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EXISTENCE RESULT FOR DEGENERATED PARABOLIC PROBLEMS IN UNBOUNDED DOMAINS

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Abstract. In this paper, we study the existence of solutions for strongly nonlinear degenerated parabolic problem $\frac{\partial u}{\partial t} - \operatorname{div} a(x,t,u,\nabla u) + g(x,t,u,\nabla u) = f$, in unbounded domains $\mathscr{O}$, where A is a classical divergence operator of Leray-Lions acting from $L^p(0,T,W_0^{1,p}(\mathscr{O},w))$ to its dual, while $g(x,t,s,\xi)$ is a nonlinear term which has a growth condition with respect to ξ and no growth with respect to s, but it satisfies a sign condition on s and $f \in L^{p'}(0,T,W^{-1,p'}(\mathscr{O},w^*))$.

Keywords: degenerated parabolic equation; weighted Sobolev spaces; unbounded domains.

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1. Introduction

Let $\mathscr{O} \subset \mathbb{R}^N$, be a domain (not necessarily bounded) whith boundary $\partial\mathscr{O}$. Let A be the classical divergence operator of Leray-lions from $L^p(0,T,W_0^{1,p}(\mathscr{O},w))$ into its dual $L^{p'}(0,T,W^{-1,p'}(\mathscr{O},w^*))$ defined as

$$Au = -div(a(x,t,u,\nabla u)),$$

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where $a : \mathscr{O} \times [0,T] \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory vector-function satisfying:

$$|a_i(x,t,s,\xi)| \leq \beta w_i^{1/p}(x) \;\; [c_1(x,t) + \sigma^{\frac{1}{p'}} |s|^{\frac{q}{p'}} + \sum_{j=1}^{N} w_j^{\frac{1}{p'}}(x) |\xi_j|^{p-1}], \tag{1}$$

for $i = 1,...,N$,

$$[a(x,t,s,\xi) - a(x,t,s,\eta)](\xi - \eta) > 0, \quad \forall \xi \neq \eta \in \mathbb{R}^N, \tag{2}$$

$$a(x,t,s,\xi).\xi \geq \alpha \sum_{i=1}^{N} w_i |\xi_i|^p, \tag{3}$$

where $c_1(x,t)$ is a positive function in $L^{p'}(0,T,\mathscr{O})$, and α, β are some strictly positive constants. Let $g(x,t,s,\xi)$ is a Carathéodory function satisfying

$$g(x,t,s,\xi)s \geq 0, \tag{4}$$

$$|g(x,t,s,\xi)| \leq b(|s|)(\sum_{i=1}^{N} w_i(x)|\xi_i|^p + c(x,t)), \tag{5}$$

where $b : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous increasing function and $c(x,t)$ is positive function which in $L^1(0,T,\mathscr{O})$.

The purpose of this paper is to prove the existence of solutions for strongly nonlinear degenerated parabolic problem in unbounded domain of the type

$$\begin{aligned} &\frac{\partial u}{\partial t} + Au + g(x,t,u,\nabla u) = f \quad \text{in } \mathscr{O} \times (0,T) \\ &u(x,t) = 0 \quad \text{on } \partial\mathscr{O} \times (0,T) \\ &u(x,0) = 0 \quad \text{in } \mathscr{O}. \end{aligned} \tag{6}$$

Similar problems for degenerate nonlinear elliptic equations have been studied in [6]. In the bounded domains Ω, it is well known that (6) is solvable by Aharouch, Azroul and Rhoudaf in [2] in the case where $f \in L^{p'}(0,T,W^{-1,p'}(\mathscr{O},w^*))$, see also [1], where A is of the form $-div(a(x,t,u,\nabla u)) + a_0(x,t,u,\nabla u)$ and $g = 0$.

The paper is organized as follows: In Section 2, we precise some basic properties of weighted Sobolev space. In Section 3, we prove some technical lemmas concerning some convergences in Weighted Sobolev space. In Section 4, we study the existence of a solution (6) in unbounded domain.

2. Preliminaries

Let $\mathcal{O} \subset \mathbb{R}^N$, be a domain (not necessarily bounded) whith boundary $\partial\mathcal{O}$ and $w = \{w_i(x) : 0 \le i \le N\}$ be a vector of weight functions, i.e. every component $w_i(x)$ is a measurable function which is strictly positive a.e. in $\mathcal{O}$. Furthermore, we suppose in all our considerations that

$$w_i \in L^1_{\mathrm{loc}}(\mathcal{O}), \tag{7}$$

$$w_i^{\frac{-1}{p-1}} \in L^1_{\mathrm{loc}}(\mathcal{O}), \tag{8}$$

for any $0 \le i \le N$. We define the weighted space $L^p(\mathcal{O},\gamma)$, where γ is a weight function on $\mathcal{O}$, by

$$L^p(\mathcal{O},\gamma) = \{u = u(x), u\gamma^{\frac{1}{p}} \in L^p(\mathcal{O})\}.$$

We denote by $W^{1,p}(\mathcal{O},w)$ the space of all real-valued functions $u \in L^p(\mathcal{O},w_0)$ such that the derivatives in the sense of distributions satisfies

$$\frac{\partial u}{\partial x_i} \in L^p(\mathcal{O},w_i) \quad \text{for } i = 1,\dots,N,$$

which is a Banach space under the norm

$$\|u\| = \Big[\int_{\mathcal{O}} |u(x)|^p w_0\,dx + \sum_{i=1}^{N}\int_{\mathcal{O}} |\frac{\partial u(x)}{\partial x_i}|^p w_i(x)\,dx\Big]^{1/p}. \tag{9}$$

Since we shall deal with the Dirichlet problem, we shall use the space

$$X = W_0^{1,p}(\mathcal{O},w)$$

defined as the closure of $C_0^\infty(\mathcal{O})$ with respect to the norm (9). Note that, $C_0^\infty(\mathcal{O})$ is dence in $W_0^{1,p}(\mathcal{O},w)$ and $(X,\|.\|_{1,p,w})$ is a reflexive Banach spaces.

We recall that the dual space of weighted Sobolev spaces $W_0^{1,p}(\mathcal{O},w)$ is equivalent to $W^{-1,p'}(\mathcal{O},w^*)$, where $w^* = \{w_i^* = w_i^{1-p'}, i = 0,\dots,N\}$ and where p' is the conjugate of p i.e. $p' = \frac{p}{p-1}$.

The expression

$$|||u||| = \Big(\sum_{i=1}^{N}\int_{\mathcal{O}} |\frac{\partial u}{\partial x_i}|^p w_i(x)\,dx\Big)^{1/p} \tag{10}$$

is a norm defined on X and it's equivalent to (9).

There exists a weight function σ on $\mathcal{O}$ and a parameter q, $1 < q < \infty$, such that the Hardy inequality:

$$\Big(\int_{\mathcal{O}} |u(x)|^q \sigma\, dx\Big)^{1/q} \leq c\Big(\sum_{i=1}^{N} \int_{\mathcal{O}} |\frac{\partial u}{\partial x_i}|^p w_i(x)\, dx\Big)^{1/p}, \tag{11}$$

holds for every $u \in X$ with a constant $c > 0$ independent of u.

The imbedding

$$X \hookrightarrow L^q(\mathcal{O}, \sigma) \tag{12}$$

determined by the inequality (11) is compact (see[4], [3], [5]).

3. Main results

Let $\mathcal{O} = \bigcup_{l=1}^{\infty} \mathcal{O}_l$, $\overline{\mathcal{O}_l} \subseteq \mathcal{O}_{l+1} \subseteq \overline{\mathcal{O}_{l+1}} \subset \mathcal{O}$, each $\mathcal{O}_l \subset \mathbb{R}^N$ is a bounded domain with boundary $\partial \mathcal{O}_l$. Let A be the classical divergence operator of Leray-lions from $L^p(0,T,W_0^{1,p}(\mathcal{O}_l, w))$ into its dual $L^{p'}(0,T,W^{-1,p'}(\mathcal{O}_l, w^*))$ defined as

$$Au = -div(a(x,t,u,\nabla u)),$$

where $a : \mathcal{O}_l \times [0,T] \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory vector-function satisfying:

$$|a_i(x,t,s,\xi)| \leq \beta w_i^{1/p}(x) \;\; [c_1(x,t) + \sigma^{\frac{1}{p'}} |s|^{\frac{q}{p'}} + \sum_{j=1}^{N} w_j^{\frac{1}{p'}}(x) |\xi_j|^{p-1}], \tag{13}$$

for $i = 1, \ldots, N$,

$$[a(x,t,s,\xi) - a(x,t,s,\eta)](\xi - \eta) > 0, \quad \forall \xi \neq \eta \in \mathbb{R}^N, \tag{14}$$

$$a(x,t,s,\xi).\xi \geq \alpha \sum_{i=1}^{N} w_i |\xi_i|^p, \tag{15}$$

where $c_1(x,t)$ is a positive function in $L^{p'}(0,T,\mathcal{O}_l)$, and α, β are some strictly positive constants. Let $g(x,t,s,\xi)$ is a Carathéodory function satisfying

$$g(x,t,s,\xi)s \geq 0, \tag{16}$$

$$|g(x,t,s,\xi)| \leq b(|s|)(\sum_{i=1}^{N} w_i(x) |\xi_i|^p + c(x,t)), \tag{17}$$

where $b : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous increasing function and $c(x,t)$ is positive function which in $L^1(0,T,\mathscr{O}_l)$.

Let $V = W_0^{1,p}(\mathscr{O}_l, w)$, $H = L^2(\mathscr{O}_l, \sigma)$, $V^* = W^{-1,p'}(\mathscr{O}_l, w^*)$, with $(2 \leq p < \infty)$ and $X = L^p(0,T,V)$. The dual space of X is $X^* = L^{p'}(0,T,V^*)$ where $\frac{1}{p'} + \frac{1}{p} = 1$ and denoting the space $W_p^1(0,T,V,H) = \{v \in X : v' \in X^*\}$ endowed with the norm

$$\|u\|_{w_p^1} = \|u\|_X + \|u'\|_{X^*}, \tag{18}$$

is a Banach space. Here u' stands for the generalized derivative of u; i.e.,

$$\int_0^T u'(t)\varphi(t)\,dt = -\int_0^T u(t)\varphi'(t)\,dt \quad \text{for all } \varphi \in C_0^\infty(0,T).$$

Lemma 3.1. [7]

- *The evolution triple $V \subseteq H \subseteq V^*$ is satisfied.*
- *The imbedding $W_p^1(0,T,V,H) \subseteq C(0,T,H)$ is continuous.*
- *The imbedding $W_p^1(0,T,V,H) \subseteq L^p(0,T,\mathscr{O}_l,\sigma)$ is compact.*

Lemma 3.2. [2] *Let $g \in L^r(0,T,\mathscr{O}_l,\gamma)$ and let $g_n \in L^r(0,T,\mathscr{O}_l,\gamma)$, with $\|g_n\|_{L^r(0,T,\mathscr{O}_l,\gamma)} \leq c, 1 < r < \infty$. If $g_n(x) \to g(x)$ a.e in $(0,T) \times \mathscr{O}_l$, then $g_n \rightharpoonup g$ in $L^r(0,T,\mathscr{O}_l,\gamma)$, where $\rightharpoonup$ denotes weak convergence and γ is a weight function on $(0,T) \times \mathscr{O}_l$.*

Lemma 3.3. [2] *Let (u_n) be a sequence in $L^p(0,T,W_0^{1,p}(\mathscr{O}_l,w))$ such that $u_n \rightharpoonup u$ weakly in $L^p(0,T,W_0^{1,p}(\mathscr{O}_l,w))$ and*

$$\int_Q [a(x,t,u_n,\nabla u_n) - a(x,t,u,\nabla u)][\nabla u_n - \nabla u]\,dt\,dx \to 0. \tag{19}$$

Then $u_n \to u$ in $L^p(0,T,W_0^{1,p}(\mathscr{O}_l,w))$.

Theorem 3.1. [2] *If $f \in L^{p'}(0,T,W^{-1,p'}(\mathscr{O}_l,w^*))$, there exists a solution of the strongly degenerated parabolic problem*

$$\begin{aligned} &\frac{\partial u}{\partial t} + A(u) + g(x,t,u,\nabla u) = f \quad \text{in } \mathscr{O}_l \times (0,T) \\ &u(x,t) = 0 \quad \text{on } \partial\mathscr{O}_l \times (0,T) \\ &u(x,0) = 0 \quad \text{in } \mathscr{O}_l. \end{aligned} \tag{20}$$

4. Existence result in unbounded domains

Theorem 4.1. *If $f \in L^{p'}(0,T,W^{-1,p'}(\mathscr{O},w^*))$, there exists a solution in unbounded domains of the problem*

$$\begin{aligned} &\frac{\partial u}{\partial t} + A(u) + g(x,t,u,\nabla u) = f \quad \text{in } \mathscr{O} \times (0,T) \\ &u(x,t) = 0 \quad \text{on } \partial\mathscr{O} \times (0,T) \\ &u(x,0) = 0 \quad \text{in } \mathscr{O}. \end{aligned} \tag{21}$$

Proof. Let $\mathscr{O} = \cup_{l=1}^{\infty} \mathscr{O}_l$, $\mathscr{O}_l \subseteq \overline{\mathscr{O}}_l \subseteq \mathscr{O}_{l+1} \subseteq \overline{\mathscr{O}}_{l+1}$ be bounded domains in $\mathscr{O}$ and $\{u_k\}$ be the sequence of solutions of (20) in $L^p(0,T,W_0^{1,p}(\mathscr{O}_k,w)$, $(k \geq 1)$. Since g verifies the sign condition, using (15) we obtain

$$\alpha \sum_{i=1}^{N} \int_{\mathscr{O}_l} w_i |\frac{\partial u_k}{\partial x_i}|^p \leq < f, u_k >, \tag{22}$$

i.e. $\alpha \|u_k\|^p \leq \|f\|_{X^*} \|u_k\|$, then

$$\|u_k\| \leq \beta_1, \tag{23}$$

for all $k \geq l$ and β_1 is independent of k. Let $\tilde{u}_k$, for $k \geq 1$, denote the extension of u_k by zero outside $\mathscr{O}_k$, which we continue to denote it by u_k. From (23), we have $\|u_k\| \leq \beta_1$, for $k \geq l$. Then, $\{u_k\}$ has a subsequence $\{u_{k_m^1}\}$ which converges weakly to u^1, as $m \to \infty$, in $L^p(0,T,W_0^{1,p}(\mathscr{O}_1,w))$. Since $\{u_{k_m^1}\}$ is bounded in $L^p(0,T,W_0^{1,p}(\mathscr{O}_2,w))$, it has a convergent subsequence $\{u_{k_m^2}\}$ converging weakly to u^2 in $L^p(0,T,W_0^{1,p}(\mathscr{O}_2,w))$. By induction, we have $\{u_{k_m^{l-1}}\}$ has a subsequence $\{u_{k_m^l}\}$ which weakly converges to u^l in $L^p(0,T,W_0^{1,p}(\mathscr{O}_l,w))$; i.e., in short, we have $u_{k_m^l} \rightharpoonup u^l$ in $L^p(0,T,W_0^{1,p}(\mathscr{O}_l,w))$, $l \geq 1$. Define $u : \mathscr{O} \to \mathbb{R}$ by $u(x) := u^l(x), \quad$ for $x \in \mathscr{O}_l$. (Here, there is no confusion since $u^l(x) = u^m(x)$ for any $m \geq l$). Let M be any fixed (but arbitrary) bounded domain such that $M \subseteq \mathscr{O}$. Then, there exists an integer l such that $M \subseteq \mathscr{O}_l$. We note that, the diagonal sequence $\{u_{k_m^m}; m \geq l\}$ converges weakly to $u = u^l$ in $L^p(0,T,W_0^{1,p}(M,w))$, as $m \to \infty$. We still need to show that u is the solution. It is sufficient to show that u is a solution of (20) for an arbitrary bounded domain M in $\mathscr{O}$. Since $u_{k_m^m} \to u^l$ in

$L^p(0,T,W_0^{1,p}(M,w))$, we can pass to the limit in

$$< \frac{\partial u_{k_m^m}}{\partial t}, v > + < Au_{k_m^m}, v > + < g(x,t,u_{k_m^m},\nabla u_{k_m^m}), v > = < f, v >$$

and we obtain

$$< \frac{\partial u}{\partial t}, v > + < Au, v > + < g(x,t,u,\nabla u), v > = < f, v >$$

as $m \to \infty$ for any $v \in L^p(0,T,W_0^{1,p}(M,w)) \cap L^\infty(0,T,M)$. This concludes the proof of Theorem 4.1.

Conflict of Interests

The author declares that there is no conflict of interests.

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