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CONVERGENCE OF WAVELET SERIES AND APPLICATIONS TO COMPUTERIZED TOMOGRAPHY

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Abstract. In computerized tomography an image must be recovered from data given by the Radon transform of the image. This data is usually in the form of sampled values of the transform. In this paper we have proved some convergence results in generalized Sobolev space using the sampling properties of the prolate spheroidal wavelets which are superier to other wavelets. Our generalized Sobolev space generalized the various spaces such as Schwartz space, Hormander space [5] and space studied by Pathak [10].

Keywords: prolate spheroidal wavelets; generalized Sobolev space; computer tomography and Radon transform; Schwartz space.

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1. Introduction

The prolate spheroidal wave functions (PSWFs) are those that are most highly localized simultaneously in both the time and frequency domain. This fact was discovered in a series of papers [7],[8],[3]-[15]. Since then the study of PSWFs has been an active area

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of research in both electrical engineering and mathematics. The PSWFs were previously known as solutions of Sturm-Liouville problem, from which many of their properties could be derived. The associated prolate spheroidal wavelets (PS wavelets) have been introduced by G.G. Walter and Xiaoping Shen [18]. Pollak and Landau discovered the connection between PSWFs and the energy concentration problem during the 1960's. The PSWFs were shown to be an important tool for analyzing some problems raised in signal processing and telecommunications [9].

Walter and Shen [18] have proposed new wavelets based on PSWFs. This wavelet family can be used in place of sinc function $S(t) = \frac{\sin \pi t}{\pi t}$to recover bandlimited signals from their sampled values and possess the same maximum energy concentration property. The corrsponding wavelet scaling function is just the first PSWF with bandwidth π. While these are not orthogonal to their integer translates, they do constitute a Riesz basis of the space of π-bandlimited signals. Shepp and Zhang [12] have used PSWFs to obtain a fast algorithm for recovering a magnetic resonance image (MRI) from its sampled values in the frequency domain. Their prolate wavelets are not ones considered here, but rather are multidimensional spheroidal wave functions. Their approach was close to optimal for imaging of brain activity.

We shall use the PS wavelets to recover the image function of an object from the sampled values of its Radon transform. To solve this fundamental problem, wavelet-based methods have been used (see[1,3,4,16,17]) but none has the combination of good time and frequency response arising with the PS wavelets. It has been noticed that approximation based on these wavelets possesses the analytic properties of the original function, something which no other wavelet systems do. Since the PSWFs do not have a closed integral forms. The author obtained it in this paper by introducing a procedure that only calls for the values of PSWFs at integers. To find these values an alternative method has been proposed for computing them without using the traditional Legendre polynomial approximations to PSWFs. Here in this paper we have obtained a convergence theorem in generalized

Sobolev space.

The work in this proper is organized as follows. Section 1, gives the introductory exposition of the topic. Section 2, deals with some properties of prolate spheroidal wave functions and associated PS wavelets. In Section 3, a brief introduction of computerized tomography has been given, Finally in Section 4, we have proved a convergence theorem in generalized Sobolev space which is more than the Schwartz space and the spaces used by Hormander[5] and Pathak [10].

2. Prolate Spheroidal Wave Functions and Associated Wavelets

In this section, we have given some related properties of the prolate spheroidal wave functions and the associated PS wavelets.

In order to recover an image from the sampled values of its Radon transform, the band limited functions play a significant role since the frequencies in such an image must be bounded. By Shannon sampling theorem [11] each function in the space β_π of π-band limited functions can be represented as

$$f(t) = \Sigma_\kappa f(\kappa) S(t-\kappa)$$

where $S(t)$ is the sinc function $S(t) = \frac{\sin \pi t}{\pi t}$. Since the sinc function has a very slow decay therefore above formula is not adequate for recovering signals with finite time duration. One of the natural solutions is to consider the set of prolate spheroidal wave functions $\{\phi_n(t)\}$which form the orthonormal basis of B_π and are highly concentrated in a time interval $[-\tau, \tau]$.

The PSWFs are also characterized as :

1. $\{\phi_n\}$ have maximum energy concentration among all $\pi-$band limited functions in the interval $[-\tau, \tau]$, i.e., ϕ_o is the function such that $\int_{-\infty}^{\infty} |\phi_o(t)|^2 dt = 1$ and $\int_{-\tau}^{\tau} |f(t)|^2 dt$ is maximized for $f = \phi_o$; ϕ_1 is the function orthogonal to $\phi_o$with the same property; ϕ_2 is ...

2. $\{\phi_n\}$ are the eigenfunctions of a differential operator

$$(\tau^2 - t^2)\frac{d^2\phi_n}{dt^2} - 2t\frac{d\phi_n}{dt} - \pi^2 t^2 \phi_n = \mu_n \phi_n, n = 0, 1, 2, ...$$

where μ_n are the eigenvalues.

3. $\{\phi_n\}$ are the eigenfunctions of an integral operator

$$\int_{-\tau}^{\tau} \phi_n(x) S(t-x)dx = \lambda_n \phi_n(t), n = 0, 1, 2, ...$$

It is also noted that each of ϕ_n has exactly n zeroes in the concentration interval $[-\tau, \tau]$ and the Fourier transform of ϕ_n is given by

$$\hat{\phi}_n(\omega) = (-1)^n \sqrt{\frac{2\tau}{\lambda_n}} \phi_n(\frac{\omega\tau}{\pi})\chi_\pi(\omega),$$

where $\chi_\pi(\omega)$ is the characteristic function of the interval $[-\pi, \pi]$.

The PS-wavelets were introduced in [18] and have as their scaling function $\phi = \phi_o/\hat{\phi}_o(o)$. It was shown that integer translates of this scaling function $\{\phi(t-n)\}$form a Riesz basis of the space B_π whatever the value of τ, just as the PSWFs do. By changing the scale by factors of 2, we obtain a multi resolution analysis (MRA)$\{V_m\}$ of subspaces of $L^2(R)$, where $f(t) \in V_m$ if and only if $f(2^{-m}t) \in V_o$. An MRA will have the following properties:

1. $...\subseteq V_{m-1} \subseteq ... \subseteq L^2(R)$,
2. $\cup \bar{V}_m = L^2(R)$,
3. $\cap V_m = \{0\}$.

Here $V_m = B_{2^m\pi}$, the Paley- Wiener space of $2^m\pi$-band limited functions; a function$f$in $L^2(R)$ may be approximated at the scale m by the series approximation of the form

$$f_m(t) = \sum_{n=-\infty}^{\infty} a_{nm}\phi(2^m t - n).$$

Several methods of calculating the coefficients a_{nm} were studied in [18]. One is obtained by using a biorthogonal series and an integral formula. The other method, which we shall use have for the most part, avoids integration and uses only the point value $f(2^{-m}n)2^{-m}$ for a_{nm}. Both methods were shown to converge to f uniformly as $m \to \infty$ on the real line for f is an appropriate Sobolev space. The former, has a more rapid rate of convergence, whereas the latter avoids Gibbs' phenomenon. This phenomenon, involving the overshoot at points of discontinuity, is present in Fourier approximation as well as in all standard wavelet approximations. It is particularly troublesome in image since it causes ripple effects at discontinuities of the image. The absence of Gibbs' phenomenon is one reason to use these wavelets in computerized Tomography.

3. Computerized Tomography

In computerized tomography, the cross-sectional image of an object in the form of a two-dimensional density function is reconstructed from data collected when the object is illuminated by X-ray beams from many different angles. As X-rays pass through the object they are attended at different rates by tissues with different densities; measurements obtained at an angle θ are recorded in the form of sampled values of the projection function $P_\theta(t)$. Let $f(x, y)$ denote the density function of the object which is often called the image function, or the object function. The projection function along the line of exposure $t = x\cos\theta + y\sin\theta$ is given by the line integral

$$P_\theta(t) = \int\int_{R^2} f(x, y)\delta(x\cos\theta + y\sin\theta - t)dxdy,$$

where δ is the one-dimensional Dirac delta-function, and $P_\theta(t)$ is the Radon transform of $f(x, y)$.

In order to recover f from P_θ relies on the Fourier Slice Theorem [12]

$$\hat{P}_\theta(w) = \hat{f}(w\cos\theta, w\sin\theta),$$

where $\hat{P}_\theta$ and $\hat{f}$ denote the Fourier transform of P_θ and f. In other words, the one-dimensional Fourier transform of the projection function gives the two-dimensional Fourier transform of the object function along a radial line. If projection is known at enough angles, the object function can be recovered by using an approximation to the inverse Fourier transform

$$f(x,y) = \frac{1}{(2\pi)^2}\int_0^{\pi}\int_{-\infty}^{\infty}\hat{P}_\theta(w)e^{iw(x\cos\theta+y\sin\theta)}|w|dwd\theta,$$

$$= \frac{1}{2\pi}\int_0^{\pi} Q_\theta(t)d\theta,$$

where $t = x\cos\theta + y\sin\theta$ and Q is the output of a filter with transfer function $|w|$, i.e.,

$$\hat{Q}_\theta(w) = \hat{P}_\theta(w)|w|,$$

followed by an averaging operator. To make inversion of $\hat{Q}$ possible, $|w|$ is usually multiplied by a smoothing window.

Since $f(x,y)$ has compact support, it cannot be band limited at the same time. However, it can belong to a Sobolev space since this would merely require that $\hat{Q}_\theta(w)$ decrease more rapidly than a negative power of w as $w \to \pm\infty$. Such functions, can be uniformly approximated by band limited functions in $V_m = B_{2^m\pi}$

4. Approximation in Generalized Sobolev Space

First we recall some definitions and properties of certain function and distribution spaces given in [2]. Let M be the set of continuous and real valued functions v on R satisfying the following conditions:

1. $0 = v(o) \leq v(\xi+\eta) \leq v(\xi)+v(\eta); \xi,\eta \in R,$

2. $\int_0^\infty \frac{v(\xi)d\xi}{(1+|\xi|)^{n+1}} < \infty,$

3. $v(\xi) \geq a + b\log(1+|\xi|), \xi \in R,$

for some real number a and positive real number b. We denote by M_c the set of all $v \in M$ satisfying condition $v(\xi) = \Omega(|\xi|)$ with a concave function Ω on $[0,\infty)$.

Let $v \in M_c$ and S_v be the set of all functions $\phi \in L^1(R)$ with the property that ϕ and $\hat{\phi} \in C^\infty$ and for each index γ and each non-negative α^* we have

$$p_{\gamma,\alpha^*}(\phi) = \sup_{x \in R} e^{\alpha^* v(x)} |D^\alpha \phi(x)| < \infty,$$

$$q_{\gamma,\alpha^*}(\phi) = \sup_{\xi \in R} e^{\alpha^* v(\xi)} |D^\alpha \hat{\phi}(\xi)| < \infty.$$

The topology of S_v is defined by the semi-norms p_{γ,α^*} and q_{γ,α^*}. The dual of S_v is denoted by $S_v^{'}$, the elements of which are called ultra-distributions. It is interesting to mention here that for $v(\xi) = \log(1 + |\xi|)$, S_v is reduced to the Schwartz space.

We denote the space D_v the set of all ϕ in $L^1(R)$ such that ϕ has compact support and $||\phi||_{\beta^*} < \infty$ for all $\beta^* > 0$ and

$$||\phi||_{\beta^*} = \int_R |\hat{\phi}(\xi)| e^{\beta^* v(\xi)} d\xi.$$

Also K_v is defined to be the set of positive functions κ in R with

$$\kappa(\xi + \eta) \leq e^{\beta^* v(-\xi)} \kappa(\eta)$$

for all $\xi, \eta \in R$. Let $v \in M_c$, $\kappa \in K_v$ and $1 \leq p < \infty$. Then generalized Sobolev space $H^v_{p,\kappa}(R)$ is defined to be the space of all ultra-distributions $f \in S_v^{'}$ such that

$$||f||_{p,\kappa} = (\int_R |\kappa(\xi) \hat{f}(\xi)|^p d\xi)^{1/p} < \infty$$

and

$$||f||_{\infty,\kappa} = ess \sup \kappa(\xi) |\hat{f}(\xi)|.$$

Remark 4.1. The space $H^v_{p,\kappa}$ is a generalization of the Hormander Space [5] and reduces to the space $H_{p,\kappa}(R)$ for $v = \log(1 + |\xi|)$. For $\kappa(\xi) = e^{sv(\xi)}$ and $1 \leq p < \infty$, $H^v_{p,\kappa} = H^{s,p}_v$, the generalized Sobolev space studied by Pathak[10].

The approximation of a projection function $P_\theta(t)$ by the sampling series at the scale of interest m is of the form

$$P_{\theta,m}(t) = \sum_{n=-\infty}^{\infty} P_\theta(n2^{-m})\phi(2^m t - n) \tag{4.1}$$

where ϕ is the scaling function of the PS-wavelet. To avoid any integrations involving ϕ we approximate the filtered projection $Q_\theta(t)$ by the series

$$Q_{\theta,m}(t) = \sum_{\kappa=-\infty}^{+\infty} a_\kappa \phi(2^m t - \kappa) \tag{4.2}$$

where the coefficients a_κ are given by $a_\kappa =< Q_\theta(t), \tilde{\phi}(2^m t - \kappa) >$ with $\tilde{\phi}$ being a function biorthogonal to ϕ. Then we have

$$\begin{aligned}
a_\kappa &= \frac{1}{2\pi}\langle |w|\hat{P}_\theta(w), 2^{-m}\hat{\tilde{\phi}}(2^{-m}w)e^{-iw\kappa 2^{-m}}\rangle \\
&= \frac{2^{-2m}}{2\pi}\int_{-\infty}^{\infty}[\Sigma_n P_\theta(n2^{-m})\hat{\phi}(2^{-m}w)e^{-iwn2^{-m}}]|w|\hat{\tilde{\phi}}(2^{-m}w)e^{-iw\kappa 2^{-m}}dw \\
&= \frac{2^{-2m}}{2\pi}\sum_{n=-\infty}^{+\infty} P_\theta(n2^{-m})\int_{-2^m\pi}^{2^m\pi}|w|e^{-iw(n+\kappa)2^{-m}}dw \\
&= \frac{1}{2\pi}\sum_{n=-\infty}^{+\infty} P_\theta(n2^{-m})g(n,\kappa)
\end{aligned}$$

where

$$\begin{aligned}
g(n,\kappa) &= 2^{-2m}\int_{-2^m\pi}^{2^m\pi}|w|e^{-iw(n+\kappa)2^{-m}}dw \\
&= \begin{cases} \pi^2, & n+\kappa = 0; \\ 0, & n+\kappa \text{ is even}; \\ -\frac{4}{(n+\kappa)^2}, & n+\kappa \text{ is odd}. \end{cases}
\end{aligned}$$

Then the approximation (4.2) gives

$$\begin{aligned}
Q_{\theta,m}(t) &= \Sigma_\kappa[\Sigma_n P_\theta(n2^{-m})g(n,\kappa)]\phi(2^m t - \kappa) \\
&= \sum_n P_\theta(n2^{-m})b_n(t)
\end{aligned}$$

where the weight functions $b_n(t)$ are given by

$$b_n(t) = \sum_{\kappa=-\infty}^{\infty} g(n,\kappa)\phi(2^m t-\kappa).$$

We now prove a convergence theorem in generalized Sobolev space.

Theorem 4.1. Let $P_\theta(t)$ have compact support and $P_\theta(t) \in H^v_{p,\kappa}(R)$, the generalized Sobolev space, for $1 \le p < \infty$, let the approximation $P_\theta(t)$ be given by a series of the form(4.1), then $P_{\theta,m}(t) \to P_\theta(t)$ uniformly in $H^v_{p.\kappa}$ for $\theta \in [0,2\pi]$ and $t \in R$ as $m \to \infty$.

Proof. We have

$$\begin{aligned}
\| P_{\theta,m}(t) - P_\theta(t) \|^p_{p,\kappa} &= \int_R |(\hat{P}_{\theta,m}(\xi) - \hat{P}_\theta(\xi))\kappa(\xi)|^p d\xi \\
&= \int_R |(\sum_{n=-\infty}^{\infty} 2^{-m} P_\theta(n2^{-m})\hat{\phi}(\xi 2^{-m})e^{-i\xi n 2^{-m}} - \hat{P}_\theta(\xi))\kappa(\xi)|^p d\xi \\
&= \int_R |(\sum_{n=-\infty}^{\infty} \hat{P}_\theta(\xi + 2^{m+1}\pi\kappa)\hat{\phi}(2^{-m}\xi) - \hat{P}_\theta(\xi))\kappa(\xi)|^p d\xi \\
&= \int_R |\hat{P}_\theta(\xi)|^p |\hat{\phi}(2^{-m}\xi) - 1|^p |\kappa(\xi)|^p d\xi + \int_R \hat{\phi}(2^{-m}\xi) \\
&\quad \sum_{\kappa\neq 0} |\hat{P}_\theta(\xi + 2\pi\kappa 2^m)|^p . |\kappa(\xi)|^p d\xi \\
&= I + II.
\end{aligned}$$

Since $\hat{\phi}$ is continuous near the origin the I integral is dominated by

$$\int_{-\infty}^{-2^m\delta} 2^p |\hat{P}_\theta(\xi)|^p |\kappa(\xi)|^p d\xi + \int_{-2^m\delta}^{2^m\delta} |\hat{P}_\theta(\xi)|^p |\hat{\phi}(2^{-m}\xi) - 1|^p |\kappa(\xi)|^p d\xi + \int_{2^m\delta}^{\infty} 2^p |\hat{P}_\theta(\xi)|^p |\kappa(\xi)|^p d\xi.$$

Here δ is the number such that $|\hat{\phi}(2^{-m}\xi) - 1|^p < \varepsilon$ for $|2^{-m}\xi| < \delta$ and we have used the fact that $||\hat{\phi}||^{\infty}_{\infty,\kappa} = 1$. The middle integral here is dominated by

$$\varepsilon \int_{-2^m\delta}^{2^m\delta} |\hat{P}_\theta(\xi)|^p |\kappa(\xi)|^p d\xi < \varepsilon \int_R |\hat{P}_\theta(\xi)|^p |\kappa(\xi)|^p d\xi.$$

The II integral is dominated by $\hat{\phi}(0) = 1$, satisfies

$$\int_{-\pi 2^m}^{\pi 2^m} \hat{\phi}(2^{-m}\xi)(\sum_{\kappa+0} |\hat{P}_\theta(\xi + 2\pi\kappa 2^m)|^p |\kappa(\xi)|^p) d\xi$$

$$\leq \int_{-\infty}^{-\pi 2^m} |\hat{P}_\theta(\xi)|^p |\kappa(\xi)|^p d\xi + \int_{\pi 2^m}^{\infty} |\hat{P}_\theta(\xi)|^p |\kappa(\xi)|^p d\xi.$$

Thus taking the limit as $m \to \infty$, all integrals converges to zero except the middle one which is a multiple of ε. This limit is arbitrarily small and hence the theorem is established.

In order to recover $f(x, y)$ we recall that

$$f(x, y) = \frac{1}{\pi} \int_0^\pi Q(x \cos\theta + y \sin\theta) d\theta.$$

Therefore we obtain the following approximation to the image function.

$$f(x, y) \approx \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \int_0^\pi P_\theta(n2^{-m}) b_n(x \cos\theta + y \sin\theta) d\theta. \tag{4.3}$$

Similar theorem as 4.1 can be proved for the image function itself approximated by series(4.3). We can show the approximation in (4.3) converges uniformly to $f(x, y)$ in $H^v_{p,\kappa}(R)$ for $(x, y) \in R$ and $m \to \infty$.

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