6a_5; d_n^{**}a_3a_2a_1) \in \mathscr{S}_A^{(2,5)g}$
a_{12}	a_5	1	$S(d_n^{**}\dots d_0^{**}a_{11}a_{10}a_9a_8a_7a_6; d_n^{**}a_4a_3a_2a_1) \in \mathscr{S}_A^{(2,5)g}$
a_{12}	a_6	3	$S(d_n^{**}\dots d_0^{**}a_{11}a_{10}a_9a_8a_7; d_n^{**}a_5a_4a_3a_2) \in \mathscr{S}_A^{(2,5)g}$
a_{12}	a_7	2	$S(d_n^{**}\dots d_0^{**}a_{11}a_{10}a_9a_8; d_n^{**}a_6a_5a_4a_3a_2a_1) \in \mathscr{S}_A^{(2,5)g}$
a_{12}	a_8	1	$S(d_n^{**}\dots d_0^{**}a_{11}a_{10}a_9; d_n^{**}a_7a_6a_5a_4a_3a_2a_1) \in \mathscr{S}_A^{(2,5)g}$
a_{12}	a_9	1	$S(d_0^{**}\dots d_n^{**}a_8a_7a_6; d_0^{**}a11a_{10}a_5a_4a_3a_2a_1) \in \mathscr{S}_A^{(2,5)g}$
a_{12}	a_{10}	1	$S(d_n^{**}\dots d_0^{**}a_6a_7a_8a_9; d_n^{**}a_{11}; a_6a_5a_4a_3a_2) \in \mathscr{S}_B^{(2,5)g}$

We note here that in the case $d_0^{**} = a_{12}, d_n^{**} = a_{10}$ we obtain a sun S such that the rotation of its beam of length 1 results in a (2,5)-good-sun S' with disc of length 8 and main beam of length 4.

In each of the possible cases, we showed there is a sun $S \in \mathscr{S}_A^{(2,5)g} \cup \mathscr{S}_B^{(2,5)g}$. We proved Proposition 3.

□

Now we can prove the following theorem.

Theorem 9. *Let G be a 3-connected $K_{1,3}B_{2,5}$-free graph. Then $\mathrm{cl}(G)$ is P_{11}-free.*

Proof. Proposition 1 implies that the class of all (2,5)-good graphs $\mathscr{H}^{(2,5)g}$ is split-closed. By the definition of $\mathscr{H}^{(2,5)g}$, every $H \in \mathscr{H}^{(2,5)g}$ contains a subgraph isomorphic to $L^{-1}(B_{2,5})$. Propositions 2 and 3 imply that every closed 3-connected $K_{1,3}B_{2,5}$-free graph is $K_{1,3}P_{11}$-free. Then, the class $\mathscr{H}^{(2,5)g}$ is a $(3, B_{2,5}, P_{11}, \mathrm{cl})$-stabilizer. Theorem 8 immediately implies that $\mathrm{cl}(G)$ is P_{11}-free. □

Corollary 2. *Every 3-connected $K_{1,3}B_{2,5}$-free graph is hamiltonian.*

2.2 $K_{1,3}B_{3,4}$-free graphs

Proposition 4. *Let G be a $K_{1,3}B_{3,4}$-free graph. Then* $\mathrm{cl}(G)$ *contains no induced subgraph F such that $L^{-1}(F) \in \mathscr{H}_A^{(3,4)g} \cup \mathscr{H}_B^{(3,4)g}$.*

Proof. If $F \overset{\mathrm{IND}}{\subset} \mathrm{cl}(G)$ is such that $L^{-1}(F) \in \mathscr{H}_A^{(3,4)g} \cup \mathscr{H}_B^{(3,4)g}$, then, by Corollary 1 and by induction, there is an $F' \overset{\mathrm{IND}}{\subset} G$ such that $L^{-1}(F') \in \mathscr{H}_A^{(3,4)g} \cup \mathscr{H}_B^{(3,4)g}$. Thus, $L^{-1}(F)$ contains a subgraph $L^{-1}(B_{3,4})$, contradicting the fact that G is $B_{3,4}$-free. □

Proposition 5. *Let G be a 3-connected closed claw-free graph. If G is not $K_{1,3}P_{11}$-free, then G contains an induced subgraph F such that $L^{-1}(F) \in \mathscr{H}_A^{(3,4)g} \cup \mathscr{H}_B^{(3,4)g}$.*

Proof of Proposition 5 is postponed to Subsection 2.3. □

Now we will prove that the closure of a 3-connected $K_{1,3}B_{3,4}$-free graph must be P_{11}-free.

Theorem 10. *Let G be a 3-connected $K_{1,3}B_{3,4}$-free graph. Then* $\mathrm{cl}(G)$ *is P_{11}-free.*

Proof. We will show that the class of all 3-good graphs $\mathscr{H}^{(3,4)g}$ is a 3-stabilizer for $B_{3,4}$ into P_{11} under closure cl. Proposition 1 implies that $\mathscr{H}^{(3,4)g}$ is split-closed. By the definition of $\mathscr{H}^{(3,4)g}$, every $H \in \mathscr{H}^{(3,4)g}$ contains a subgraph isomorphic to $L^{-1}(B_{3,4})$. Propositions 4 and 5 implies that every closed 3-connected $K_{1,3}B_{3,4}$-free graph is $K_{1,3}P_{11}$-free. Then, the class $\mathscr{H}^{(3,4)g}$ is a $(3, B_{3,4}, P_{11}, \mathrm{cl})$-stabilizer. Theorem 8 immediately implies that $\mathrm{cl}(G)$ is P_{11}-free.

□

Corollary 3. *Every 3-connected $K_{1,3}B_{3,4}$-free graph is hamiltonian.*

2.3 Proof of Proposition 5

Proof. Let G be a 3-connected closed claw-free graph and suppose that G is not P_{11}-free. By Theorem 1, there is a graph $H = L^{-1}(G)$. Since G is not P_{11}-free, H contains a subgraph $T = L^{-1}(P_{11})$ (not necessarily induced). We will use the labeling of vertices of T as indicated in Figure 5. The graph G is 3-connected and hence H is essentially 3-edge-connected. Thus, the set $R = \{a_5a_6, a_7a_8\}$ cannot be a cut-set of H, implying there is a path $P = d_0d_1 \dots d_k$, $k \geq 1$, such that, up to symmetry, $d_0 = a_6$, $d_k \in \{a_1, a_2, a_3, a_4, a_5, a_8, a_9, a_{10}, a_{11}, a_{12}\}$ and $d_i \in V(H) \setminus V(T)$ for $1 \leq i \leq k-1$. We will show that in each of the possible cases H contains either a sun $S \in \mathscr{S}_A^{(3,4)g} \cup \mathscr{S}_B^{(3,4)g}$ or $W \in \mathscr{H}_A^{(3,4)g} \cup \mathscr{H}_B^{(3,4)g}$. We list all possible cases in the following table in which the first two columns describe the case, in the third column we give minimum length of the path P which follows from the fact that H is triangle free, and in the last column we indicate the sun obtained in this case. In the following cases we obtain a sun S containing a copy of $L^{-1}(B_{3,4})$.

Case d_0	d_k	Min. k	The sun S containing a copy of $L^{-1}(B_{3,4})$	
a_6	a_1	1	$S(d_0 \dots d_k a_2 a_3 a_4 a_5; d_0 a_7 \dots a_{11}) \in \mathscr{S}_A^{(3,4)g}$	
a_6	a_2	1	$S(d_0 \dots d_k a_3 a_4 a_5; d_0 a_7 \dots a_{11}) \in \mathscr{S}_A^{(3,4)g}$	if $k \geq 2$
a_6	a_3	1	$S(d_0 \dots d_k a_4 a_5; d_0 a_7 \dots a_{11}; d_k a_2 a_1) \in \mathscr{S}_A^{(3,4)g}$	if $k \geq 2$
a_6	a_4	2	$S(d_0 \dots d_k a_5; d_0 a_7 \dots a_{11}; d_k a_3 a_2) \in \mathscr{S}_A^{(3,4)g}$	
a_6	a_5	3	$S(d_0 \dots d_k; d_0 a_7 \dots a_{11}; d_k a_4 a_3 a_2) \in \mathscr{S}_A^{(3,4)g}$	
a_6	a_8	2	$S(d_0 \dots d_k a_7; d_0 a_5 a_4 a_3 a_2 a_1; d_k a_9 a_{10}) \in \mathscr{S}_A^{(3,4)g}$	
a_6	a_9	1	$S(d_0 \dots d_k a_8 a_7; d_0 a_5 a_4 a_3 a_2 a_1; d_k a_{10} a_{11}) \in \mathscr{S}_A^{(3,4)g}$	
a_6	a_{10}	1	$S(d_0 \dots d_k a_9 a_8 a_7; d_0 a_5 a_4 a_3 a_2 a_1; d_k a_{11} a_{12}) \in \mathscr{S}_A^{(3,4)g}$	if $k \geq 2$
a_6	a_{11}	1	$S(d_0 \dots d_k a_{10} a_9 a_8 a_7; d_0 a_5 a_4 a_3 a_2 a_1) \in \mathscr{S}_A^{(3,4)g}$	
a_6	a_{11}	1	$S(d_0 \dots d_k a_{11} a_{10} a_9 a_8 a_7; d_0 a_5 a_4 a_3 a_2 a_1) \in \mathscr{S}_A^{(3,4)g}$	

The remaining possibilities are:

$$d_0 = a_6, \quad d_k = a_2, \quad k = 1;$$
$$d_0 = a_6, \quad d_k = a_{10}, \quad k = 1;$$
$$d_0 = a_6, \quad d_k = a_3, \quad k = 1.$$

We consider these cases separately.

Case 1. $d_0 = a_6,\ d_k = a_2$ and $k = 1$. We define $T' = (V(T'), E(T'))$ such that $V(T') = V(T), \quad E(T') = E(T) \cup \{a_6 a_2\}$. Since H is essentially 3-edge-connected, the set $R' = \{a_2 a_3, a_4 a_5\}$ cannot be a cut-set of H, implying there is a path $P' = d'_0 d'_1 \dots d'_t,\ t \geq 1$, such that $d'_0 \in \{a_3, a_4\}$, $d'_t \in \{a_1, a_2, a_5, a_6, a_7, a_8, a_9, a_{10}, a_{11}, a_{12}\}$ and $d'_i \in V(H) \setminus V(T')$ for $1 \leq i \leq t-1$. There are the following possibilities.

Case d'_0	d'_t	Min. t	The sun S containing a copy of $L^{-1}(B_{3,4})$	
a_3	a_1	2	$S(a_6a_5a_4d'_0\dots d'_ta_2; a_6a_7a_8a_9a_{10}a_{11}) \in \mathscr{S}_A^{(3,4)g}$	
a_3	a_2	3	$S(a_6a_5a_4d'_0\dots d'_t; a_6a_7a_8a_9a_{10}a_{11}) \in \mathscr{S}_A^{(3,4)g}$	
a_3	a_5	2	$S(a_6d'_t\dots d'_0a_2; a_6a_7a_8a_9a_{10}a_{11}; d'_0a_4) \in \mathscr{S}_A^{(3,4)g}$	
a_3	a_6	2	$S(d'_t\dots d'_0a_4a_5; d'_ta_7a_8a_9a_{10}a_{11}; d'_0a_2a_1) \in \mathscr{S}_A^{(3,4)g}$	
a_3	a_7	1	$S(d'_t\dots d'_0a_4a_5a_6; d'_ta_8a_9a_{10}a_{11}a_{12}) \in \mathscr{S}_A^{(3,4)g}$	if $k \geq 2$
a_3	a_8	1	$S(d'_t\dots d'_0a_2a_6a_7; d'_ta_9a_{10}a_{11}a_{12}; a_6a_5a_4) \in \mathscr{S}_A^{(3,4)g}$	
a_3	a_9	1	$S(d'_0\dots d'_ta_8a_7a_6a_5a_4; d'_0a_2a_1; d'_ta_{10}a_{11}a_{12}) \in \mathscr{S}_A^{(3,4)g}$	
a_3	a_{10}	1	$S(d'_0\dots d'_ta_9a_8a_7a_6a_5a_4; d'_0a_2a_1; d'_ta_{11}a_{12}) \in \mathscr{S}_A^{(3,4)g}$	if $k \geq 2$
a_3	a_{11}	1	$S(a_6a_5a_4d'_0\dots d'_ta_{10}a_9a_8a_7; a_6a_2a_1; d'_ta_{12}) \in \mathscr{S}_A^{(3,4)g}$	
a_3	a_{12}	1	$S(a_6a_5a_4d'_0\dots d'_ta_{10}a_9a_8a_7; a_6a_2a_1; d'_ta_{12}) \in \mathscr{S}_A^{(3,4)g}$	
a_4	a_1	1	$S(a_6a_2a_3d'_0a_5; a_6a_7a_8a_9a_{10}a_{11}; d'_0d'_1) \in \mathscr{S}_A^{(3,4)g}$	
a_4	a_2	2	$S(a_6a_2a_3d'_0a_5; a_6a_7a_8a_9a_{10}a_{11}; d'_0d'_1) \in \mathscr{S}_A^{(3,4)g}$	
a_4	a_5	3	$S(a_6a_2a_3d'_0\dots d'_t; a_6a_7a_8a_9a_{10}a_{11}) \in \mathscr{S}_A^{(3,4)g}$	
a_4	a_6	2	$S(d'_t\dots d'_0a_5; d'_ta_7a_8a_9a_{10}a_{11}; d'_0a_3a_2a_1) \in \mathscr{S}_A^{(3,4)g}$	
a_4	a_7	1	$S(d'_t\dots d'_0a_5a_6; d'_ta_8a_9a_{10}a_{11}a_{12}; d'_0a_3a_2a_1) \in \mathscr{S}_A^{(3,4)g}$	
a_4	a_8	1	$S(d'_0a_3a_2a_6a_5; d'_0\dots d'_ta_9a_{10}a_{11}a_{12}; a_6a_7) \in \mathscr{S}_A^{(3,4)g}$	
a_4	a_9	1	$S(d'_0a_3a_2a_6a_5; d'_0\dots d'_ta_{10}a_{11}a_{12}; a_6a_7a_8) \in \mathscr{S}_A^{(3,4)g}$	
a_4	a_{10}	1	$S(d'_t\dots d'_0a_4a_5a_6a_7a_8a_9; d'_ta_{11}a_{12}; d'_0a_3a_2a_1) \in \mathscr{S}_A^{(3,4)g}$	
a_4	a_{11}	1	$S(a_6a_2a_3a_4a_5; a_6a_7a_8a_9a_{10}; d'_0\dots d'_ta_{12}) \in \mathscr{S}_A^{(3,4)g}$	
a_4	a_{11}	1	$S(a_6a_2a_3a_4a_5; a_6a_7a_8a_9a_{10}a_{11}; d'_0\dots d'_t) \in \mathscr{S}_A^{(3,4)g}$	

The remaining possibilities are:

$$d'_0 = a_3, \quad d'_t = a_7, \quad t = 1;$$
$$d'_0 = a_3, \quad d'_t = a_{10}, \quad t = 1.$$

Subcase 1.1 $d'_0 = a_3,\ d'_t = a_7$ and $t = 1$. We set $T'' = (V(T''), E(T''))$ such that $V(T'') = V(T'),\ \ E(T') = E(T'') \cup \{a_3a_7\}$. Since H is essentially 3-edge-connected, the set $R'' = \{a_3a_4, a_5a_6\}$ cannot be a cut-set of H, implying there is a path $P'' = d''_0d''_1\dots d''_s$, $s \geq 1$, such that $d''_0 \in \{a_4, a_5\}$, $d''_s \in \{a_1, a_2, a_3, a_6, a_7, a_8, a_9, a_{10}, a_{11}, a_{12}\}$ and $d''_i \in V(H) \setminus V(T'')$ for $1 \leq i \leq s-1$. Symmetrically as above, there is no path such that $d''_0 = a_4$ and $d''_s = a_i,\ i = 1, 2, 3, 6, 7, \dots, 12$, thus, all possible paths begin at $d''_0 = a_5$. There are the following possibilities.

Case d''_0	d''_s	Min. s	The sun S containing a copy of $L^{-1}(B_{3,4})$
a_5	a_1	1	$S(a_7a_3a_4d''_0a_6; a_7a_8a_9a_{10}a_{11}a_{12}; d''_0 \dots d''_s a_2) \in \mathscr{S}_A^{(3,4)g}$
a_5	a_2	2	$S(a_7a_3a_4d''_0a_6; a_7a_8a_9a_{10}a_{11}a_{12}; d''_0 \dots d''_s a_1) \in \mathscr{S}_A^{(3,4)g}$
a_5	a_3	2	$S(a_7a_3a_4d''_0a_6; a_7a_8a_9a_{10}a_{11}a_{12}; d''_0 \dots d''_1 \in \mathscr{S}_A^{(3,4)g}$
a_5	a_6	3	$S(a_7a_3a_4d''_0 \dots d''_s; a_7a_8a_9a_{10}a_{11}a_{12} \in \mathscr{S}_A^{(3,4)g}$
a_5	a_7	2	$S(d''_s \dots d''_0a_6; d''_sa_8a_9a_{10}a_{11}a_{12}; d''_0a_4a_3a_2 \in \mathscr{S}_A^{(3,4)g}$
a_5	a_8	1	$S(d''_s \dots d''_0a_6a_7; d''_sa_9a_{10}a_{11}a_{12}; a_6a_2a_3a_4 \in \mathscr{S}_A^{(3,4)g}$
a_5	a_9	1	$S(d''_0a_6a_2a_3a_4; d''_0 \dots d''_sa_{10}a_{11}a_{12}; a_3a_7a_8 \in \mathscr{S}_A^{(3,4)g}$
a_5	a_{10}	1	$S(d''_s \dots d''_0a_6a_7a_8a_9; d''_sa_{11}a_{12}; d''_0a_4a_3a_2a_1 \in \mathscr{S}_A^{(3,4)g}$
a_5	a_{11}	1	$S(d''_0 \dots d''_sa_{10}a_9a_8a_7a_6; d''_0a_4a_3a_2a_1 \in \mathscr{S}_A^{(3,4)g}$
a_5	a_{12}	1	$S(d''_0 \dots d''_sa_{11}a_{10}a_9a_8a_7a_6; d''_0a_4a_3a_2a_1 \in \mathscr{S}_A^{(3,4)g}$

Subcase 1.2 $d'_0 = a_3,\ d'_t = a_{10}$ and $t = 1$. We redefine $T'' = (V(T''), E(T''))$ such that $V(T'') = V(T'),\quad E(T'') = E(T') \cup \{a_3a_{10}\}$. Since H is essentially 3-edge-connected, the set $R'' = \{a_3a_4, a_5a_6\}$ cannot be a cut-set of H, implying there is a path $P'' = d''_0d''_1 \dots d''_s$, $s \geq 1$, such that $d''_0 \in \{a_4, a_5\}$, $d''_s \in \{a_1, a_2, a_3, a_6, a_7, a_8, a_9, a_{10}, a_{11}, a_{12}\}$ and $d''_i \in V(H) \setminus V(T'')$ for $1 \leq i \leq s-1$. By Case 1, there is no path beginning at $d''_0 = a_4$. Thus all possible paths begin at $d''_0 = a_5$. There are the following possibilities.

Case d''_0	d''_s	Min. s	The sun S containing a copy of $L^{-1}(B_{3,4})$	
a_5	a_1	1	$S(a_{10}a_3a_4d''_0a_6a_7a_8a_9; a_{10}a_{11}a_{12}; d''_0 \dots d''_sa_2) \in \mathscr{S}_A^{(3,4)g}$	
a_5	a_2	2	$S(a_{10}a_3a_4d''_0a_6a_7a_8a_9; a_{10}a_{11}a_{12}; d''_0 \dots d''_sa_1) \in \mathscr{S}_A^{(3,4)g}$	
a_5	a_3	2	$S(a_{10}a_3a_4d''_0a_6a_7a_8a_9; a_{10}a_{11}a_{12}; d''_0 \dots d''_1 \in \mathscr{S}_A^{(3,4)g}$	
a_5	a_6	3	$S(d''_s \dots d''_0; d''_sa_7a_8a_9a_{10}a_{11}; d''_0a_4a_3a_2 \in \mathscr{S}_A^{(3,4)g}$	
a_5	a_7	2	$S(d''_s \dots d''_0a_6; d''_sa_8a_9a_{10}a_{11}a_{12}; d''_0a_4a_3a_2 \in \mathscr{S}_A^{(3,4)g}$	
a_5	a_8	1	$S(d''_s \dots d''_0a_6a_7; d''_sa_9a_{10}a_{11}a_{12}; d''_0a_4a_3a_2a_1 \in \mathscr{S}_A^{(3,4)g}$	
a_5	a_9	1	$S(a_{10}d''_s \dots d''_0a_4a_3; a_{10}a_{11}a_{12}; a_3a_2a_6a_7a_8) \in \mathscr{H}_A^{(3,4)g}$	if $s \geq 2$
a_5	a_{10}	1	$S(d''_s \dots d''_0a_6a_7a_8a_9; d''_sa_{11}a_{12}; d''_0a_4a_3a_2a_1 \in \mathscr{S}_A^{(3,4)g}$	
a_5	a_{11}	1	$S(d''_0 \dots d''_sa_{10}a_9a_8a_7a_6; d''_0a_4a_3a_2a_1 \in \mathscr{S}_A^{(3,4)g}$	
a_5	a_{12}	1	$S(d''_0 \dots d''_sa_{11}a_{10}a_9a_8a_7a_6; d''_0a_4a_3a_2a_1 \in \mathscr{S}_A^{(3,4)g}$	

We note here that in the case $d''_0 = a_5,\ d''_s = a_9,\quad s = 1$ we define a good $H = (V(H), E(H)) \in \mathscr{H}_B^{(3,4)g}$ as follows: $V(H) = V(T''),\ E(H) = E(T'') \cup \{a_5a_9\}$ and the critical edge is a_2a_1. After rotation of its critical edge $H' \in \mathscr{H}_A^{(3,4)g}$.

Case 2. $d_0 = a_6,\ d_k = a_{10}$ and $k = 1$. We define $T' = (V(T'), E(T'))$ such that $V(T') = V(T),\quad E(T') = E(T) \cup \{a_6a_{10}\}$. Since H is essentially 3-edge-connected, the set $R' = \{a_6a_7, a_8a_9\}$ cannot be a cut-set of H, implying there is a path $P' = d'_0d'_1 \dots d'_t$, $t \geq 1$, such that $d'_0 \in \{a_7, a_8\}$, $d'_t \in \{a_1, a_2, a_3, a_4, a_5, a_6, a_9, a_{10}, a_{11}, a_{12}\}$ and $d'_i \in V(H) \setminus V(T')$ for $1 \le i \le t-1$. There are the following possibilities.

Case d'_0	Case d'_t	Min. t	The sun S containing a copy of $L^{-1}(B_{3,4})$
a_8	a_1	1	$S(d'_0a_9a_{10}a_6a_5; d'_0\dots d'_ta_2a_3a_4; a_{10}a_{11}a_{12}) \in \mathscr{S}_A^{(3,4)g}$
a_8	a_2	1	$S(d'_0a_9a_{10}a_6a_5; d'_0\dots d'_ta_3a_4a_5; a_{10}a_{11}a_{12}) \in \mathscr{S}_A^{(3,4)g}$
a_8	a_3	1	$S(d'_t\dots d'_0a_7a_6a_5a_4; d'_0a_2a_1; d'_0a_9a_{10}a_{11}a_{12}) \in \mathscr{S}_A^{(3,4)g}$
a_8	a_4	1	$S(d'_0a_9a_{10}a_6a_5; d'_0\dots d'_ta_3a_2a_1; a_{10}a_{11}a_{12}) \in \mathscr{S}_A^{(3,4)g}$
a_8	a_5	1	$S(d'_0a_9a_{10}a_6a_5; d'_0\dots d'_ta_4a_3a_2a_1; a_{10}a_{11}a_{12}) \in \mathscr{S}_A^{(3,4)g}$
a_8	a_6	2	$S(d'_t\dots d'_0a_7; d'_ta_5a_4a_3a_2a_1; d'_0a_9a_{10}) \in \mathscr{S}_A^{(3,4)g}$
a_8	a_9	3	$S(d'_0\dots d'_t; d'_0a_7a_6a_5a_4a_3; d'_ta_{10}a_{11}a_{12}) \in \mathscr{S}_A^{(3,4)g}$
a_8	a_{10}	2	$S(d'_0\dots d'_ta_9; d'_0a_7a_6a_5a_4a_3; d'_ta_{11}a_{12}) \in \mathscr{S}_A^{(3,4)g}$
a_8	a_{11}	1	$S(a_6a_5d'_0a_9a_{10}; a_6a_5a_4a_3a_2a_1; d'_0\dots d'_ta_{12}) \in \mathscr{S}_A^{(3,4)g}$
a_8	a_{12}	1	$S(a_6a_5d'_0a_9a_{10}; a_6a_5a_4a_3a_2a_1; d'_0\dots d'_ta_{11}) \in \mathscr{S}_A^{(3,4)g}$

By Case 1, there are only three paths beginning at the vertex a_7 and these are:

$$d'_0 = a_7,\quad d'_t = a_{11},\quad t = 1;$$
$$d'_0 = a_7,\quad d'_t = a_3,\quad t = 1;$$
$$d'_0 = a_7,\quad d'_t = a_{10},\quad t = 1.$$

We note here that, in the third case we have a triangle $a_6a_7a_{10}$, a contradiction with the fact that H is triangle-free.

Subcase 2.1 $d'_0 = a_7$, $d'_t = 11$ and $t = 1$. We set $T'' = (V(T''), E(T''))$ such that $V(T'') = V(T')$, $E(T'') = E(T') \cup \{a_7a_{11}\}$. Since H is essentially 3-edge-connected, the set $R'' = \{a_7a_8, a_9a_{10}\}$ cannot be a cut-set of H, implying there is a path $P'' = d''_0d''_1\dots d''_s$, $s \ge 1$, such that $d''_0 \in \{a_8, a_9\}$, $d''_s \in \{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_{10}, a_{11}, a_{12}\}$ and $d''_i \in V(H) \setminus V(T'')$ for $1 \le i \le s-1$. By Case 2, there is no path beginning at $d''_0 = a_8$. Thus, all possible paths begin at $d''_0 = a_9$. There are the following possibilities.

Case d''_0	d''_s	Min. s	The sun S containing a copy of $L^{-1}(B_{3,4})$
a_9	a_1	1	$S(d''_0a_8a_7a_6a_{10}; d''_0\dots d''_sa_2a_3a_4; a_7a_{11}a_{12}) \in \mathscr{S}_A^{(3,4)g}$
a_9	a_2	1	$S(d''_0a_8a_7a_6a_{10}; d''_0\dots d''_sa_3a_4a_5; a_7a_{11}a_{12}) \in \mathscr{S}_A^{(3,4)g}$
a_9	a_3	1	$S(d''_s\dots d''_0a_8a_7a_{11}a_{10}a_6a_5a_4; d''_sa_2a_1; a_{11}a_{12}) \in \mathscr{S}_A^{(3,4)g}$
a_9	a_4	1	$S(d''_0a_{10}a_6a_7a_8; d''_0\dots d''_sa_3a_2a_1; a_7a_{11}a_{12}) \in \mathscr{S}_A^{(3,4)g}$
a_9	a_5	1	$S(d''_s\dots d''_0a_8a_7a_{11}a_{10}a_6; d''_sa_4a_3a_2a_1; a_{11}a_{12}) \in \mathscr{S}_A^{(3,4)g}$
a_9	a_6	2	$S(d''_s\dots d''_0a_8a_7; d''_sa_5a_4a_3a_2a_1; d''_0a_{10}a_{11}a_{12}) \in \mathscr{S}_A^{(3,4)g}$
a_9	a_7	2	$S(d''_s\dots d''_0a_8; d''_sa_6a_5a_4a_3a_2; d''_0a_{10}a_{11}a_{12}) \in \mathscr{S}_A^{(3,4)g}$
a_9	a_{10}	3	$S(a_6a_7a_8d''_0\dots d''_s; a_6a_5a_4a_3a_2a_1) \in \mathscr{S}_A^{(3,4)g}$
a_9	a_{11}	2	$S(a_6a_7a_8d''_0\dots d''_sa_{10}; a_6a_5a_4a_3a_2a_1) \in \mathscr{S}_A^{(3,4)g}$
a_9	a_{12}	1	$S(a_6a_7a_8d''_0\dots d''_sa_{10}a_{11}; a_6a_5a_4a_3a_2a_1) \in \mathscr{S}_A^{(3,4)g}$

Subcase 2.2 $d'_0 = a_7,\ d'_t = a_3$ and $t = 1$. We redefine $T'' = (V(T''), E(T''))$ such that $V(T'') = V(T'),\ \ E(T'') = E(T') \cup \{a_7a_3\}$. Again since H is essentially 3-edge-connected, the set $R'' = \{a_7a_8, a_9a_{10}\}$ cannot be a cut-set of H, implying there is a path $P'' = d''_0d''_1\dots d''_s$, $s \geq 1$, such that $d''_0 \in \{a_8, a_9\}$, $d''_s \in \{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_{10}, a_{11}, a_{12}\}$ and $d''_i \in V(H) \setminus V(T'')$ for $1 \leq i \leq s-1$. By Case 2, there is no path beginning at $d''_0 = a_8$. Thus, all paths begin at $d''_0 = a_9$. There are the following possibilities.

Case d''_0	d''_s	Min. s	The sun S containing a copy of $L^{-1}(B_{3,4})$	
a_9	a_1	1	$S(a_3a_7a_6a_5a_4; a_3a_2d''_s\dots d''_0a_8; a_6a_{10}a_{11}a_{12}) \in \mathscr{S}_A^{(3,4)g}$	
a_9	a_2	1	$H = (V(H), E(H)) \in \mathscr{H}_B^{(3,4)g}$	
a_9	a_3	1	$S(d''_s\dots d''_0a_8a_7a_6a_5a_4; d''_sa_2a_1; d''_0a_{10}a_{11}a_{12}) \in \mathscr{S}_A^{(3,4)g}$	
a_9	a_4	1	$S(d''_0\dots d''_sa_5a_6a_7a_8; d''_0a_{10}a_{11}a_{12}; d''_sa_3a_2a_1) \in \mathscr{S}_A^{(3,4)g}$	if $k \geq 2$
a_9	a_5	1	$S(d''_s\dots d''_0a_8a_7a_3a_4; d''_sa_6a_{10}a_{11}a_{12}; a_3a_2a_1) \in \mathscr{S}_A^{(3,4)g}$	
a_9	a_6	2	$S(d''_s\dots d''_0a_8a_7; d''_sa_5a_4a_3a_2a_1; d''_0a_{10}a_{11}a_{12}) \in \mathscr{S}_A^{(3,4)g}$	
a_9	a_7	2	$S(d''_s\dots d''_0a_8; d''_sa_6a_5a_4a_3a_2; d''_0a_{10}a_{11}a_{12}) \in \mathscr{S}_A^{(3,4)g}$	
a_9	a_{10}	3	$S(a_6a_7a_8d''_0\dots d''_s; a_6a_5a_4a_3a_2a_1) \in \mathscr{S}_A^{(3,4)g}$	
a_9	a_{11}	2	$S(a_6a_7a_8d''_0\dots d''_sa_{10}; a_6a_5a_4a_3a_2a_1) \in \mathscr{S}_A^{(3,4)g}$	
a_9	a_{12}	1	$S(a_6a_7a_8d''_0\dots d''_sa_{10}a_{11}; a_6a_5a_4a_3a_2a_1) \in \mathscr{S}_A^{(3,4)g}$	

We note here that in the second case $d''_0 = a_9,\ d''_s = a_2$ we define a good $H = (V(H), E(H)) \in \mathscr{H}_B^{(3,4)g}$ as follows: $V(H) = V(T''),\ E(H) = E(T'') \cup \{d''_0d''_1, \dots, d''_{s-1}d''_s\}$ and the critical edge is a_2a_1. After rotation of its critical edge we have a graph $H' \in \mathscr{H}_A^{(3,4)g}$.

The only remaining case is $d''_0 = a_9;\ d''_s = a_4$ an $s = 1$. We define $T''' = (V(T'''), E(T'''))$ such that $V(T''') = V(T''),\ \ E(T'') = E(T'') \cup \{a_9a_4\}$.

Again since H is essentially 3-edge-connected, the edge a_2a_3 cannot be a cut-edge of H, implying there is a path $P''' = d_0'''d_1''' \dots d_\ell'''$, $\ell \geq 1$, such that $d_0''' \in \{a_1, a_2\}$, $d_\ell''' \in \{a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, a_{11}, a_{12}\}$ and $d_i''' \in V(H) \setminus V(T''')$ for $1 \leq i \leq \ell - 1$. There are the following possibilities.

Case d_0'''	d_ℓ'''	Min. ℓ	The sun S containing a copy of $L^{-1}(B_{3,4})$
a_1	a_3	2	$S(a_7a_6a_5a_4d_\ell'''; a_7a_8a_9a_{10}a_{11}a_{12}; d_\ell''' \dots d_0'''a_2) \in \mathscr{S}_A^{(3,4)g}$
a_1	a_4	1	$S(a_7a_3a_2d_0''' \dots d_\ell'''a_9a_8; a_7a_6a_{10}a_{11}a_{12}) \in \mathscr{S}_A^{(3,4)g}$
a_1	a_5	1	$S(a_7a_6a_{10}a_9a_8; a_7a_3a_2d_0''' \dots d_\ell'''a_4; a_{10}a_{11}a_{12}) \in \mathscr{S}_A^{(3,4)g}$
a_1	a_6	1	$S(a_3a_2d_0''' \dots d_\ell'''a_5a_4; a_3a_7a_8; a_4a_9a_{10}a_{11}a_{12}) \in \mathscr{S}_A^{(3,4)g}$
a_1	a_7	1	$S(d_\ell''' \dots d_0'''a_2a_3a_4a_5a_6; d_\ell'''a_8a_9; a_6a_{10}a_{11}a_{12}) \in \mathscr{S}_A^{(3,4)g}$
a_1	a_8	1	$S(a_4a_5a_6a_7d_\ell''' \dots d_0'''a_2a_3; a_4a_9a_{10}a_{11}a_{12}) \in \mathscr{S}_A^{(3,4)g}$
a_1	a_9	1	$S(a_3a_7a_6a_5a_4; a_3a_2d_0''' \dots d_\ell'''a_8; a_6a_{10}a_{11}a_{12}) \in \mathscr{S}_A^{(3,4)g}$
a_1	a_{10}	1	$S(d_\ell''' \dots d_0'''a_2a_3a_4a_5a_6; d_\ell'''a_{11}a_{12}; a_6a_7a_8a_9) \in \mathscr{S}_A^{(3,4)g}$
a_1	a_{11}	1	$S(a_3a_7a_6a_5a_4; a_3a_2d_0''' \dots d_\ell'''a_{12}; a_6a_{10}a_9a_8) \in \mathscr{S}_A^{(3,4)g}$
a_1	a_{12}	1	$S(a_3a_7a_6a_5a_4; a_3a_2d_0''' \dots d_\ell'''a_{11}; a_6a_{10}a_9a_8) \in \mathscr{S}_A^{(3,4)g}$
a_2	a_3	3	$S(a_9a_8a_7a_6a_5a_4; a_9a_{10}a_{11}a_{12}; a_4d_\ell''' \dots d_0'''a_1) \in \mathscr{S}_A^{(3,4)g}$
a_2	a_4	2	$S(a_{10}a_9a_8a_7a_3d_0''' \dots d_\ell'''a_5a_6; a_{10}a_{11}a_{12}) \in \mathscr{S}_A^{(3,4)g}$
a_2	a_5	1	$S(a_{10}a_9a_8a_7a_3a_4d_\ell'''a_6; a_{10}a_{11}a_{12}; d_\ell''' \dots d_0'''a_1) \in \mathscr{S}_A^{(3,4)g}$
a_2	a_6	1	$H = (V(H), E(H)) \in \mathscr{H}_B^{(3,4)g}$
a_2	a_7	2	$S(a_{10}a_9a_8d_\ell''' \dots d_0'''a_3a_4a_5a_6; a_{10}a_{11}a_{12}) \in \mathscr{S}_A^{(3,4)g}$
a_2	a_8	1	$S(a_4a_5a_6a_7d_\ell''' \dots d_0'''a_3; a_4a_9a_{10}a_{11}a_{12}) \in \mathscr{S}_A^{(3,4)g}$
a_2	a_9	1	$S(a_7a_6a_{10}d_\ell'''a_8; a_7a_3a_4d_\ell''' \dots d_0'''a_1; a_{10}a_{11}a_{12}) \in \mathscr{S}_A^{(3,4)g}$
a_2	a_{10}	1	$S(a_4a_5a_6a_7a_8a_9; a_4a_3d_0''' \dots d_\ell'''a_{11}a_{12}) \in \mathscr{S}_A^{(3,4)g}$
a_2	a_{11}	1	$S(a_7a_6a_{10}a_9a_8; a_7a_3d_0''' \dots d_\ell'''a_{12}; a_9a_4a_5) \in \mathscr{S}_A^{(3,4)g}$
a_2	a_{12}	1	$S(a_{10}a_9a_8a_7a_6; a_{10}a_{11}d_\ell''' \dots d_0'''a_1; a_7a_3a_4) \in \mathscr{S}_A^{(3,4)g}$

Note here that in the case $d_0''' = a_2$, $d_\ell''' = a_6$ we define a good $H \in \mathscr{H}_B^{(3,4)g}$ in the following way: $V(H) = V(T''')$, $E(H) = E(T''') \cup \{d_0'''d_1''', \dots, d_{\ell-1}'''d_\ell'''\}$ and then the critical edge is a_2a_1. After rotation of the critical edge the resulting graph is $H' \in \mathscr{H}_A^{(3,4)g}$.

Case 3. $d_0 = a_6$, $d_k = a_3$ and $k = 1$. We define $T' = (V(T'), E(T'))$ such that $V(T') = V(T)$, $E(T') = E(T) \cup \{a_6a_3\}$. Since H is essentially 3-edge-connected, the set $R' = \{a_6a_7, a_8a_9\}$ cannot be a cut-set of H, implying there is a path $P' = d_0'd_1' \dots d_t'$, $t \geq 1$, such that $d_0' \in \{a_7, a_8\}$, $d_t' \in \{a_1, a_2, a_3, a_4, a_5, a_6, a_9, a_{10}, a_{11}, a_{12}\}$ and $d_i' \in V(H) \setminus V(T')$ for $1 \leq i \leq t - 1$. There are the following possibilities.

Case d'_0	Case d'_t	Min. t	The sun S containing a copy of $L^{-1}(B_{3,4})$	
a_8	a_1	1	$S(d'_0 \dots d'_t a_2 a_3 a_4 a_5 a_6 a_7; d'_0 a_9 a_{10} a_{11} a_{12}) \in \mathscr{S}_A^{(3,4)g}$	
a_8	a_2	1	$S(d'_0 \dots d'_t a_3 a_4 a_5 a_6 a_7; d'_0 a_9 a_{10} a_{11} a_{12}) \in \mathscr{S}_A^{(3,4)g}$	
a_8	a_3	1	$S(d'_t \dots d'_0 a_7 a_6 a_5 a_4; d'_t a_2 a_1; d'_0 a_9 a_{10} a_{11} a_{12})) \in \mathscr{S}_A^{(3,4)g}$	
a_8	a_4	1	$S(d'_0 \dots d'_t a_5 a_6 a_7; d'_0 a_9 a_{10} a_{11} a_{12}; a_6 a_3 a_2 a_1) \in \mathscr{S}_A^{(3,4)g}$	
a_8	a_5	1	$S(d'_0 \dots d'_t a_6 a_7; d'_0 a_9 a_{10} a_{11} a_{12}; a_6 a_3 a_2 a_1) \in \mathscr{S}_A^{(3,4)g}$	
a_8	a_6	2	$S(d'_t \dots d'_0 a_7; d'_t a_5 a_4 a_3 a_2 a_1; d'_0 a_9 a_{10} a_{11} a_{12}) \in \mathscr{S}_A^{(3,4)g}$	
a_8	a_9	3	$S(d'_0 \dots d'_t; d'_0 a_7 a_6 a_5 a_4 a_3; d'_t a_{10} a_{11} a_{12}) \in \mathscr{S}_A^{(3,4)g}$	
a_8	a_{10}	2	$S(d'_0 \dots d'_t a_9; d'_0 a_7 a_6 a_5 a_4 a_3; d'_t a_{11} a_{12}) \in \mathscr{S}_A^{(3,4)g}$	
a_8	a_{11}	1	$S(d'_0 \dots d'_t a_{10} a_9; d'_0 a_7 a_6 a_5 a_4 a_3; d'_t a_{12}) \in \mathscr{S}_A^{(3,4)g}$	if $t \geq 2$
a_8	a_{12}	1	$S(d'_0 \dots d'_t a_9 a_{11} a_{10} a_9; d'_0 a_7 a_6 a_5 a_4 a_3) \in \mathscr{S}_A^{(3,4)g}$	if $t \geq 2$

By Case 1, there are only three paths beginning at the vertex a_7, and these are $d'_0 = a_7, d'_t = a_{11}, t = 1$; $d'_0 = a_7, d'_t = a_3, t = 1$ and $d'_0 = a_7, d'_t = a_{10}, t = 1$. The remaining possibilities are:

$$\begin{aligned} d'_0 &= a_8, & d'_t &= a_{11}, & t &= 1; \\ d'_0 &= a_8, & d'_t &= a_{12}, & t &= 1; \\ d'_0 &= a_7, & d'_t &= a_3, & t &= 1; \\ d'_0 &= a_7, & d'_t &= a_{10}, & t &= 1; \\ d'_0 &= a_7, & d'_t &= a_{11}, & t &= 1. \end{aligned}$$

We note here that in the third case we have a triangle $a_6 a_7 a_3$, contradicting the fact that H is a triangle-free graph.

Again since H is essentially 3-edge-connected, the set $R'' = \{a_3 a_4, a_5 a_6\}$ cannot be a cut-set of H, implying there is a path $P'' = d''_0 d''_1 \dots d''_s$, $s \geq 1$, such that $d''_0 \in \{a_4, a_5\}$, $d''_s \in \{a_1, a_2, a_3, a_6, a_7, a_8, a_9, a_{10}, a_{11}, a_{12}\}$ and $d''_i \in V(H) \setminus V(T')$ for $1 \leq i \leq s-1$. In some cases, if some edge is needed in order to have $H \in \mathscr{H}_B^{(3,4)g}$, we give this edge in the last column "Extra edge".

Case		Min.	The sun S containing		Extra
d''_0	d''_s	s	a copy of $L^{-1}(B_{3,4})$		edge
a_4	a_1	1	$S(a_6a_3d''_0a_5; a_6a_7a_8a_9a_{10}a_{11}; d''_0 \dots d''_sa_2) \in \mathscr{S}_A^{(3,4)g}$		
a_4	a_2	2	$S(a_6a_3d''_0a_5; a_6a_7a_8a_9a_{10}a_{11}; d''_0 \dots d'_sa_1) \in \mathscr{S}_A^{(3,4)g}$		
a_4	a_3	3	$S(a_6d''_0 \dots d''_s; a_6a_7a_8a_9a_{10}a_{11}) \in \mathscr{S}_A^{(3,4)g}$		
a_4	a_6	2	$S(d''_s \dots d''_0a_5; d'_ta_7a_8a_9a_{10}a_{11}; d''_0a_3a_2a_1) \in \mathscr{S}_A^{(3,4)g}$		
a_4	a_7	1	$S(d''_s \dots d''_0a_5a_6; d''_sa_8a_9a_{10}a_{11}a_{12}; d''_0a_3a_2a_1) \in \mathscr{S}_A^{(3,4)g}$		
a_4	a_8	1	$S(d''_s \dots d''_0a_5a_6a_7; d''_sa_9a_{10}a_{11}a_{12}; a_6a_3a_2a_1) \in \mathscr{S}_A^{(3,4)g}$		
a_4	a_9	1	$H_1 = (V(H_1), E(H_1)) \in \mathscr{H}_B^{(3,4)g}$		a_7a_{11},
			$S(a_3a_6a_7a_8a_{12}a_{11}a_{10}d''_s \dots d''_0; a_3a_2a_1) \in \mathscr{S}_A^{(3,4)g}$	if $s \geq 2$,	a_8a_{12},
			$S(a_3a_6a_7a_8d''_s \dots d''_0; a_3a_2a_1; a_7a_{10}a_{11}a_{12}) \in \mathscr{S}_A^{(3,4)g}$	if $s \geq 2$,	a_7a_{10},
a_4	a_{10}	1	$S(d''_s \dots d''_0a_5a_6a_7a_8a_9; d''_sa_{11}a_{12}; d''_0a_3a_2a_1) \in \mathscr{S}_A^{(3,4)g}$		
a_4	a_{11}	1	$H_2 = (V(H_2), E(H_2)) \in \mathscr{H}_B^{(3,4)g}$		
a_4	a_{12}	1	$S(a_{10}a_7a_8a_9; a_{10}a_{11}d''_s \dots d''_0a_5; a_7a_6a_3a_2a_1) \in \mathscr{S}_A^{(3,4)g}$		a_7a_{10},
			$S(a_{11}a_8a_9a_{10}; a_{11}d''_s \dots d''_0a_3a_2a_1; a_8a_7a_6a_5) \in \mathscr{S}_A^{(3,4)g}$		a_8a_{11},
			$S(d''_0 \dots d''_sa_{11}a_{10}a_9a_8a_7a_6a_5; d''_0a_3a_2a_1) \in \mathscr{S}_A^{(3,4)g}$	if $s \geq 2$	
a_5	a_1	1	$S(a_6a_3a_4d''_0; a_6a_7a_8a_9a_{10}a_{11}; d''_0 \dots d''_sa_2) \in \mathscr{S}_A^{(3,4)g}$	if $s \geq 2$	
a_5	a_2	1	$S(a_6a_3a_4d''_0; a_6a_7a_8a_9a_{10}a_{11}; d''_0 \dots d''_sa_1) \in \mathscr{S}_A^{(3,4)g}$	if $t \geq 2$	
a_5	a_3	2	$S(d''_0 \dots d''_sa_4; d''_0a_6a_7a_8a_9a_{10}a_{11}; d''_sa_2a_1) \in \mathscr{S}_A^{(3,4)g}$		
a_5	a_6	3	$S(d''_0 \dots d''_s; d''_0a_6a_7a_8a_9a_{10}a_{11}; d''_sa_3a_2a_1) \in \mathscr{S}_A^{(3,4)g}$		
a_5	a_7	2	$S(d''_s \dots d''_0a_6; d''_sa_8a_9a_{10}a_{11}a_{12}; d''_0a_4a_3a_2a_1) \in \mathscr{S}_A^{(3,4)g}$		
a_5	a_8	1	$S(d''_s \dots d''_0a_6a_7; d''_sa_9a_{10}a_{11}a_{12}; d''_0a_4a_3a_2a_1) \in \mathscr{S}_A^{(3,4)g}$		
a_5	a_9	1	$S(d''_0 \dots d''_sa_9a_{10}a_{11}a_8a_7a_6; d''_0a_4a_3a_2a_1) \in \mathscr{S}_A^{(3,4)g}$		a_8a_{11}
			$S(d''_0 \dots d''_sa_{10}a_{11}a_{12}a_8a_7a_6; d''_0a_4a_3a_2a_1) \in \mathscr{S}_A^{(3,4)g}$		a_8a_{12}
			$S(d''_0 \dots d''_sa_8a_7a_6; d''_0a_4a_3a_2a_1; a_7a_{10}a_{11}) \in \mathscr{S}_A^{(3,4)g}$		a_7a_{10}
			$S(d''_0 \dots d''_sa_8a_7a_6; d''_0a_4a_3a_2a_1; a_7a_{11}a_{12}) \in \mathscr{S}_A^{(3,4)g}$		a_7a_{11}
a_5	a_{10}	1	$S(d''_s \dots d''_0a_6a_7a_8a_9; d''_sa_{11}a_{12}; d''_0a_4a_3a_2a_1) \in \mathscr{S}_A^{(3,4)g}$		
a_5	a_{11}	1	$S(d''_0 \dots d''_sa_{10}a_9a_8a_7a_6; d''_0a_4a_3a_2a_1) \in \mathscr{S}_A^{(3,4)g}$		
a_5	a_{12}	1	$S(d''_0 \dots d''_sa_{11}a_{10}a_9a_8a_7a_6; d''_0a_4a_3a_2a_1) \in \mathscr{S}_A^{(3,4)g}$		

Note here that in the case $d''_0 = a_4,\ d''_s = a_9$ we define a good $H_1 \in \mathscr{H}_B^{(3,4)g}$ as follows: $V(H_1) = V(T'),\ E(H_1) = E(T') \cup \{a_7a_{11}, d''_0d''_1, \dots, d''_{s-1}d''_s\}$ and the critical edge is $a_{11}a_{12}$. After rotation of its critical edge we have $(H_1)' \in \mathscr{H}_A^{kg}$. In the case $d''_0 = a_4,\ d''_s = a_{11}$ we define a good $H_2 \in \mathscr{H}_B^{(3,4)g}$ in the following way $V(H_2) = V(T'),\ E(H_2) = E(T') \cup \{d''_0d''_1, \dots, d''_{s-1}d''_s\}$ and the critical edge is $a_{11}a_{12}$. After rotation of its critical edge we are in the case $d''_0 = a_4,\ d''_s = a_{12}$.

Remaining possibilities are:

$$
\begin{array}{lllllll}
d''_0 = a_4, & d''_s = a_9, & s = 1; & d'_0 = a_8, & d'_t = a_{11}, & t = 1; \\
 & & & d'_0 = a_8, & d'_t = a_{12}, & t = 1; \\
 & & & d'_0 = a_7, & d'_t = a_{10}, & t = 1; \\
d''_0 = a_4, & d''_s = a_{12}, & s = 1; & d'_0 = a_8, & d'_t = a_{12}, & t = 1; \\
 & & & d'_0 = a_7, & d'_t = a_{10}, & t = 1; \\
d''_0 = a_5, & d''_s = a_1, & s = 1; & d'_0 = a_8, & d'_t = a_{11}, & t = 1; \\
 & & & d'_0 = a_8, & d'_t = a_{12}, & t = 1; \\
 & & & d'_0 = a_7, & d'_t = a_{10}, & t = 1; \\
 & & & d'_0 = a_7, & d'_t = a_{11}, & t = 1; \\
d''_0 = a_5, & d''_s = a_2, & s = 1; & d'_0 = a_8, & d'_t = a_{11}, & t = 1; \\
 & & & d'_0 = a_8, & d'_t = a_{12}, & t = 1; \\
 & & & d'_0 = a_7, & d'_t = a_{10}, & t = 1; \\
 & & & d'_0 = a_7, & d'_t = a_{11}, & t = 1;
\end{array}
$$

Subcase 3.1 $d''_0 = a_4,\ d''_s = a_9$ and $s = 1$. We define $T'' = (V(T''), E(T''))$ such that $V(T'') = V(T'),\ \ E(T'') = E(T') \cup \{a_4a_9\}$. Again since H is essentially 3-edge-connected, the edge a_2a_3 cannot be a cut-edge of H, implying there is a path $P''' = d'''_0 d'''_1 \dots d'''_\ell$, $\ell \geq 1$, such that $d'''_0 \in \{a_1, a_2\},\ d'''_\ell \in \{a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, a_{11}, a_{12}\},\ d'''_i \in V(H) \setminus V(T'')$ for $1 \leq i \leq \ell - 1$. There are the following possibilities.

Case d'''_0	Case d'''_ℓ	Min. ℓ	The sun S containing a copy of $L^{-1}(B_{3,4})$		Extra edge
a_1	a_3	2	$S(a_6d'''_\ell a_4a_5; a_6a_7a_8a_9a_{10}a_{11}; d'''_\ell \dots d'''_0 a_2) \in \mathscr{S}_A^{(3,4)g}$		
a_1	a_4	1	$S(a_6a_3d'''_\ell a_5; a_6a_7a_8a_9a_{10}a_{11}; d'''_\ell \dots d'''_0 a_2) \in \mathscr{S}_A^{(3,4)g}$		
a_1	a_5	1	$S(d'''_\ell \dots d'''_0 a_2a_3a_4; d'''_\ell a_6a_7a_8a_9a_{10}) \in \mathscr{S}_A^{(3,4)g}$	if $\ell \geq 2$	
a_1	a_6	1	$S(a_4a_5d'''_\ell \dots d'''_0 a_2a_3; a_4a_9a_{10}a_{11}a_{12}; d'''_\ell a_7a_8) \in \mathscr{S}_A^{(3,4)g}$		
a_1	a_7	1	$S(a_4a_5a_6d'''_\ell \dots d'''_0 a_2a_3; a_4a_9a_{10}a_{11}a_{12}) \in \mathscr{S}_A^{(3,4)g}$		
a_1	a_8	1	$S(a_4a_5a_6a_7d'''_\ell \dots d'''_0 a_2a_3; a_4a_9a_{10}a_{11}a_{12}) \in \mathscr{S}_A^{(3,4)g}$		
a_1	a_9	1	$S(a_3a_2d'''_0 \dots d'''_\ell a_4; a_3a_6a_7a_8a_{11}a_{12}; d'''_\ell a_{10}) \in \mathscr{S}_A^{(3,4)g}$		a_8a_{11}
			$S(a_3a_2d'''_0 \dots d'''_\ell a_4; a_3a_6a_7a_8a_{12}a_{11}; d'''_\ell a_{10}) \in \mathscr{S}_A^{(3,4)g}$		a_8a_{12}
			$S(a_3a_2d'''_0 \dots d'''_\ell a_4; a_3a_6a_7a_8a_{12}a_{11}; d'''_\ell a_8) \in \mathscr{S}_A^{(3,4)g}$		a_7a_{10}
a_1	a_{10}	1	$S(a_4a_9a_8a_7a_6a_5; a_4a_3a_2d'''_0 \dots d'''_\ell a_{11}a_{12}) \in \mathscr{S}_A^{(3,4)g}$		
a_1	a_{11}	1	$S(a_4a_9a_8a_7a_6a_5; a_4a_3a_2d'''_0 \dots d'''_\ell a_{10}) \in \mathscr{S}_A^{(3,4)g}$		
a_1	a_{12}	1	$S(a_4a_9a_8a_7a_6a_5; a_4a_3a_2d'''_0 \dots d'''_\ell a_{11}a_{10}) \in \mathscr{S}_A^{(3,4)g}$		
a_2	a_3	3	$S(a_6d'''_\ell a_4a_5; a_6a_7a_8a_9a_{10}a_{11}; d'''_\ell \dots d'''_0 a_1) \in \mathscr{S}_A^{(3,4)g}$		
a_2	a_4	2	$S(a_6a_3d'''_\ell a_5; a_6a_7a_8a_9a_{10}a_{11}; d'''_\ell \dots d'''_0 a_1) \in \mathscr{S}_A^{(3,4)g}$		
a_2	a_5	1	$H_1 = (V(H_1), E(H_1)) \in \mathscr{H}_B^{(3,4)g}$		
a_2	a_6	2	$S(d'''_\ell \dots d'''_0 a_3a_4a_5; d'''_\ell a_7a_8a_9a_{10}a_{11}a_{12}) \in \mathscr{S}_A^{(3,4)g}$		
a_2	a_7	1	$S(d'''_\ell \dots d'''_0 a_3a_4a_5a_6; d'''_\ell a_8a_9a_{10}a_{11}a_{12}) \in \mathscr{S}_A^{(3,4)g}$		
a_2	a_8	1	$S(d'''_\ell \dots d'''_0 a_3a_4a_5a_6a_7; d'''_\ell a_9a_{10}a_{11}a_{12}) \in \mathscr{S}_A^{(3,4)g}$		
a_2	a_9	1	$H_2 = (V(H_2), E(H_2)) \in \mathscr{H}_B^{(3,4)g}$		
a_2	a_{10}	1	$S(a_6a_5a_4a_9a_8a_7; a_6a_3d'''_0 \dots d'''_\ell)a_{11}a_{12}) \in \mathscr{S}_A^{(3,4)g}$		
a_2	a_{11}	1	$H_3 = (V(H_3), E(H_3)) \in \mathscr{H}_B^{(3,4)g}$		
a_2	a_{12}	1	$S(a_6a_5a_4a_9a_8a_7; a_6a_3d'''_0 \dots d'''_\ell a_{11}a_{10}) \in \mathscr{S}_A^{(3,4)g}$		

Note here that in the case $d'''_0 = a_2,\ d'''_\ell = a_5$ we define a good $H_1 \in \mathscr{H}_B^{(3,4)g}$ as follows: $V(H_1) = V(T''),\ E(H_1) = E(T'') \cup \{d'''_0 d'''_1, \dots, d'''_{\ell-1} d'''_\ell\}$ and the critical edge is a_1a_2. After rotation of its critical edge we are in the case $d'''_0 = a_1,\ d'''_\ell = a_5$. In the case $d'''_0 = a_2,\ d'''_\ell = a_9$ we define a good $H_2 \in \mathscr{H}_B^{(3,4)g}$ in the following way: $V(H_2) = V(T'');\ \ E(H_2) = E(T'') \cup \{d'''_0 d'''_1, \dots, d'''_{\ell-1} d'''_\ell\}$ and the critical edge is a_1a_2. After rotation of its critical edge we are in the case $d'''_0 = a_1,\ d'''_\ell = a_9$. In the case $d'''_0 = a_2,\ d'''_\ell = a_{11}$ we define a good $H_3 \in \mathscr{H}_B^{(3,4)g}$ as follows: $V(H_3) = V(T''),\ E(H_3) = E(T'') \cup \{d'''_0 d'''_1, \dots, d'''_{\ell-1} d'''_\ell\}$ and the critical edge is a_1a_2. After rotation of its critical edge we are in the case $d'''_0 = a_1,\ d'''_\ell = a_{11}$.

The remaining case is $d'''_0 = a_1,\ d'''_\ell = a_5$ and $\ell = 1$. We define $T''' = (V(T''), E(T''))$ such that $V(T''') = V(T''),\ \ E(T''') = E(T'') \cup \{a_1a_5\}$. Hence H is essentially 3-edge-connected, the set $R''' = \{a_1a_5, a_2a_3\}$ cannot be a cut-set of H, implying there is a path $\bar{P} = \bar{d}_0\bar{d}_1 \dots \bar{d}_r$, $r \geq 1$, such that $\bar{d}_0 \in \{a_1, a_2\}$, $\bar{d}_r \in \{a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, a_{11}, a_{12}\}$, $\bar{d}_i \in V(H) \setminus V(T''')$ for $1 \leq i \leq r-1$. By the subcase 3.1, there is only one possible case and this is $\bar{d}_0 = a_1,\ \bar{d}_r = a_5$ and $r = 1$. But this contradicts the fact that H is a simple graph (i.e. a_1a_5 is multi-edge).

Subcase 3.2 $d''_0 = a_4,\ d''_s = a_{12}$ and $s = 1$. We redefine $T'' = (V(T''), E(T''))$ such that $V(T'') = V(T'),\ \ E(T'') = E(T')\cup\{a_4a_{12}\}$. Again since H is essentially 3-edge-connected, the edge a_2a_3 cannot be a cut-edge of H, implying there is a path $P''' = d'''_0d'''_1 \dots d'''_\ell$, $\ell \geq 1$, such that $d'''_0 \in \{a_1, a_2\}$, $d'''_\ell \in \{a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, a_{11}, a_{12}\}$, $d'''_i \in V(H) \setminus V(T'')$ for $1 \leq i \leq \ell - 1$. There are the following possibilities.

Case d'''_0	Case d'''_ℓ	Min. ℓ	The sun S containing a copy of $L^{-1}(B_{3,4})$	Extra edge
a_1	a_3	2	$S(a_6d'''_\ell a_4a_5; a_6a_7a_8a_9a_{10}a_{11}; d'''_\ell \dots d'''_0 a_2) \in \mathscr{S}_A^{(3,4)g}$	
a_1	a_4	1	$S(a_6a_3d'''_\ell a_5; a_6a_7a_8a_9a_{10}a_{11}; d'''_\ell \dots d'''_0 a_2) \in \mathscr{S}_A^{(3,4)g}$	
a_1	a_5	1	$S(d'''_\ell \dots d'''_0 a_2a_3a_4; d'''_\ell a_6a_7a_8a_9a_{10}) \in \mathscr{S}_A^{(3,4)g}$	if $\ell \geq 2$
a_1	a_6	1	$S(d'''_\ell \dots d'''_0 a_2a_3a_4a_5a_6; d'''_\ell a_7a_8a_9a_{10}a_{11}a_{12}) \in \mathscr{S}_A^{(3,4)g}$	
a_1	a_7	1	$S(d'''_\ell \dots d'''_0 a_2a_3a_4a_5a_6; d'''_\ell a_8a_9a_{10}a_{11}a_{12}) \in \mathscr{S}_A^{(3,4)g}$	
a_1	a_8	1	$S(d'''_\ell \dots d'''_0 a_2a_3a_4a_5a_6a_7; d'''_\ell a_9a_{10}a_{11}a_{12}) \in \mathscr{S}_A^{(3,4)g}$	
a_1	a_9	1	$S(a_9a_{10}a_{11}a_{12}a_4a_3a_6a_7a_8; d'''_\ell \dots d'''_0 a_2; a_6a_6) \in \mathscr{S}_A^{(3,4)g}$	
a_1	a_{10}	1	$S(a_3a_6a_5a_4; a_3a_2d'''_0 \dots d'''_\ell a_{11}a_{12}; a_6a_7a_8) \in \mathscr{S}_A^{(3,4)g}$	
a_1	a_{11}	1	$S(a_6a_3a_4a_5; a_6a_7a_8a_9a_{10}; a_3a_2d'''_0 \dots d'''_\ell a_{12}) \in \mathscr{S}_A^{(3,4)g}$	
a_1	a_{12}	1	$S(a_6a_3a_4a_5; a_6a_7a_8a_9a_{10}a_{11}; a_3a_2d'''_0 \dots d'''_\ell) \in \mathscr{S}_A^{(3,4)g}$	
a_2	a_3	3	$S(a_6d'''_\ell a_4a_5; a_6a_7a_8a_9a_{10}a_{11}; d'''_\ell \dots d'''_0 a_1) \in \mathscr{S}_A^{(3,4)g}$	
a_2	a_4	2	$S(a_6a_3d'''_\ell a_5; a_6a_7a_8a_9a_{10}a_{11}; d'''_\ell \dots d'''_0 a_1) \in \mathscr{S}_A^{(3,4)g}$	
a_2	a_5	1	$H_1 = (V(H_1), E(H_1)) \in \mathscr{H}_B^{(3,4)g}$	
a_2	a_6	2	$S(d'''_\ell \dots d'''_0 a_3a_4a_5; d'''_\ell a_7a_8a_9a_{10}a_{11}a_{12}) \in \mathscr{S}_A^{(3,4)g}$	
a_2	a_7	1	$S(d'''_\ell \dots d'''_0 a_3a_4a_5a_6; d'''_\ell a_8a_9a_{10}a_{11}a_{12}) \in \mathscr{S}_A^{(3,4)g}$	
a_2	a_8	1	$S(d'''_\ell \dots d'''_0 a_3a_4a_5a_6a_7; d'''_\ell a_9a_{10}a_{11}a_{12}) \in \mathscr{S}_A^{(3,4)g}$	
a_2	a_9	1	$S(a_6a_3a_4a_5; a_6a_7a_8d'''_\ell \dots d'''_0 a_1; a_4a_{12}a_{11}) \in \mathscr{S}_A^{(3,4)g}$	
a_2	a_{10}	1	$S(a_6a_3a_4a_5; a_6a_7a_8a_9d'''_\ell \dots d'''_0 a_1; a_4a_{12}a_{11}) \in \mathscr{S}_A^{(3,4)g}$	
a_2	a_{11}	1	$H_2 = (V(H_2), E(H_2)) \in \mathscr{H}_B^{(3,4)g}$	
a_2	a_{12}	1	$S(d'''_\ell \dots d'''_0 a_3a_6a_5a_4; d'''_\ell a_{11}a_{10}a_9a_8a_7) \in \mathscr{S}_A^{(3,4)g}$	

Note here that in the case $d'''_0 = a_2,\ d'''_\ell = a_5$ we define a good $H_1 \in \mathscr{H}_B^{(3,4)g}$ as follows: $V(H_1) = V(T''),\ E(H_1) = E(T'') \cup \{d'''_0d'''_1, \dots, d'''_{\ell-1}d'''_\ell\}$ and the critical edge is a_1a_2. After rotation of its critical edge we are in the case $d'''_0 = a_1,\ d'''_\ell = a_5$. In the case $d'''_0 = a_2,\ d'''_\ell = a_{11}$ we define a good $H_2 \in \mathscr{H}_B^{(3,4)g}$ in the following way: $V(H_2) = V(T'');\ \ E(H_2) = E(T'')\cup\{d'''_0d'''_1, \dots, d'''_{\ell-1}d'''_\ell\}$ and the critical edge is a_1a_2. After rotation of its critical edge we are in the case $d'''_0 = a_1,\ d'''_\ell = a_{11}$.

The remaining case is $d'''_0 = a_1,\ d'''_\ell = a_5$ and $\ell = 1$. We define $T''' = (V(T''), E(T''))$ such that $V(T''') = V(T''),\ \ E(T''') = E(T'') \cup \{a_1a_5\}$. Hence H is essentially 3-edge-connected, the set $R''' = \{a_1a_5, a_2a_3\}$ cannot be a cut-set of H, implying there is a path

$\bar{P} = \bar{d}_0\bar{d}_1 \dots \bar{d}_r$, $r \geq 1$, such that $\bar{d}_0 \in \{a_1, a_2\}$, $\bar{d}_r \in \{a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, a_{11}, a_{12}\}$, $\bar{d}_i \in V(H) \setminus V(T''')$ for $1 \leq i \leq r-1$. Again, by the subcase 3.1, there is only one possible case and this case is $\bar{d}_0 = a_1$, $\bar{d}_r = a_5$ and $r = 1$. But this contradicts the fact that H is a simple graph (i.e. a_1a_5 is multi-edge).

Subcase 3.3 $d''_0 = a_5$, $d''_s = a_1$ and $s = 1$. We redefine $T'' = (V(T''), E(T''))$ such that $V(T'') = V(T')$, $E(T'') = E(T') \cup \{a_5a_1\}$. Since H is essentially 3-edge-connected, the set $R''' = \{a_1a_5, a_2a_3\}$ cannot be a cut-set of H, implying there is a path $P''' = d'''_0d'''_1 \dots d'''_\ell$, $\ell \geq 1$, such that $d'''_0 \in \{a_1, a_2\}$, $d'''_\ell \in \{a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, a_{11}, a_{12}\}$, $d'''_i \in V(H) \setminus V(T'')$ for $1 \leq i \leq \ell - 1$. There are the following possibilities.

Case d'''_0	Case d'''_ℓ	Min. ℓ	The sun S containing a copy of $L^{-1}(B_{3,4})$	Extra edge
a_1	a_3	2	$S(a_6d'''_\ell a_4a_5; a_6a_7a_8a_9a_{10}a_{11}; d'''_\ell \dots d'''_0a_2) \in \mathscr{S}_A^{(3,4)g}$	
a_1	a_4	2	$S(a_6a_3d'''_\ell a_5; a_6a_7a_8a_9a_{10}a_{11}; d'''_\ell \dots d'''_0a_2) \in \mathscr{S}_A^{(3,4)g}$	
a_1	a_5	3	$S(d'''_\ell \dots d'''_0a_2a_3a_4; d'''_\ell a_6a_7a_8a_9a_{10}) \in \mathscr{S}_A^{(3,4)g}$	
a_1	a_6	1	$S(d'''_\ell \dots d'''_0a_2a_3a_4a_5a_6; d'''_\ell a_7a_8a_9a_{10}a_{11}a_{12}) \in \mathscr{S}_A^{(3,4)g}$	
a_1	a_7	1	$S(d'''_\ell \dots d'''_0a_2a_3a_4a_5a_6; d'''_\ell a_8a_9a_{10}a_{11}a_{12}) \in \mathscr{S}_A^{(3,4)g}$	
a_1	a_8	1	$S(d'''_\ell \dots d'''_0a_2a_3a_4a_5a_6a_7; d'''_\ell a_9a_{10}a_{11}a_{12}) \in \mathscr{S}_A^{(3,4)g}$	
a_1	a_9	1	$S(d'''_0a_2a_3a_6a_5; d'''_0 \dots d'''_\ell a_{11}a_{12}; a_6a_7a_8) \in \mathscr{S}_A^{(3,4)g}$	
a_1	a_{10}	1	$S(a_3a_6a_5a_4; a_3a_2d'''_0 \dots d'''_\ell a_{11}a_{12}; a_6a_7a_8a_9) \in \mathscr{S}_A^{(3,4)g}$	
a_1	a_{11}	1	$S(a_6a_3a_4a_5; a_6a_7a_8a_9a_{10}; a_3a_2d'''_0 \dots d'''_\ell a_{12}) \in \mathscr{S}_A^{(3,4)g}$	
a_1	a_{12}	1	$S(a_6a_3a_4a_5; a_6a_7a_8a_9a_{10}a_{11}; a_3a_2d'''_0 \dots d'''_\ell) \in \mathscr{S}_A^{(3,4)g}$	
a_2	a_3	3	$S(a_6d'''_\ell a_4a_5; a_6a_7a_8a_9a_{10}a_{11}; d'''_\ell \dots d'''_0a_1) \in \mathscr{S}_A^{(3,4)g}$	
a_2	a_4	2	$S(a_6a_3d'''_\ell a_5; a_6a_7a_8a_9a_{10}a_{11}; d'''_\ell \dots d'''_0a_1) \in \mathscr{S}_A^{(3,4)g}$	
a_2	a_5	2	$S(a_6a_3a_4d'''_\ell; a_6a_7a_8a_9a_{10}a_{11}; d'''_\ell \dots d'''_0a_1) \in \mathscr{S}_A^{(3,4)g}$	
a_2	a_6	2	$S(d'''_\ell \dots d'''_0a_3a_4a_5; d'''_\ell a_7a_8a_9a_{10}a_{11}a_{12}) \in \mathscr{S}_A^{(3,4)g}$	
a_2	a_7	1	$S(d'''_\ell \dots d'''_0a_3a_4a_5a_6; d'''_\ell a_8a_9a_{10}a_{11}a_{12}) \in \mathscr{S}_A^{(3,4)g}$	
a_2	a_8	1	$S(d'''_\ell \dots d'''_0a_3a_4a_5a_6a_7; d'''_\ell a_9a_{10}a_{11}a_{12}) \in \mathscr{S}_A^{(3,4)g}$	
a_2	a_9	1	$S(d'''_0a_1a_5a_4a_3; d'''_0 \dots d'''_\ell a_{10}a_{11}a_{12}; a_5a_6a_7a_8) \in \mathscr{S}_A^{(3,4)g}$	
a_2	a_{10}	1	$S(d'''_\ell \dots d'''_0a_1a_5a_4a_3a_6a_7a_8a_9; d'''_\ell a_{11}a_{12}) \in \mathscr{S}_A^{(3,4)g}$	
a_2	a_{11}	1	$S(a_6a_3a_4a_5; a_6a_7a_8a_9a_{10}; a_5a_1d'''_0 \dots d'''_\ell a_{12}) \in \mathscr{S}_A^{(3,4)g}$	
a_2	a_{12}	1	$S(a_6a_3a_4a_5; a_6a_7a_8a_9a_{10}; a_3a_2a_1d'''_0 \dots d'''_\ell a_{11}) \in \mathscr{S}_A^{(3,4)g}$	

Subcase 3.4 $d''_0 = a_5$, $d''_s = a_2$ and $s = 1$. We redefine $T'' = (V(T''), E(T''))$ such that $V(T'') = V(T')$, $E(T'') = E(T') \cup \{a_5a_2\}$. Since H is essentially 3-edge-connected, the set $R''' = \{a_2a_5, a_2a_3\}$ cannot be a cut-set of H, implying there is a path $P''' = d'''_0d'''_1 \dots d'''_\ell$, $\ell \geq 1$, such that $d'''_0 \in \{a_1, a_2\}$, $d'''_\ell \in \{a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, a_{11}, a_{12}\}$, $d'''_i \in V(H) \setminus V(T'')$ for $1 \leq i \leq \ell - 1$. There are the following possibilities.

Case d_0'''	d_ℓ'''	Min. ℓ	The sun S containing a copy of $L^{-1}(B_{3,4})$		Extra edge
a_1	a_3	2	$S(a_6d_\ell'''a_4a_5;a_6a_7a_8a_9a_{10}a_{11};d_\ell'''\dots d_0'''a_2)\in\mathscr{S}_A^{(3,4)g}$		
a_1	a_4	1	$S(a_6a_3d_\ell'''a_5;a_6a_7a_8a_9a_{10}a_{11};d_\ell'''\dots d_0'''a_2)\in\mathscr{S}_A^{(3,4)g}$		
a_1	a_5	2	$S(d_\ell'''\dots d_0'''a_2a_3a_4;d_\ell'''a_6a_7a_8a_9a_{10})\in\mathscr{S}_A^{(3,4)g}$		
a_1	a_6	1	$S(d_\ell'''\dots d_0'''a_2a_3a_4a_5a_6;d_\ell'''a_7a_8a_9a_{10}a_{11}a_{12})\in\mathscr{S}_A^{(3,4)g}$		
a_1	a_7	1	$S(d_\ell'''\dots d_0'''a_2a_3a_4a_5a_6;d_\ell'''a_8a_9a_{10}a_{11}a_{12})\in\mathscr{S}_A^{(3,4)g}$		
a_1	a_8	1	$S(d_\ell'''\dots d_0'''a_2a_3a_4a_5a_6a_7;d_\ell'''a_9a_{10}a_{11}a_{12})\in\mathscr{S}_A^{(3,4)g}$		
a_1	a_9	1	$S(a_2a_3a_4a_5;a_2d_0'''\dots d_\ell'''a_{10}a_{11}a_{12};a_5a_6a_7a_8)\in\mathscr{S}_A^{(3,4)g}$		
a_1	a_{10}	1	$S(a_3a_6a_5a_4;a_3a_2d_0'''\dots d_\ell'''a_{11}a_{12};a_6a_7a_8a_9)\in\mathscr{S}_A^{(3,4)g}$		
a_1	a_{11}	1	$S(a_6a_3a_4a_5;a_6a_7a_8a_9a_{10};a_3a_2d_0'''\dots d_\ell'''a_{12})\in\mathscr{S}_A^{(3,4)g}$		
a_1	a_{12}	1	$S(a_6a_3a_4a_5;a_6a_7a_8a_9a_{10}a_{11};a_3a_2d_0'''\dots d_\ell''')\in\mathscr{S}_A^{(3,4)g}$		
a_2	a_3	3	$S(a_6d_\ell'''a_4a_5;a_6a_7a_8a_9a_{10}a_{11};d_\ell'''\dots d_0'''a_1)\in\mathscr{S}_A^{(3,4)g}$		
a_2	a_4	2	$S(a_6a_3d_\ell'''a_5;a_6a_7a_8a_9a_{10}a_{11};d_\ell'''\dots d_0'''a_1)\in\mathscr{S}_A^{(3,4)g}$		
a_2	a_5	3	$S(a_6a_3a_4d_\ell''';a_6a_7a_8a_9a_{10}a_{11};d_\ell'''\dots d_0'''a_1)\in\mathscr{S}_A^{(3,4)g}$		
a_2	a_6	2	$S(d_\ell'''\dots d_0'''a_3a_4a_5;d_\ell'''a_7a_8a_9a_{10}a_{11}a_{12})\in\mathscr{S}_A^{(3,4)g}$		
a_2	a_7	1	$S(d_\ell'''\dots d_0'''a_3a_4a_5a_6;d_\ell'''a_8a_9a_{10}a_{11}a_{12})\in\mathscr{S}_A^{(3,4)g}$		
a_2	a_8	1	$S(d_\ell'''\dots d_0'''a_3a_4a_5a_6a_7;d_\ell'''a_9a_{10}a_{11}a_{12})\in\mathscr{S}_A^{(3,4)g}$		
a_2	a_9	1	$H_1=(V(H_1),E(H_1))\in\mathscr{H}_B^{(3,4)g}$		
a_2	a_{10}	1	$S(d_\ell'''a_{11}a_{12}a_8a_9;d_\ell'''\dots d_0'''a_5a_4a_3;a_8a_7a_6)\in\mathscr{S}_A^{(3,4)g}$		a_8a_{12}
			$H^*=(V(H^*),E(H^*))\in\mathscr{H}_B^{(3,4)g}$		a_8a_{11}
			$S(a_7a_{11}d_\ell'''a_9a_8;a_7a_6a_5a_4a_3;d_\ell'''\dots d_0'''a_1)\in\mathscr{S}_A^{(3,4)g}$		a_7a_{11}
			$S(d_0'''a_5a_6a_3;d_0'''\dots d_\ell'''a_{11}a_{12};a_6a_7a_8a_9)\in\mathscr{S}_A^{(3,4)g}$	if $\ell\geq 2$	
a_2	a_{11}	1	$H_2=(V(H_2),E(H_2))\in\mathscr{H}_B^{(3,4)g}$		
a_2	a_{12}	1	$H_3=(V(H_3),E(H_3))\in\mathscr{H}_B^{(3,4)g}$		

We note here that in the cases $d_0'''=a_2,\ d_\ell'''=a_9$; $d_0'''=a_2,\ d_\ell'''=a_{11}$ and $d_0'''=a_2,\ d_\ell'''=a_{12}$ we define a good $H_i=(V(H_i),E(H_i))\in\mathscr{H}_B^{(3,4)g}$ as follows: $V(H_i)=V(T'')$, $E(H_i)=E(T'')\cup\{d_0'''d_1''',\dots,d_{\ell-1}'''d_\ell'''\}$ and the critical edge is a_1a_2, for $i=1,2,3$ respectively. After rotation of its critical edge the resulting graph is $(H_i)'\in\mathscr{H}_A^{(3,4)g}$, $(i=1,2,3)$ respectively. In the case $d_0'''=a_2,\ d_\ell'''=a_{10}$ we define a good $H^*=(V(H^*),E(H^*))\in\mathscr{H}_B^{(3,4)g}$ in the following way $V(H^*)=V(T'')$, $E(H^*)=E(T'')\cup\{a_8a_{11},d_0'''d_1''',\dots,d_{\ell-1}'''d_\ell'''\}$ and the critical edge is a_2a_1. After rotation of its critical edge we have a $(H^*)'\in\mathscr{H}_A^{(3,4)g}$.

The remaining possibility is:

$$d_0'''=a_2,\quad d_\ell'''=a_{10},\quad \ell=1;\quad d_0'=a_7,\quad d_t'=a_{10},\quad t=1.$$

3.4-1 $d_0'''=a_2,d_\ell'''=a_{10},\ell=1$; $d_0'=a_7,d_t'=a_{10},t=1$. We redefine $T'''=(V(T'''),E(T'''))$ such that $V(T''')=V(T'')$, $E(T''')=E(T'')\cup\{a_2a_{10},a_7a_{10}\}$. Since H is essentially 3-edge-connected, the edge $a_{10}a_{11}$ cannot be a cut-edge of H, implying there is a path $\bar{P} = \bar{d}_0\bar{d}_1 \dots \bar{d}_r$, $r \geq 1$, such that $\bar{d}_0 = \{a_{11}, a_{12}\}$, $\bar{d}_r \in \{a_1, a_2, a_3, a_5, a_6, a_7, a_8, a_9, a_{10}\}$, $\bar{d}_i \in V(H) \setminus V(T''')$ for $1 \leq i \leq r-1$. There are the following possibilities.

Case		Min.	The sun S containing
$\bar{d}_0$	$\bar{d}_r$	r	a copy of $L^{-1}(B_{3,4})$
a_{12}	a_{10}	2	$S(a_7\bar{d}_ra_9a_8; a_7a_6a_5a_4a_3a_2; \bar{d}_r \dots \bar{d}_0a_{11}) \in \mathscr{S}_A^{(3,4)g}$
a_{12}	a_9	1	$S(a_7a_{10}bard_ra_8; a_7a_6a_5a_4a_3a_2; \bar{d}_r \dots \bar{d}_0a_{11}) \in \mathscr{S}_A^{(3,4)g}$
a_{12}	a_8	1	$S(a_{10}a_{11}bard_0 \dots \bar{d}_ra_9; a_{10}a_2a_3a_4a_5; \bar{d}_ra_7a_6) \in \mathscr{S}_A^{(3,4)g}$
a_{12}	a_7	1	$S(\bar{d}_r \dots \bar{d}_0a_{11}a_{10}a_9a_8; \bar{d}_ra_6a_5a_4a_3a_2a_1) \in \mathscr{S}_A^{(3,4)g}$
a_{12}	a_6	1	$S(\bar{d}_r \dots \bar{d}_0a_{11}a_{10}a_9a_8a_7; \bar{d}_ra_5a_4a_3a_2a_1) \in \mathscr{S}_A^{(3,4)g}$
a_{12}	a_5	1	$S(\bar{d}_r \dots \bar{d}_0a_{11}a_{10}a_9a_8a_7a_6; \bar{d}_ra_4a_3a_2a_1) \in \mathscr{S}_A^{(3,4)g}$
a_{12}	a_4	1	$S(a_{10}a_2a_5a_6a_7a_8a_9; a_{10}a_{11}\bar{d}_0 \dots \bar{d}_ra_3) \in \mathscr{S}_A^{(3,4)g}$
a_{12}	a_3	1	$S(\bar{d}_r \dots \bar{d}_0a_{11}a_{10}a_9a_8a_7a_6a_5a_4; \bar{d}_ra_2a_1) \in \mathscr{S}_A^{(3,4)g}$
a_{12}	a_2	1	$S(a_{10}a_7a_8a_9; a_{10}a_{11}\bar{d}_0 \dots \bar{d}_ra_1; a_7a_6a_5a_4a_3) \in \mathscr{S}_A^{(3,4)g}$
a_{12}	a_1	1	$S(a_{10}a_7a_8a_9; a_{10}a_{11}\bar{d}_0 \dots \bar{d}_ra_2; a_7a_6a_5a_4a_3) \in \mathscr{S}_A^{(3,4)g}$
a_{11}	a_{10}	3	$S(a_7\bar{d}_ra_9a_8; a_7a_6a_5a_4a_3a_2; \bar{d}_r \dots \bar{d}_0a_{12}) \in \mathscr{S}_A^{(3,4)g}$
a_{11}	a_9	2	$S(a_7a_{10}bard_ra_8; a_7a_6a_5a_4a_3a_2; \bar{d}_r \dots \bar{d}_0a_{12}) \in \mathscr{S}_A^{(3,4)g}$
a_{11}	a_8	1	$S(a_{10}a_7bard_ra_9; \bar{d}_r \dots \bar{d}_0a_{12}; a_{10}a_2a_3a_4a_5a_6) \in \mathscr{S}_A^{(3,4)g}$
a_{11}	a_7	2	$S(\bar{d}_r \dots \bar{d}_0a_{10}a_9a_8; \bar{d}_ra_6a_5a_4a_3a_2a_1) \in \mathscr{S}_A^{(3,4)g}$
a_{11}	a_6	1	$S(\bar{d}_r \dots \bar{d}_0a_{10}a_9a_8a_7; \bar{d}_ra_5a_4a_3a_2a_1) \in \mathscr{S}_A^{(3,4)g}$
a_{11}	a_5	1	$S(\bar{d}_r \dots \bar{d}_0a_{10}a_9a_8a_7a_6; \bar{d}_ra_4a_3a_2a_1) \in \mathscr{S}_A^{(3,4)g}$
a_{11}	a_4	1	$H_1 = (V(H_1), E(H_1)) \in \mathscr{H}_B^{(3,4)g}$
a_{11}	a_3	1	$H_2 = (V(H_2), E(H_2)) \in \mathscr{H}_B^{(3,4)g}$
a_{11}	a_2	2	$S(a_3a_6a_5a_4; a_3\bar{d}_r \dots \bar{d}_0a_{12}; a_6a_7a_8a_9a_{10}) \in \mathscr{S}_A^{(3,4)g}$
a_{11}	a_1	1	$S(a_3a_6a_5a_4; a_3a_2\bar{d}_r \dots \bar{d}_0a_{12}; a_6a_7a_8a_9a_{10}) \in \mathscr{S}_A^{(3,4)g}$

We note here that in the case $\bar{d}_0 = a_{11}$, $\bar{d}_r = a_4$ we define a good $H_1 \in \mathscr{H}_B^{(3,4)g}$ as follows: $V(H_1) = V(T''')$, $E(H_1) = E(T''') \cup \{\bar{d}_0\bar{d}_1, \dots, \bar{d}_{r-1}\bar{d}_r\}$ and the critical edge is $a_{11}a_{12}$. After rotation of its critical edge we are in the case $\bar{d}_0 = a_{12}$, $\bar{d}_r = a_4$. In the case $\bar{d}_0 = a_{11}$, $\bar{d}_r = a_3$ we define a good $H_2 \in \mathscr{H}_B^{(3,4)g}$ in the following way $V(H_2) = V(T''')$, $E(H_2) = E(T''') \cup \{\bar{d}_0\bar{d}_1, \dots, \bar{d}_{r-1}\bar{d}_r\}$ and the critical edge is $a_{11}a_{12}$. After rotation of its critical edge we are in the case $\bar{d}_0 = a_{12}$, $\bar{d}_r = a_3$.

In each of the possible cases, we showed there is either $H \in \mathscr{H}_A^{(3,4)g} \cup \mathscr{H}_B^{(3,4)g}$ or $S \in \mathscr{S}_A^{(3,4)g} \cup \mathscr{S}_B^{(3,4)g}$. We proved Proposition 5. □

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