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# ON SYMMETRICAL FUNCTIONS WITH BOUNDED BOUNDARY ROTATION

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**Abstract.** The object of the present paper is to derive the integral representation for classes involving the notion of $(m,n)$-symmetrical functions with bounded boundary rotation and bounded radius rotation. Some more properties like radius of univalent and starlike are also investigated.

**Keywords**: convex functions, starlike functions, functions of bounded boundary rotation, bounded radius rotation, $(m,n)$-symmetric points.

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## 1. Introduction-preliminaries

Let $\mathscr{A}$ denote the class of functions of form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are analytic in the open unit disk $\mathscr{U} = \{z : z \in \mathbb{C} \; and \; |z| < 1\}$. Let $\mathscr{S}$ denote the subclass of $\mathscr{A}$ consisting of all functions which are univalent in $\mathscr{U}$. We also denote by $\mathscr{S}^*, \mathscr{K}$ the familiar subclasses of it consisting of functions which are respectively starlike and convex in

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$\mathscr{U}$. It is known that $f(z) \in \mathscr{S}^*$ if and only if

$$f(z) = z\ \exp\left\{\int_0^z -\log(1-ze^{-it})dm(t)\right\},$$

for some $m(t) \in M_2$.

Pinchuk [1] generalized the class $\mathscr{S}^*$ by allowing $m(t)$ to range over the class $M_k$. More precisely a function $f(z)$ is said to be in the class $U_k$ if $f(z) = z\ \exp\left\{\int_0^z -\log(1-ze^{-it})dm(t)\right\}$, $m(t) \in M_k$ i.e, $m(t)$ is a real valued function of bounded variation on $[0,2\pi]$ satisfying the conditions.

$$\int_0^{2\pi} dm(t) = 2,\ \int_0^{2\pi} |dm(t)| \leq k. \tag{2}$$

Geometrically the condition is that the total variation of the angle which the radius vector $f(re^{i\theta})$ makes whit the positive real axis is bounded above by $\pi k$ as $z$ describes the circle $|z| = r$ for $|z| < 1$. Thus $U_k$ the class of functions with radius rotation bounded by $\pi k$. Similarly $V_k$ denotes the class of functions $f$ defined on $\mathscr{U}$ which map conformally onto a image domain of boundary rotation at most $k\pi$. Hence $f(z) \in V_k$, if and only if

$$f'(z) = \exp\int_0^{2\pi} -\log(1-ze^{-it})dm(t),\ m(t) \in M_k.$$

It is easy to see that $U_2$ is the class of starlike functions and $V_2$ is the class of convex functions.

Let $\mathscr{P}_k$ denote the class of functions which are analytic in $\mathscr{U}$ and have the representation

$$p(z) = \frac{1}{2}\int_0^{2\pi} \frac{1+ze^{-it}}{1-ze^{-it}} dm(t), \tag{3}$$

where $m(t) \in M_k$. Clearly we have $p_2 = p$ and $f \in U_k$ *and* $V_k$ if and only if $\frac{zf'}{f}$ and $1+\frac{zf''}{f'}$ belong to $\mathscr{P}_k$. For $p \in \mathscr{P}_k$, then it has the following properties

(1) $p(0) = 1$,

(2) $\int_0^{2\pi} |\Re\{p(z)\}| d\theta \leq k\pi$, where $k \geq 2$ and $z = re^{i\theta}, 0 \leq r < 1$.

Liczberski and Polubinki [4] introduce the notion of $(m,n)$-symmetrical functions $(n = 1,2,3,...,m = 0,1,..,n-1)$ which is generalization of notions of even odd and $n$ symmetrical functions. They also generalized the known result that each function defined in symmetrical subset can be uniquely represented as the sum of an even function and odd function.

**Definition 1.1.** Let $\varepsilon = (e^{\frac{2\pi i}{n}})$ and $m = 0,1,2,..,n-1$ where $n \geq 2$ is a natural number. A function $f : \mathscr{U} \mapsto \mathbb{C}$ is called $(m,n)$-symmetrical if

$$f(\varepsilon z) = \varepsilon^m f(z),\ z \in \mathscr{U}.$$

The family of all $(m,n)$-symmetrical functions is denoted be $\mathscr{S}^{(m,n)}$ . $\mathscr{S}^{(0,2)}$, $\mathscr{S}^{(1,2)}$ and $\mathscr{S}^{(1,n)}$ are respectively the classes of even, odd and $n$-symmetric functions. We have the following decomposition theorem.

**Theorem 1.2.** [4] *For every mapping $f : \mathscr{U} \mapsto \mathbb{C}$, there exists exactly the sequence of $(m,n)$-symmetrical functions $f_{m,n}$,*

$$f(z) = \sum_{m=0}^{n-1} f_{m,n}(z),$$

*where*

$$f_{m,n}(z) = \frac{1}{n}\sum_{v=0}^{n-1} \varepsilon^{-vm} f(\varepsilon^v z). \tag{4}$$

$$(f \in \mathscr{A};\ n = 1,2,...;m = 0,1,2,...,n-1).$$

The following identities follow directly from (4)

$$f'_{m,n}(z) = \frac{1}{n}\sum_{v=0}^{n-1} \varepsilon^{v-vm} f'(\varepsilon^v z),\ f''_{m,n}(z) = \frac{1}{n}\sum_{v=0}^{n-1} \varepsilon^{2v-vm} f''(\varepsilon^v z), \tag{5}$$

$$f_{m,n}(\varepsilon^v z) = \varepsilon^{vm} f_{m,n}(z),\ f'_{m,n}(\varepsilon^v z) = \varepsilon^{vm-v} f'_{m,n}(z). \tag{6}$$

**Definition 1.3.** Let $\mathbf{U}_k(m,n)$ denote the class of functions $f \in \mathscr{A}$ satisfies $f(0) = 0,\ f'(0) = 1$ and,

$$\frac{zf'(z)}{f_{m,n}(z)} \in \mathscr{P}_k,$$

where $f_{m,n}(z)$ is defined by (4).

**Definition 1.4.** Let $\mathbf{V}_k(m,n)$ denote the class of functions $f \in \mathscr{A}$ satisfies $f(0) = 0,\ f'(0) = 1$ and

$$\frac{(zf'(z))'}{f'_{m,n}(z)} \in \mathscr{P}_k,$$

where $f_{m,n}(z)$ is defined by (4).

**Remark 1.5.** *$f \in \mathbf{V}_k(m,n)$ if and only if $zf' \in \mathbf{U}_k(m,n)$.* Spacial cases

(i) For $k = 1, m = 1$ we get Singh and Tygel in [8].

(ii)For $m = n = 1$ we get paatero in [2].

(iii)For $k = 2, m = 1, n = 2$ we get Sakaguchi in [13].

In our paper, we also need the the following lemmas.

**Lemma 1.6.** [3] *Suppose $p(z) \in \mathscr{P}_k$. Then*

$$\Re\left\{\frac{zp'(z)}{p(z)}\right\} \geq \frac{-r(k-4r+kr^2)}{(1-r^2)(1-kr+r^2)},\ wher|z| = r, k \geq 4$$

*and*

$$|z| < R_0 = \frac{k-\sqrt{k^2-4}}{2}.\ For 2 \leq k \leq 4,$$

$$\Re\left\{\frac{zp'(z)}{p(z)}\right\} \geq \frac{-2kr+(8-4k+k^2)r^2-2kr^3}{2(1-r^2)(1-kr+r^2)}.$$

*The above inequality is sharp for function $p(z) = \frac{1-kz+z^2}{1-z^2}$.*

## 2. Main results

**Theorem 2.1.** *A function $f \in \mathscr{A}$ belongs to $\mathbf{U}_k(m,n)$, then*

$$f_{m,n}(z) = z\ exp\left\{-\frac{1}{n}\sum_{v=0}^{n-1}\int_0^{2\pi}\log(1-ze_m^{-i(t-\frac{2\pi v}{n})})dm(t)\right\}. \tag{7}$$

*where $f_{m,n}(z)$ is defined by* (4) *and $m(t)$ is defined* (2).

**Proof.** Suppose that $f \in \mathbf{U}_k(m,n)$. It follows that

$$\frac{zf'(z)}{f_{m,n}(z)} = p_m(z), \tag{8}$$

where

$$p_m(z) = \frac{1}{2}\int_0^{2\pi}\frac{1+ze_m^{-it}}{1-ze_m^{-it}}dm(t). \tag{9}$$

Substituting $z$ by $\varepsilon_m^v z$ in (8) respectively

$$\frac{z\varepsilon_m^v f'(\varepsilon_m^v z)}{f_{m,n}(\varepsilon_m^v z)} = p_m(\varepsilon_m^v z). \tag{10}$$

Then

$$\frac{z\varepsilon_m^{v-vm} f'(\varepsilon_m^v z)}{f_{m,n}(z)} = \frac{1}{2}\int_0^{2\pi} \frac{1+z\varepsilon_m^v e_m^{-it}}{1-z\varepsilon_m^v e_m^{-it}} dm(t), \tag{11}$$

or

$$\frac{z\varepsilon_m^{v-vm} f'(\varepsilon_m^v z)}{f_{m,n}(z)} = \frac{1}{2}\int_0^{2\pi} \frac{1+ze_m^{-i(t-\frac{2\pi v}{n})}}{1-ze_m^{-i(t-\frac{2\pi v}{n})}} dm(t). \tag{12}$$

Let $(v = 0,1,2,....n-1)$ in (12) and summing them we get

$$\frac{f'_{m,n}(z)}{f_{m,n}(z)} - \frac{1}{z} = \frac{1}{2nz}\sum_{v=0}^{n-1}\int_0^{2\pi} \frac{1+ze_m^{-i(t-\frac{2\pi v}{n})}}{1-ze_m^{-i(t-\frac{2\pi v}{n})}} dm(t) - \frac{1}{z}, \tag{13}$$

by integral (13) we have

$$\log\left(\frac{f_{m,n}(z)}{z}\right) = \frac{1}{n}\sum_{v=0}^{n-1}\int_0^{2\pi} -\log[1-ze_m^{-i(t-\frac{2\pi v}{n})}]dm(t), \tag{14}$$

from (14) we get (7). Hence the proof is complete.

**Theorem 2.2.** *A function $f \in \mathscr{A}$ belongs to $\mathbf{U}_k(m,n)$, then*

$$f(z) = \frac{1}{2}\int_0^z \left\{ exp\left[-\frac{1}{n}\sum_{v=0}^{n-1}\int_0^{2\pi} \log(1-ye_m^{-i(t-\frac{2\pi v}{n})})dm(t)\right] . \int_0^{2\pi} \frac{1+ye_m^{-it}}{1-ye_m^{-it}} dm(t)\right\} dy \tag{15}$$

*where $f_{m,n}(z)$ is defined by* (4) *and $m(t)$ is defined* (2).

**Proof.** Suppose that $f \in \mathbf{U}_k(m,n)$. It follows that

$$\frac{zf'(z)}{f_{m,n}(z)} = p_m(z). \tag{16}$$

Then

$$zf'(z) = f_{m,n}(z)p_m(z).$$

By using Theorem 2.1, we get

$$f'(z) = exp\left\{-\frac{1}{n}\sum_{v=0}^{n-1}\int_0^{2\pi} \log(1-ze_m^{-i(t-\frac{2\pi v}{n})})dm(t)\right\} . \frac{1}{2}\int_0^{2\pi} \frac{1+ze_m^{-it}}{1-ze_m^{-it}} dm(t), \tag{17}$$

from (17) we get (15). Hence the proof is complete.

**Corollary 2.3.** *For $m = 1$ and $n = 1$ in Theorem 2.1 we get Paatero [2].*

By using the same method in Theorem 2.1, we have the following corollaries.

**Corollary 2.4.** *A function $f \in \mathscr{A}$ belongs to $\mathbf{V}_k(m,n)$, then*

$$f'_{m,n}(z) = exp\left\{-\frac{1}{n}\sum_{v=0}^{n-1}\int_0^{2\pi}\log(1 - ze_m^{-i(t-\frac{2\pi v}{n})})dm(t)\right\}, \tag{18}$$

*where $f_{m,n}(z)$ is defined by* (4) *and $m(t)$ is defined* (2).

**Corollary 2.5.** *A function $f \in \mathscr{A}$ belongs to $\mathbf{V}_k(m,n)$. Then*

$$f'(z) = \frac{1}{2z}\int_0^z\left\{exp\left[-\frac{1}{n}\sum_{v=0}^{n-1}\int_0^{2\pi}\log(1 - ye_m^{-i(t-\frac{2\pi v}{n})})dm(t)\right]\int_0^{2\pi}\frac{1+ye_m^{-it}}{1-ye_m^{-it}}dm(t)\right\}dy, \tag{19}$$

*where $f_{m,n}(z)$ is defined by* (4) *and $m(t)$ is defined* (2).

**Theorem 2.6.** *A function $f \in \mathscr{A}$ belongs to $\mathbf{U}_k(m,n)$. Then $f_{m,n}(z)$ in $\mathbf{U}_k$.*

**Proof.** Suppose that $f \in \mathbf{U}_k(m,n)$. It follows that

$$\frac{zf'(z)}{f_{m,n}(z)} = p_m(z). \tag{20}$$

Substituting $z$ by $\varepsilon_m^v z$ in (20) respectively

$$\frac{z\varepsilon_m^v f'(\varepsilon_m^v z)}{f_{m,n}(\varepsilon_m^v z)} = p_m(\varepsilon_m^v z). \tag{21}$$

Now let $(v = 0,1,2,....n-1)$ in (21) and summing them we get

$$\frac{zf'_{m,n}(z)}{f_{m,n}(z)} = \frac{1}{n}\sum_{v=0}^{m-1}p_m(\varepsilon_m^v z). \tag{22}$$

It is vivid that $\frac{1}{n}\sum_{v=0}^{m-1}p_m(\varepsilon_m^v z)$ be bongs to $\mathscr{P}_k$. Hence the proof is complete.

**Theorem 2.7.** *Let $f \in \mathbf{U}_k(m,n)$ and let $F(z) = zf'(z)$. Then $F(z)$ is starlike for $|z| < r_2$, where $r_2$ is the least positive root of the equation*

$$1 - 3kr + (k^2+6)r^2 - 3kr^3 + r^4 = 0,$$

*where $|z| = r$ and $k \geq 4$. For $2 \leq k \leq 4$, then $F(z)$ is starlike for $|z| < r_3$ where $r_3$ is the least positive root of the equation*

$$2 - 6kr + (12 - 4k + 3k^2)r^2 - 4kr^3 + 2r^4 = 0.$$

*However the bound $r_3$ is not sharp when $2 \leq k < 4$.*

**Proof.** Let $f \in \mathbf{U}_k(m,n)$. Then

$$F(z) = zexp\left\{-\frac{1}{n}\sum_{v=0}^{n-1}\int_0^{2\pi}\log(1-ze_m^{-i(t-\frac{2\pi v}{n})})dm(t)\right\}.p_m(z).$$

It follows that

$$\frac{zF'(z)}{F(z)} = 1+\frac{1}{n}\sum_{v=0}^{n-1}\int_0^{2\pi}\frac{ze_m^{-i(t-\frac{2\pi v}{n})}}{1-ze_m^{-i(t-\frac{2\pi v}{n})}}dm(t)+\frac{zp_m'(z)}{p_m(z)}, \tag{23}$$

or

$$\frac{zF'(z)}{F(z)} = \frac{1}{n}\sum_{v=0}^{n-1}p_m(\varepsilon_m^v z)+\frac{zp_m'(z)}{p_m(z)}. \tag{24}$$

Hence

$$\Re\left\{\frac{zF'(z)}{F(z)}\right\} = \Re\left\{\frac{1}{n}\sum_{v=0}^{n-1}p_m(\varepsilon_m^v z)\right\}+\Re\left\{\frac{zp_m'(z)}{p_m(z)}\right\}. \tag{25}$$

Therefore, we have

$$\Re\left\{\frac{zp_m'(z)}{p_m(z)}\right\} \geq \frac{-r(k-4r+kr^2)}{(1-r^2)(1-kr+r^2)},\ where\ |z| = r, k \geq 4,$$

and

$$\Re\left\{\frac{1}{n}\sum_{v=0}^{n-1}p_m(\varepsilon_m^v z)\right\} \geq \frac{1-kr+r^2}{(1-r^2)},\ where\ |z| = r, k \geq 4.$$

Then

$$\Re\left\{\frac{zF'(z)}{F(z)}\right\} \geq \frac{1-kr+r^2}{(1-r^2)}+\frac{-r(k-4r+kr^2)}{(1-r^2)(1-kr+r^2)}$$

$$\geq \frac{(1-kr+r^2)^2-r(k-4r+kr^2)}{(1-r^2)(1-kr+r^2)},$$

where $|z| = r < R_0 = \frac{k-\sqrt{k^2-4}}{2}$. Hence $\Re\left\{\frac{zF'(z)}{F(z)}\right\} \geq 0$ provided $Q(r) = 1-3kr+(k^2+6)r^2-3kr^3+r^4 > 0$. The equation $Q(r) = 0$ has a unique positive root in $(0,R_0)$. For $2 \leq k \leq 4$, by using (25), we have

$$\Re\left\{\frac{zF'(z)}{F(z)}\right\} \geq \frac{1-kr+r^2}{(1-r^2)}+\frac{-2kr+(8-4k+k^2)r^2-2kr^3}{2(1-r^2)(1-kr+r^2)},$$

where $|z| = r < R_0 = \frac{k-\sqrt{k^2-4}}{2}$. Hence $\Re\left\{\frac{zF'(z)}{F(z)}\right\} > 0$ provided

$$D(r) = 2-6kr+(12-4k+3k^2)r^2-4kr^3+2r^4 > 0.$$

Also $D(r) = 0$ has a root in $(0,R_0)$.

**Corollary 2.8.** *Let $f \in \mathbf{U}_k(m,n)$. Then $f$ is convex for $|z| < r_2$, where $r_2$ is the least positive root of the equation*

$$1 - 3kr + (k^2+6)r^2 - 3kr^3 + r^4 = 0,$$

*where $|z| = r$ and $k \geq 4$. For $2 \leq k \leq 4$, then $f(z)$ is convex for $|z| < r_3$, where $r_3$ is the least positive root of the equation*

$$2 - 6kr + (12 - 4k + 3k^2)r^2 - 4kr^3 + 2r^4 = 0.$$

*However the bound $r_3$ is not sharp when $2 \leq k < 4$.*

**Theorem 2.9.** *Let $f \in \mathbf{U}_k(m,n)$ and let $F(z) = \int_0^z \frac{\{-f_{m,n}(t)f_{m,n}(-t)\}^{\frac{1}{2}}}{t}dt$. Then $F(z)$ is in $V_k$.*

**Proof.** Since $f \in \mathbf{U}_k(m,n)$, we have

$$F'(z) = \frac{\{-f_{m,n}(z)f_{m,n}(-z)\}^{\frac{1}{2}}}{z},$$

and

$$f_{m,n}(z) = z\, exp\left\{-\frac{1}{n}\sum_{v=0}^{n-1}\int_0^{2\pi}\log(1 - ze_m^{-i(t-\frac{2\pi v}{n})})dm(t)\right\}.$$

Then

$$\frac{(zF'(z))'}{F'(z)} = 1 + \frac{1}{2n}\sum_{v=0}^{n-1}\int_0^{2\pi}\frac{ze_m^{-i(t-\frac{2\pi v}{n})}}{1 - ze_m^{-i(t-\frac{2\pi v}{n})}}dm(t) - \frac{1}{2n}\sum_{v=0}^{n-1}\int_0^{2\pi}\frac{ze_m^{-i(t-\frac{2\pi v}{n})}}{1 + ze_m^{-i(t-\frac{2\pi v}{n})}}dm(t)$$

or

$$\frac{(zF'(z))'}{F'(z)} = \frac{1}{2}\left\{\frac{1}{n}\sum_{v=0}^{n-1}\int_0^{2\pi}\frac{1 + ze_m^{-i(t-\frac{2\pi v}{n})}}{1 - ze_m^{-i(t-\frac{2\pi v}{n})}}dm(t) + \frac{1}{n}\sum_{v=0}^{n-1}\int_0^{2\pi}\frac{1 - ze_m^{-i(t-\frac{2\pi v}{n})}}{1 + ze_m^{-i(t-\frac{2\pi v}{n})}}dm(t)\right\}$$

$$= \frac{1}{2}\left\{\frac{1}{n}\sum_{v=0}^{n-1}p_m(\varepsilon_m^v z) + \frac{1}{n}\sum_{v=0}^{n-1}p_m(-\varepsilon_m^v z)\right\}.$$

Since $p_m(z) \in \mathscr{P}_k$ so $\frac{1}{n}\sum_{v=0}^{n-1}p_m(\varepsilon_m^v z)$ also in $\mathscr{P}_k$, by setting $q(z) = \frac{1}{n}\sum_{v=0}^{n-1}p_m(\varepsilon_m^v z)$, we have

$$\frac{(zF'(z))'}{F'(z)} = \frac{1}{2}\{q(z) + q(-z)\} \text{ where } q(z) \in \mathscr{P}_k,$$

$$= \frac{1}{2}\left\{\left(\frac{k+2}{4}\right)q_1(z) - \left(\frac{k-2}{4}\right)q_2(z)\right\} + \frac{1}{2}\left\{\left(\frac{k+2}{4}\right)q_1(-z) - \left(\frac{k-2}{4}\right)q_2(-z)\right\}$$

$$\frac{(zF'(z))'}{F'(z)} = \left(\frac{k+2}{4}\right)\left\{\frac{q_1(z) + q_1(-z)}{2}\right\} - \left(\frac{k-2}{4}\right)\left\{\frac{q_2(z) + q_2(-z)}{2}\right\},$$

where $q_i(z) \in \mathscr{P}_2,\ i = 1,2$, also $\frac{q_i(z)+q_i(-z)}{2} \in \mathscr{P}_2,\ i = 1,2$. Hence

$$\frac{(zF'(z))'}{F'(z)} = \left(\frac{k+2}{4}\right) w_1(z) - \left(\frac{k-2}{4}\right) w_2(z),$$

where $w_i(z) \in \mathscr{P}_2,\ i = 1,2$. Hence

$$\frac{(zF'(z))'}{F'(z)} \in \mathscr{P}_k,$$

which means $F(z) \in V_k$.

## Conflict of Interests

The authors declare that there is no conflict of interests.

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