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ALEXANDER FUZZY TOPOLOGIES INDUCED BY MAPS

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Abstract. In this paper, we investigate the properties of upper approximation operators and Alexander fuzzy topologies induced by maps in complete residuated lattices. We give their examples.

Keywords: complete residuated lattices; fuzzy preorder; upper approximation operators; Alexander fuzzy topologies.

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1. Introduction

Pawlak [11,12] introduced the rough set theory as a formal tool to deal with imprecision and uncertainty in the data analysis. The relationship between rough set theory and topological spaces was investigated in sets [8], on left-continuous t-norm [13]. Hájek [2] introduced a complete residuated lattice which is an algebraic structure for many valued logic. It is an important mathematical tool for algebraic structure of fuzzy contexts [1-7, 9-10, 13-18]. By using the concepts of lower and upper approximation operators, information systems and decision rules are investigated in complete residuated lattices [1,2,9,10,13,14]. Kim [7] investigated

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between fuzzy rough set and fuzzy quasi-uniform spaces in complete residuated lattices. Kim [6] investigated the properties of upper approximation operators in complete residuated lattices.

In this paper, we investigate the properties of upper approximation operators and Alexander fuzzy topologies induced by maps in complete residuated lattices. We give their examples.

2. Preliminaries

Definition 2.1. [1,2] A structure $(L,\vee,\wedge,\odot,\to,\perp,\top)$ is called a *complete residuated lattice* iff it satisfies the following properties:

(L1) $(L,\vee,\wedge,\perp,\top)$ is a complete lattice where $\perp$ is the bottom element and $\top$ is the top element;

(L2) $(L,\odot,\top)$ is a monoid;

(L3) an adjointness property holds,i.e., for all $x,y,z\in X$,

$$x\le y\to z \text{ iff } x\odot y\le z.$$

A operator $^*: L\to L$ defined by $a^*=a\to\perp$ is called *strong negations* if $a^{**}=a$. For $\alpha\in L, A\in L^X$, we denote $(\alpha\to A),(\alpha\odot A),\bar{\alpha},\top_x,\top_x^*\in L^X$ as

$$(\alpha\to A)(x)=\alpha\to A(x),\ (\alpha\odot A)(x)=\alpha\odot A(x),\ \bar{\alpha}(x)=\alpha,$$
$$\top_x(y)=\begin{cases}\top, & \text{if } y=x,\\ \perp, & \text{otherwise,}\end{cases}\quad \top_x^*(y)=\begin{cases}\perp, & \text{if } y=x,\\ \top, & \text{otherwise.}\end{cases}$$

In this paper, we assume that $(L,\vee,\wedge,\odot,\to,^*,\perp,\top)$ be a complete residuated lattice with a strong negation *.

Definition 2.2. [16,17] Let X be a set. A function $e_X: X\times X\to L$ is called a fuzzy preorder if it satisfies the following conditions:

(E1) $e_X(x,x)=1$ for all $x\in X$,

(E2) $e_X(x,y)\odot e_X(y,z)\le e_X(x,z)$, for all $x,y,z\in X$.

Example 2.3. (1) We define a function $e_L: L\times L\to L$ as $e_L(x,y)=x\to y$. Then (L,e_L) is a fuzzy preorder.

(2) We define a function $e_{L^X} : L^X \times L^X \to L$ as $e_{L^X}(A,B) = \bigwedge_{x\in X}(A(x) \to B(x))$. Then (L^X, e_{L^X}) is a fuzzy preorder.

Definition 2.4. [6] An operator $\mathbf{T} : L^X \to L$ is called an *Alexander fuzzy topology* on X iff it satisfies the following conditions: for all $\alpha \in L, A, A_i \in L^X$,

(T1) $\mathbf{T}(\overline{\alpha}) = \top$,

(T2) $\mathbf{T}(\bigwedge_{i\in\Gamma} A_i) \geq \bigwedge_{i\in\Gamma} \mathbf{T}(A_i)$ and $\mathbf{T}(\bigvee_{i\in\Gamma} A_i) \geq \bigwedge_{i\in\Gamma} \mathbf{T}(A_i)$,

(T3) $\mathbf{T}(\alpha \odot A) \geq \mathbf{T}(A)$,

(T4) $\mathbf{T}(\alpha \to A) \geq \mathbf{T}(A)$.

A map $f : (X, \mathbf{T}_X) \to (Y, \mathbf{T}_Y)$ is fuzzy continuous if $\mathbf{T}_X(f^{-1}(B)) \geq \mathbf{T}_Y(B)$ for all $B \in L^Y$.

Definition 2.5.[6] A map $\Phi : L^X \to L^X$ is called an *upper approximation operator* iff it satisfies the following conditions

(H1) $\Phi(\alpha \odot A) = \alpha \odot \Phi(A)$ for all $A \in L^X$ and $\alpha \in L$.

(H2) $\Phi(\bigvee_{i\in I} A_i) = \bigvee_{i\in I} \Phi(A_i)$ for all $A_i \in L^X$.

(H3) $A \leq \Phi(A)$,

(H4) $\Phi(\Phi(A)) \leq \Phi(A)$, for all $A \in L^X$.

Lemma 2.6. [1,2] *Let* $(L, \vee, \wedge, \odot, \to, ^*, \bot, \top)$ *be a complete residuated lattice with a strong negation* *. *For each* $x, y, z, x_i, y_i \in L$, *the following properties hold.*

(1) *If* $y \leq z$, *then* $x \odot y \leq x \odot z$.

(2) *If* $y \leq z$, *then* $x \to y \leq x \to z$ *and* $z \to x \leq y \to x$.

(3) $x \to y = \top$ *iff* $x \leq y$.

(4) $x \to \top = \top$ *and* $\top \to x = x$.

(5) $x \odot y \leq x \wedge y$.

(6) $x \odot (\bigvee_{i\in\Gamma} y_i) = \bigvee_{i\in\Gamma}(x \odot y_i)$ *and* $(\bigvee_{i\in\Gamma} x_i) \odot y = \bigvee_{i\in\Gamma}(x_i \odot y)$.

(7) $x \to (\bigwedge_{i\in\Gamma} y_i) = \bigwedge_{i\in\Gamma}(x \to y_i)$ *and* $(\bigvee_{i\in\Gamma} x_i) \to y = \bigwedge_{i\in\Gamma}(x_i \to y)$.

(8) $\bigvee_{i\in\Gamma} x_i \to \bigvee_{i\in\Gamma} y_i \geq \bigwedge_{i\in\Gamma}(x_i \to y_i)$ *and* $\bigwedge_{i\in\Gamma} x_i \to \bigwedge_{i\in\Gamma} y_i \geq \bigwedge_{i\in\Gamma}(x_i \to y_i)$.

(9) $(x \to y) \odot x \leq y$ *and* $(y \to z) \odot (x \to y) \leq (x \to z)$.

(10) $x \to y \leq (y \to z) \to (x \to z)$ *and* $x \to y \leq (z \to x) \to (z \to y)$.

(11) $\bigwedge_{i\in\Gamma} x_i^* = (\bigvee_{i\in\Gamma} x_i)^*$ *and* $\bigvee_{i\in\Gamma} x_i^* = (\bigwedge_{i\in\Gamma} x_i)^*$.

(12) $(x \odot y) \to z = x \to (y \to z) = y \to (x \to z)$ *and* $(x \odot y)^* = x \to y^*$.

(13) $x^* \to y^* = y \to x$ *and* $(x \to y)^* = x \odot y^*$.

(14) $y \to z \le x \odot y \to x \odot z$.

Theorem 2.7. [6] *Let* $\mathbf{T} : L^X \to L$ *be an Alexander fuzzy topology. Define* $\mathbf{T}^*(A) = \mathbf{T}(A^*)$. *Then* $\mathbf{T}^*$ *is an Alexander fuzzy topology.*

Theorem 2.8 [6] *Let* $\mathbf{T}_X$ *be an Alexandrov fuzzy topology on* X. *Define* $\Phi_{T_X} : L^X \times L \to L^X$ *as follows*

$$\Phi_{T_X}(A,r) = \bigwedge\{B \in L^X \mid A \le B,\ \mathbf{T}_X(B) \ge r^*\}$$

Then we have the following properties.

(1) $\Phi_{T_X}(-,r) : L^X \to L^X$ *is an upper approximation operator.*

(2) *If* $r \le s$, *then* $\Phi_{T_X}(A,s) \le \Phi_{T_X}(A,r)$ *for all* $A \in L^X$.

(3) *There exists a fuzzy preorder* $R^r_{T_X} \in L^{X \times X}$ *such that*

$$\Phi_{T_X}(A,r) = \bigvee_{x \in X} (A(x) \odot R^r_{T_X}(x,y)).$$

(4) *If* $r \le s$, *then* $R^r_{T_X} \ge R^s_{T_X}$ *for all* $A \in L^X$.

(5) *If* $\Phi_{T_X}(A,r_i) = B$ *for all* $i \in \Gamma \ne \emptyset$, *then* $\Phi_{T_X}(A, \bigwedge_{i \in \Gamma} r_i) = B$.

(6) *Define* $\mathbf{T}_{H_{T_X}} : L^X \to L$ *as*

$$\mathbf{T}_{H_{T_X}}(A) = \bigvee\{r_i^* \in L \mid \Phi_{T_X}(A,r_i) = A\}$$

Then $\mathbf{T}_{H_{T_X}} = \mathbf{T}_X$ *is an Alexandrov fuzzy topology on* X.

(7) *There exists an Alexandrov fuzzy topology* $\mathbf{T}^r_X$ *such that*

$$\mathbf{T}^r_X(A) = e_{L^X}(\Phi_{T_X}(A,r),A).$$

(8) *If* $r \le s$, *then* $\mathbf{T}^r_X \le \mathbf{T}^s_X$ *for all* $A \in L^X$.

(9) *Define* $\mathbf{T}_{T_X} : L^X \to L$ *as*

$$\mathbf{T}_{T_X}(A) = \bigvee\{r^* \in L \mid \mathbf{T}^r_X(A) = \top\}.$$

Then $\mathbf{T}_{T_X} = \mathbf{T}^*_X = \mathbf{T}_{H_{T_X}}$ *is an Alexandrov fuzzy topology on* X.

3. Alexander fuzzy topologies induced by maps

Theorem 3.1. *Let $\mathbf{T}_Y$ be an Alexandrov fuzzy topology on Y. Then $f : X \to Y$ be a map. Define $\mathbf{T}_X, \mathbf{T}^*_X : L^X \to L$ as follows*

$$\mathbf{T}_X(A) = \begin{cases} \bigvee\{\mathbf{T}_Y(B) \mid A = f^{-1}(B)\}, & \text{if } A = f^{-1}(B), \\ \perp, & \text{if } A \neq f^{-1}(B), \end{cases}$$

$$\mathbf{T}^*_X(A) = \begin{cases} \bigvee\{\mathbf{T}^*_Y(B) \mid A = f^{-1}(B)\}, & \text{if } A = f^{-1}(B). \\ \perp, & \text{if } A \neq f^{-1}(B), \end{cases}$$

Then the following properties hold.

(1) *$\mathbf{T}_X$ is the coarsest Alexandrov fuzzy topology on X for each $f : (X, \mathbf{T}) \to (Y, \mathbf{T}_Y)$ is fuzzy continuous.*

(2) *$\mathbf{T}^*_X$ is the coarsest Alexandrov fuzzy topology on X for each $f : (X, \mathbf{T}) \to (Y, \mathbf{T}^*_Y)$ is fuzzy continuous. Moreover, $\mathbf{T}^*_X(A) = \mathbf{T}_X(A^*)$ for each $A \in L^X$.*

(3) *$f^{-1}(\Phi_{T_Y}(\top_{f(x)}, r)) \geq \Phi_{T_X}(\top^*_x, r)$ for all $x \in X$. If f is surjective, then the equality holds.*

(4) *$f^{-1}(\Phi_{T^*_Y}(\top_{f(x)}, r)) \geq \Phi_{T^*_X}(\top_x, r)$ for all $x \in X$. If f is surjective, then the equality holds.*

(5) *There exists fuzzy preorder $R^r_{T_X} \in L^{X \times X}$ and $R^r_{T_Y} \in L^{Y \times Y}$ such that*

$$R^r_{T_X}(x, y) \leq R^r_{T_Y}(f(x), f(y)).$$

If f is surjective, then the equality holds.

(6) *There exists fuzzy preorder $R^r_{T^*_X} \in L^{X \times X}$ and $R^r_{T^*_Y} \in L^{Y \times Y}$ such that*

$$R^r_{T^*_X}(x, y) \leq R^r_{T^*_Y}(f(x), f(y)).$$

If f is surjective, then the equality holds.

(7) *$f^{-1}(\Phi_{T_Y}(B, r))(y) \geq \Phi_{T_X}(f^{-1}(B), r)(y)$ for all $y \in X$ and $B \in L^Y$. If f is surjective, then the equality holds.*

(8) *$f^{-1}(\Phi_{T^*_Y}(B, r))(y) \geq \Phi_{T^*_X}(f^{-1}(B), r)(y)$ for all $y \in X$ and $B \in L^Y$. If f is surjective, then the equality holds.*

(9) *$\mathbf{T}^r_{T_X}(f^{-1}(B)) \geq \mathbf{T}^r_{T_Y}(B)$ for all $B \in L^Y$ and $r \in L$. If f is surjective, then the equality holds.*

(10) *$\mathbf{T}^r_{T^*_X}(f^{-1}(B)) \geq \mathbf{T}^r_{T^*_Y}(B)$ for all $B \in L^Y$ and $r \in L$. If f is surjective, then the equality holds.*

(11) $\mathbf{T}^r_{T_Y}(B) = \mathbf{T}^r_{T_Y^*}(B^*)$ *for all* $B \in L^Y$ *and* $r \in L$ *iff* $\Phi_{T_Y}(\top^*_x, r)(y) = \Phi_{T_Y^*}(\top^*_y, r)(x)$ *for all* $x, y \in Y$ *iff* $R^r_{T_Y^*}(x,y) = R^r_{T_X}(y,x)$ *for all* $x, y \in Y$.

(12) *Define* $\mathbf{T}_{T_X} : L^X \to L$ *as*

$$\mathbf{T}_{T_X}(A) = \bigvee\{r^* \in L \mid \mathbf{T}^r_{T_X}(A) = \top\}.$$

Then $\mathbf{T}_{T_X} = \mathbf{T}^*_X = \mathbf{T}_{H_{T_X}}$ *is an Alexandrov fuzzy topology on* X.

(13) *Define* $\mathbf{T}_{T_X^*} : L^X \to L$ *as*

$$\mathbf{T}_{T_X^*}(A) = \bigvee\{r^* \in L \mid \mathbf{T}_{T_X^*}(A) = \top\}.$$

Then $\mathbf{T}_{T_X^*} = \mathbf{T}_X = \mathbf{T}_{H_{T_X^*}}$ *is an Alexandrov fuzzy topology on* X.

Proof (1) (T1) $\mathbf{T}_X(\overline{\alpha}) = \bigvee\{\mathbf{T}_Y(B) \mid \overline{\alpha} = f^{-1}(B)\} \geq \mathbf{T}_Y(\overline{\alpha}) = \top$.

(T2)

$$\begin{aligned}
\textstyle\bigwedge_{i\in\Gamma} \mathbf{T}_X(A_i) &= \textstyle\bigwedge_{i\in\Gamma}(\bigvee\{\mathbf{T}_Y(B_i) \mid A_i = f^{-1}(B_i)\}) \\
&= \textstyle\bigvee(\{\bigwedge_{i\in\Gamma} \mathbf{T}_Y(B_i) \mid A_i = f^{-1}(B_i)\}) \\
&\leq \textstyle\bigvee(\{\mathbf{T}_Y(\bigwedge_{i\in\Gamma} B_i) \mid \bigwedge_{i\in\Gamma} A_i = f^{-1}(\bigwedge_{i\in\Gamma} B_i)\}) \\
&\leq \mathbf{T}_X(\textstyle\bigwedge_{i\in\Gamma} A_i).
\end{aligned}$$

$$\begin{aligned}
\textstyle\bigwedge_{i\in\Gamma} \mathbf{T}_X(A_i) &= \textstyle\bigwedge_{i\in\Gamma}(\bigvee\{\mathbf{T}_Y(B_i) \mid A_i = f^{-1}(B_i)\}) \\
&= \textstyle\bigvee(\{\bigwedge_{i\in\Gamma} \mathbf{T}_Y(B_i) \mid A_i = f^{-1}(B_i)\}) \\
&\leq \textstyle\bigvee(\{\mathbf{T}_Y(\bigvee_{i\in\Gamma} B_i) \mid \bigvee_{i\in\Gamma} A_i = f^{-1}(\bigvee_{i\in\Gamma} B_i)\}) \\
&\leq \mathbf{T}_X(\textstyle\bigvee_{i\in\Gamma} A_i).
\end{aligned}$$

(T3) $\mathbf{T}(\alpha \odot A) \geq \mathbf{T}(A)$,

$$\begin{aligned}
\mathbf{T}_X(A) &= \textstyle\bigvee\{\mathbf{T}_Y(B) \mid A = f^{-1}(B)\} \\
&\leq \textstyle\bigvee\{\mathbf{T}_Y(\alpha \odot B) \mid \alpha \odot A = f^{-1}(\alpha \odot B)\} \\
&\leq \mathbf{T}(\alpha \odot A).
\end{aligned}$$

(T4) $\mathbf{T}(\alpha \to A) \geq \mathbf{T}(A)$.

$$\begin{aligned}\mathbf{T}_X(A) &= \bigvee\{\mathbf{T}_Y(B) \mid A = f^{-1}(B)\}\\ &\leq \bigvee\{\mathbf{T}_Y(\alpha \to B) \mid \alpha \to A = f^{-1}(\alpha \to B)\}\\ &\leq \mathbf{T}(\alpha \to A).\end{aligned}$$

Let $f : (X, \mathbf{T}) \to (Y, \mathbf{T}_Y)$ be fuzzy continuous. Then $\mathbf{T}(f^{-1}(B)) \geq \mathbf{T}_Y(B)$ for each $B \in L^Y$.

$$\begin{aligned}\mathbf{T}_X(A) &= \bigvee\{\mathbf{T}_Y(B) \mid A = f^{-1}(B)\}\\ &\leq \bigvee\{\mathbf{T}(f^{-1}(B)) \mid A = f^{-1}(B)\} = \mathbf{T}(A).\end{aligned}$$

(2) $\mathbf{T}_X(A^*) = \bigvee\{\mathbf{T}_Y(B) \mid A^* = f^{-1}(B)\} = \bigvee\{\mathbf{T}_Y(B) \mid A = f^{-1}(B^*)\} = \bigvee\{\mathbf{T}^*_Y(B^*) \mid A = f^{-1}(B^*)\} = \mathbf{T}^*_X(A)$. Other cases are similarly proved as (1).

(3) Since $\mathbf{T}_X(f^{-1}(B)) \geq \mathbf{T}_Y(B)$ for all $B \in L^Y$, we have

$$\begin{aligned}&f^{-1}(\Phi_{T_Y}(\top_{f(x)}, r))(y) = \Phi_{T_Y}(\top_{f(x)}, r)(f(y))\\ &= \bigwedge\{B(f(y)) \mid \top_{f(x)} \leq B,\ \mathbf{T}_Y(B) \geq r^*\}\\ &\geq \bigwedge\{f^{-1}(B)(y) \mid \top_x \leq f^{-1}(B),\ \mathbf{T}_X(f^{-1}(B)) \geq r^*\}\\ &= \Phi_{T_X}(\top_x, r).\end{aligned}$$

If f is surjective and $f^{-1}(B_1)(x) = f^{-1}(B_2)(x)$ for $x \in X$, then $B_1 = B_2$. Thus, $f^{-1}(\Phi_{T_Y}(\top_{f(x)}, r))(y) = \Phi_{T_X}(\top_x, r)(y)$ for all $x \in X$.

(5) By Theorem 2.8(3) and (3), there exists fuzzy preorder $R^r_{T_X} \in L^{X\times X}$ and $R^r_{T_Y} \in L^{Y\times Y}$ such that

$$\begin{aligned}&f^{-1}(\Phi_{T_Y}(\top_{f(x)}, r))(y) \geq \Phi_{T_X}(\top_x, r)(y)\\ &\text{iff } \Phi_{T_Y}(\top_{f(x)}, r))(f(y)) = R^r_{T_Y}(f(x), f(y))\\ &\geq \Phi_{T_X}(\top_x, r)(y) = R^r_{T_X}(x, y).\end{aligned}$$

(7) By (3), for $B = \bigvee_{z\in Y}(B(z) \odot \top_z)$, we have

$$\begin{aligned}&f^{-1}(\Phi_{T_Y}(B, r))(y) = \Phi_{T_Y}(B, r)(f(y))\\ &= \Phi_{T_Y}(\bigvee_{z\in Y}(B(z) \odot \top_z), r)(f(y))\\ &= \bigvee_{z\in Y}(B(z) \odot \Phi_{T_Y}(\top_z, r)(f(y))\\ &\geq \bigvee_{x\in X}(B(f(x)) \odot \Phi_{T_Y}(\top_{f(x)}, r)(f(y))\\ &\geq \bigvee_{x\in X}((f^{-1}(B)(x) \odot \Phi_{T_X}(\top_x, r)(y)\\ &= \Phi_{T_X}(f^{-1}(B), r)(y).\end{aligned}$$

If f is surjective

$$
\begin{aligned}
& f^{-1}(\Phi_{T_Y}(B,r))(y) = \bigvee_{z\in Y}(B(z)\odot \Phi_{T_Y}(\top_z,r)(f(y)) \\
& = \bigvee_{x\in X}(B(f(x))\odot \Phi_{T_Y}(\top_{f(x)},r)(f(y)) \\
& = \bigvee_{x\in X}((f^{-1}(B)(x)\odot \Phi_{T_X}(\top_x,r)(y) \\
& = \Phi_{T_X}(f^{-1}(B),r)(y).
\end{aligned}
$$

(9)

$$
\begin{aligned}
& \mathbf{T}^r_X(f^{-1}(B))) = e_{L^X}(\Phi_{T_X}(f^{-1}(B),r),f^{-1}(B)) \\
& = \bigwedge_{x\in X}(\Phi_{T_X}(f^{-1}(B),r)(x)\to f^{-1}(B)(x)) \\
& \geq \bigwedge_{x\in X}(f^{-1}(\Phi_{T_Y}(B,r))(x)\to B(f(x))) \\
& \geq \bigwedge_{y\in Y}(f^{-1}(\Phi_{T_Y}(B,r))(y)\to B(y)) \\
& = \mathbf{T}^r_{T_Y}(B).
\end{aligned}
$$

If f is surjective, then the equality holds.

(11) Let $\Phi_{T_Y}(\top_x,r)(y) = \Phi_{T_Y^*}(\top_y,r)(x)$ for all $x,y\in Y$. For $B=\bigvee_{y\in X}(B(y)\odot\top_y)$, we have

$$
\begin{aligned}
\mathbf{T}^r_{T_Y}(B) \quad & = \bigwedge_{x\in Y}(\Phi_{T_Y}(B,r)(x)\to B(x)) \\
& = \bigwedge_{x\in Y}(\Phi_{T_Y}(\bigvee_{y\in Y}(B(y)\odot\top_y),r)(x)\to B(x)) \\
& = \bigwedge_{x\in X}(\bigvee_{y\in X}(B(y)\odot\Phi_{T_Y}(\top_y,r)(x)\to B(x)) \\
& = \bigwedge_{x,y\in Y}(\Phi_{T_Y}(\top_y,r)(x)\to (B(y)\to B(x)) \\
& = \bigwedge_{x,y\in Y}(\Phi_{T_Y^*}(\top_x,r)(y)\to (B^*(x)\to B^*(y)) \\
& = \mathbf{T}^r_{T_Y^*}(B^*).
\end{aligned}
$$

Let $\mathbf{T}^r_{T_Y}(B) = \mathbf{T}^r_{T_Y^*}(B^*)$ be given.

$$
\begin{aligned}
\mathbf{T}^r_{T_Y}(B) \quad & = \bigwedge_{x,y\in Y}(\Phi_{T_Y}(\top_y,r)(x)\to (B(y)\to B(x)) \\
\mathbf{T}^r_{T_Y^*}(B^*) \quad & = \bigwedge_{x,y\in Y}(\Phi_{T_Y^*}(\top_x,r)(y)\to (B^*(x)\to B^*(y)) \\
& = \bigwedge_{x,y\in Y}(\Phi_{T_Y^*}(\top_x,r)(y)\to (B(y)\to B(x)).
\end{aligned}
$$

Put $B=\top_y$. Then $\Phi^*_{T_Y}(\top_y,r)(x) = \Phi^*_{T_Y^*}(\top_x,r)(y)$. Hence $\Phi_{T_Y}(\top_y,r)(x) = \Phi_{T_Y^*}(\top_x,r)(y)$.

(12) Since $\mathbf{T}^r_{T_X}(A) = e_{L^X}(\Phi_{T_X}(A,r),A) = \top$ iff $A=\Phi_{T_X}(A,r)$, by (6), the result holds.

Example 3.2. Let $Y = \{x,y,z\}$ be a set and $(L = [0,1], \odot, \to, ^*)$ be a complete residuated lattice with a strong negation defined by

$$x \odot y = (x+y-1) \vee 0,\ \ x \to y = (1-x+y) \wedge 1,\ x^* = 1-x.$$

(1) Let $Y = \{x,y,z\}$ be a set. Define a map $\mathbf{T}_Y : [0,1]^Y \to [0,1]$ as

$$\mathbf{T}_Y(A) = (1-A(x)+A(z)) \wedge 1 = A(x) \to A(z).$$

Trivially, $\mathbf{T}_Y(\overline{\alpha}) = \alpha$.

Since $\alpha \odot A(x) \to \alpha \odot A(z) \geq A(x) \to A(z)$, $\mathbf{T}_Y(\alpha \odot A) \geq \mathbf{T}_Y(A)$. Since $(\alpha \to A(x)) \to (\alpha \to A(z)) \geq A(x) \to A(z)$, $\mathbf{T}_Y(\alpha \to A) \geq \mathbf{T}_Y(A)$. By Lemma 2.10 (8), $\mathbf{T}_Y(\bigvee_{i\in\Gamma} A_i) \geq \bigwedge_{i\in\Gamma} \mathbf{T}_Y(A_i)$ and $\mathbf{T}_Y(\bigwedge_{i\in\Gamma} A_i) \geq \bigwedge_{i\in\Gamma} \mathbf{T}_Y(A_i)$. Hence $\mathbf{T}_Y$ is an Alexandrov fuzzy topology.

By Theorem 2.8 (1), we obtain an upper approximation operator $\Phi_{T_Y}(-,r) : L^Y \to L^Y$ as follows:

$$\Phi_{T_Y}(1_x,r)(z) = \bigwedge\{B(z) \mid B \geq 1_x,\ \mathbf{T}_Y(B) \geq r^*\}.$$

Since $B(x) = 1$ and $\mathbf{T}_Y(B) = 1-1+B(z) \geq 1-r$, then $B(z) \geq 1-r$. We have $\Phi_{T_Y}(1_x,r)(z) = 1-r$.

$$\Phi_{T_Y}(1_x,r)(x) = \bigwedge\{B(x) \mid B \geq 1_x,\ \mathbf{T}_Y(B) \geq r^*\} = 1,$$

$$\Phi_{T_Y}(1_x,r)(y) = \bigwedge\{B(y) \mid B \geq 1_x,\ \mathbf{T}_Y(B) \geq r^*\} = 0,$$

$$\Phi_{T_Y}(1_z,r)(x) = \bigwedge\{B(x) \mid B \geq 1_z,\ \mathbf{T}_Y(B) \geq r^*\}.$$

Since $B(z) = 1$ and $\mathbf{T}_Y(B) = (1-B(x)+1) \wedge 1 = 1$, then $\Phi_{T_Y}(1_z,r)(x) = 0$.

$$\begin{pmatrix} \Phi_{T_Y}(1_x,r)(x) = 1 & \Phi_{T_Y}(1_x,r)(y) = 0 & \Phi_{T_Y}(1_x,r)(z) = 1-r \\ \Phi_{T_Y}(1_y,r)(x) = 0 & \Phi_{T_Y}(1_y,r)(y) = 1 & \Phi_{T_Y}(1_y,r)(z) = 0 \\ \Phi_{T_Y}(1_z,r)(x) = 0 & \Phi_{T_Y}(1_z,r)(y) = 0 & \Phi_{T_Y}(1_z,r)(z) = 1 \end{pmatrix}.$$

For $A = \bigvee_{x\in X}(A(x) \odot \top_x)$, we have

$$\Phi_{T_Y}(A,r)(y) = \bigvee_{x\in X}(A(x) \odot \Phi_{T_Y}(\top_x,r)(y))$$

$$\Phi_{T_Y}(A,r) = (A(x), A(y), A(z) \vee (A(x)-r)).$$

If $A(x)-r \le A(z)$, then $\Phi_{T_Y}(A,r)=A$. Thus

$$\begin{aligned}\mathbf{T}_{H_{T_Y}}(A) &= \bigvee\{r^* \in L \mid \Phi_{T_Y}(A,r)=A\}\\ &= (1-A(x)+A(z))\wedge 1 = \mathbf{T}_Y(A).\end{aligned}$$

Moreover, we obtain

$$\begin{aligned}\mathbf{T}^r_Y(A) &= \bigwedge_{x\in X}(\Phi_{T_Y}(A,r)(x)\to A(x))\\ &= A(z)\vee (A(x)-r)\to A(z)\\ &= (A(z)\to A(z))\wedge((A(x)-r)\to A(z))\\ &= (1+r-A(x)+A(z))\vee 0.\end{aligned}$$

$$\begin{aligned}\mathbf{T}_{T_Y}(A) &= \bigvee\{1-r\in L \mid \mathbf{T}^r_Y(A)=1\}\\ &= (1-A(x)+A(z))\wedge 1.\end{aligned}$$

Hence $\mathbf{T}_{T_Y}=\mathbf{T}_{H_{T_Y}}=\mathbf{T}_Y$. Since $R^r_{T_Y}(x,y)=\Phi_{T_Y}(1_x,r)(y)$, then $\Phi_{T_Y}(A,r)(y)=\bigvee_{x\in X}(A(x)\odot R^r_{T_Y}(x,y))$ with

$$R^r_{T_Y} = \begin{pmatrix} 1 & 0 & 1-r\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$$

(2) Let $X=\{a,b,c,d\}, Y=\{x,y,z\}$ be a set and $f: X\to Y$ be a map as follows:

$$f(a)=f(b)=x, f(c)=y, f(d)=z.$$

Since f is surjective and $f^{-1}(B_1)(x)=f^{-1}(B_2)(x)$ for $x\in X$, then $B_1=B_2$. By Theorem 3.1, we obtain a map $\mathbf{T}_X:[0,1]^X\to[0,1]$ as

$$\mathbf{T}_X(A)=\begin{cases}(1-B(f(a))+B(f(d)))\wedge 1, & \text{if } A=f^{-1}(B),\\ 0, & \text{if } A\ne f^{-1}(B).\end{cases}$$

For $A_1(a)=A_1(b)=0.7, A_1(c)=0.8, A_1(d)=0.5$, we have $B_1(x)=0.7, B_1(y)=0.8, B_1(z)=0.5$ such that $A_1=f^{-1}(B_1)$. Hence $\mathbf{T}_X(A_1)=(1-B_1(x)+B_1(z))\wedge 1=0.8$.

For $A_2(a)=0.6, A_2(b)=0.3, A_1(b)=0.8, A_1(c)=0.5$, $A_2\ne f^{-1}(B)$. Hence $\mathbf{T}_X(A_2)=0$.

We obtain an upper approximation operator $\Phi_{T_X}(-,r): L^X\to L^X$ as follows:

$$\Phi_{T_X}(1_a,r)(d)=\bigwedge\{f^{-1}(B)(d)\mid f^{-1}(B)\ge 1_a,\ \mathbf{T}_Y(B)\ge r^*\}.$$

Since $B(f(a)) = B(x) = 1$ and $\mathbf{T}_Y(B) = 1-1+B(f(d)) \geq 1-r$, then $B(f(d)) \geq 1-r$. We have $\Phi_{T_X}(1_a,r)(d) = 1-r$. Similarly, $\Phi_{T_X}(1_b,r)(d) = 1-r$,

$$\Phi_{T_X}(1_a,r)(a) = \Phi_{T_X}(1_a,r)(b) = 1, \Phi_{T_X}(1_a,r)(c) = 0,$$

$$\Phi_{T_X}(1_b,r)(a) = \Phi_{T_X}(1_b,r)(b) = 1, \Phi_{T_X}(1_b,r)(c) = 0,$$

$$\Phi_{T_X}(1_c,r)(a) = \Phi_{T_X}(1_c,r)(b) = \Phi_{T_X}(1_c,r)(d) = 0, \Phi_{T_X}(1_c,r)(c) = 1,$$

$$\Phi_{T_X}(1_d,r)(a) = \bigwedge\{f^{-1}(B)(a) \mid f^{-1}(B) \geq 1_d,\ \mathbf{T}_Y(B) \geq r^*\}.$$

Since $B(f(d)) = B(z) = 1$ and $\mathbf{T}_Y(B) = 1 \geq 1-r$, then $\Phi_{T_X}(1_d,r)(a) = 0$. Similarly, $\Phi_T(1_d,r)(b) = 0$,

$$\Phi_{T_X}(1_d,r)(c) = 0, \Phi_{T_X}(1_a,r)(d) = 1.$$

Since $R^r_{T_X}(a,b) = \Phi_{T_X}(1_a,r)(b)$, then $\Phi_{T_X}(A,r)(b) = \bigvee_{x\in X}(A(x) \odot R^r_{T_X}(x,b))$ with

$$R^r_{T_X} = \begin{pmatrix} 1 & 1 & 0 & 1-r \\ 1 & 1 & 0 & 1-r \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

For $A = \bigvee_{a\in X}(A(a) \odot \top_a)$, we have

$$\Phi_{T_X}(A,r)(b) = \bigvee_{a\in X}(A(a) \odot \Phi_{T_X}(\top_a,r)(b)),$$

$$\begin{aligned} \Phi_{T_X}(A,r) &= (A(a) \vee A(b), A(a) \vee A(b), \\ &\quad A(c), A(d) \vee (A(a)-r) \vee (A(b)-r)). \end{aligned}$$

If $A(a) = A(b)$ and $A(a) - r \leq A(d)$, then $\Phi_{T_X}(A,r) = A$. Thus

$$\begin{aligned} \mathbf{T}_{H_{T_X}}(A) &= \bigvee\{r^* \in L \mid \Phi_{T_X}(A,r) = A\} \\ &= (1 - A(a) + A(d)) \wedge 1 = \mathbf{T}_X(A). \end{aligned}$$

Moreover, we obtain

$$\begin{aligned} \mathbf{T}^r_X(A) &= \textstyle\bigwedge_{x\in X}(\Phi_{T_X}(A,r)(x) \to A(x)) \\ &= (A(b) \to A(a)) \wedge (A(a) \to A(b)) \\ &\wedge \Big(((A(a)-r) \to A(d)) \wedge ((A(b)-r) \to A(d))\Big). \end{aligned}$$

For $B(x)=0.9, B(y)=0.3, B(z)=0.2$, $f^{-1}(B)(a)=f^{-1}(B)(b)=0.9, f^{-1}(c)=0.3, f^{-1}(d)=0.2$. Then

$$\begin{aligned}\mathbf{T}_Y^{0.5}(B) &= (1+r-B(x)+B(z))\vee 0 = 0.8\\ \mathbf{T}_X^{0.5}(f^{-1}(B)) &= (f^{-1}(B)(a)-0.5)\to f^{-1}(B)(d))\\ &\wedge((f^{-1}(B)(b)-0.5)\to f^{-1}(B)(d)) = 0.8.\end{aligned}$$

If $\mathbf{T}_X^r(A)=1$, then $A(a)=A(b), A(a)-r\le A(d)$. So, $1-r\le 1-A(a)+A(d)$. Thus,

$$\begin{aligned}\mathbf{T}_T(A) &= \bigvee\{1-r\in L \mid \mathbf{T}^r(A)=1\}\\ &= (1-A(a)+A(d))\wedge 1.\end{aligned}$$

Hence $\mathbf{T}_{T_X}=\mathbf{T}_{H_{T_X}}=\mathbf{T}_X$.

(3) Let $X=\{a,b,c,d\}, Y=\{x,y,z\}$ be a set and $g: X\to Y$ be a map as follows:

$$g(a)=g(b)=x, g(c)=g(d)=y.$$

For each $B\in L^Y$ with $B(z)=1$ and $A=g^{-1}(B)$, we have $\mathbf{T}_X(A)=1$. Thus, we obtain a map $\mathbf{T}_X:[0,1]^X\to[0,1]$ as

$$\mathbf{T}_X(A)=\begin{cases}1, & \text{if } A=g^{-1}(B),\\ 0, & \text{if } A\neq g^{-1}(B).\end{cases}$$

For $A_1(a)=A_1(b)=0.3, A_1(c)=A_1(d)=0.5$, we have $B_1(x)=0.3, B_1(y)=0.5, B_1(z)=1$ such that $A_1=g^{-1}(B_1)$. Hence $\mathbf{T}_X(A_1)=(1-B_1(x)+B_1(z))\wedge 1=1$.

For $A_2(a)=A_2(b)=0.3, A_1(b)=0.8, A_1(c)=0.5, A_2\neq g^{-1}(B)$. Hence $\mathbf{T}_X(A_2)=0$.

We obtain an upper approximation operator $\Phi_{T_X}(-,r): L^X\to L^X$ as follows:

$$\Phi_{T_X}(1_a,r)(d)=\bigwedge\{g^{-1}(B)(d)\mid g^{-1}(B)\ge 1_a,\ \mathbf{T}_Y(B)\ge r\}.$$

Since $B(f(a))=B(x)=1$ for $B(z)=1$ and $\mathbf{T}_Y(B)=(1-1+B(f(d))+1)\wedge 1=1$, then $\Phi_{T_X}(1_a,r)(d)=0$. Similarly, $\Phi_{T_X}(1_b,r)(d)=0$,

$$\Phi_{T_X}(1_a,r)(a)=\Phi_{T_X}(1_a,r)(b)=1, \Phi_{T_X}(1_a,r)(c)=0,$$

$$\Phi_{T_X}(1_b,r)(a)=\Phi_{T_X}(1_b,r)(b)=1, \Phi_{T_X}(1_b,r)(c)=0,$$

$$\Phi_{T_X}(1_c,r)(a)=\Phi_{T_X}(1_c,r)(b)=0, \Phi_{T_X}(1_c,r)(c)=\Phi_{T_X}(1_c,r)(d)=1,$$

$$\Phi_{T_X}(1_d,r)(a)=\bigwedge\{g^{-1}(B)(a)\mid g^{-1}(B)\ge 1_d,\ \mathbf{T}_Y(B)\ge r\}.$$

Since $B(f(d)) = B(z) = 1$ and $\mathbf{T}_Y(B) = 1 \geq 1-r$, then $\Phi_{T_X}(1_d, r)(a) = 0$. Similarly, $\Phi_{T_X}(1_d, r)(b) = 0$,

$$\Phi_{T_X}(1_d, r)(c) = 0, \Phi_{T_X}(1_a, r)(d) = 1.$$

Since $R^r_{T_X}(a,b) = \Phi_{T_X}(1_a, r)(b)$, then $\Phi_{T_X}(A, r)(b) = \bigvee_{x\in X}(A(x) \odot R^r_{T_X}(x,b))$ with

$$R^r_{T_X} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

For $A = \bigvee_{a\in X}(A(a) \odot \top^*_a)$, we have

$$\Phi_{T_X}(A, r)(b) = \bigvee_{a\in X}(A(a) \odot \Phi_{T_X}(\top_a, r)(b)),$$

$$\Phi_{T_X}(A, r) = (A(a) \vee A(b), A(a) \vee A(b), A(c) \vee A(d), A(c) \vee A(d)).$$

If $A(a) = A(b)$ and $A(c) = A(d)$, then $A = g^{-1}(B)$ and $\Phi_{T_X}(A, r) = A$. Thus

$$\begin{aligned} \mathbf{T}_{H_{T_X}}(A) &= \bigvee\{r^* \in L \mid \Phi_{T_X}(A, r) = A\} \\ &= 1 = \mathbf{T}_X(A). \end{aligned}$$

Moreover, we obtain

$$\begin{aligned} \mathbf{T}^r_X(A) &= \bigwedge_{x\in X}(\Phi_{T_X}(A, r)(x) \to A(x)) \\ &= (A(a) \to A(b)) \wedge (A(b) \to A(a)) \\ &\wedge (A(c) \to A(d)) \wedge (A(d) \to A(c)). \end{aligned}$$

For $B(x) = 0.9, B(y) = 0.3, B(z) = 0.2$, $g^{-1}(B)(a) = g^{-1}(B)(b) = 0.9, g^{-1}(c) = g^{-1}(d) = 0.3$. Then

$$\begin{aligned} \mathbf{T}^{0.5}_Y(B) &= (1 + r - B(x) + B(z)) \vee 0 = 0.8 \\ \mathbf{T}^{0.5}_X(g^{-1}(B)) &= 1. \end{aligned}$$

If $\mathbf{T}^r_X(A) = 1$, then $A(a) = A(b)$, $A(c) = A(d)$. Thus,

$$\mathbf{T}_T(A) = \bigvee\{1 - r \in L \mid \mathbf{T}^r_X(A) = 1\} = 1.$$

Hence $\mathbf{T}_{T_X} = \mathbf{T}_{H_{T_X}} = \mathbf{T}_X$.

(4) By (1), we obtain a map $\mathbf{T}^*_Y : [0,1]^Y \to [0,1]$ as

$$\mathbf{T}^*_Y(A) = (1 - A^*(x) + A^*(z)) \wedge 1 = (1 - A(z) + A(x)) \wedge 1.$$

We obtain an upper approximation operator $\Phi_{T^*_Y}(-,r) : L^X \to L^X$ as follows:

$$\Phi_{T^*_Y}(1_x, r)(z) = \bigwedge\{B(z) \in L^X \mid B \geq 1_x,\ \mathbf{T}^*_Y(B) \geq r^*\}.$$

Since $B(x) = 1$ and $\mathbf{T}^*_Y(B) = (1 - B(z) + 1) \wedge 1 = 1$, then $\Phi_{T^*_Y}(1_x, r)(z) = 0$,

$$\Phi_{T^*_Y}(1_z, r)(y) = \bigwedge\{B(y) \in L^X \mid B \geq 1_z,\ \mathbf{T}^*_Y(B) \geq r^*\} = 0,$$

$$\Phi_{T^*_Y}(1_y, r)(y) = \bigwedge\{B(y) \in L^X \mid B \geq 1_y,\ \mathbf{T}^*_Y(B) \geq r^*\} = 1,$$

$$\Phi_{T^*_Y}(1_z, r)(x) = \bigwedge\{B(x) \in L^X \mid B \geq 1_z,\ \mathbf{T}^*_Y(B) \geq r^*\}.$$

Since $B(z) = 1$ and $\mathbf{T}^*_Y(B) = 1 - 1 + B(x) \geq 1 - r$, then $B(x) \geq 1 - r$. We have $\Phi_{T^*_Y}(1_z, r)(x) = 1 - r$.

$$\begin{pmatrix} \Phi_{T^*_Y}(1_x,r)(x) = 1 & \Phi_{T^*_Y}(1_x,r)(y) = 0 & \Phi_{T^*_Y}(1_x,r)(z) = 0 \\ \Phi_{T^*_Y}(1_y,r)(x) = 0 & \Phi_{T^*_Y}(1_y,r)(y) = 1 & \Phi_{T^*_Y}(1_y,r)(z) = 0 \\ \Phi_{T^*_Y}(1_z,r)(x) = 1-r & \Phi_{T^*_Y}(1_z,r)(y) = 0 & \Phi_{T^*_Y}(1_z,r)(z) = 1 \end{pmatrix}.$$

For $A = \bigvee_{x \in X}(A(x) \odot \top_x)$, we have

$$\Phi_{T^*_Y}(A, r)(y) = \bigvee_{x \in X} (A(x) \odot \Phi_{T^*_Y}(\top_x, r)(y)),$$

$$\Phi_{T^*_Y}(A, r) = (A(x) \vee (A(z) - r), A(y), A(z)).$$

If $A(z) - r \leq A(x)$, then $\Phi_{T^*_Y}(A, r) = A$. Thus

$$\begin{aligned} \mathbf{T}_{H_{T^*_Y}}(A) &= \bigvee\{r^* \in L \mid \Phi_{T^*_Y}(A, r) = A\} \\ &= (1 - A(z) + A(x)) \wedge 1 = \mathbf{T}^*_Y(A). \end{aligned}$$

Moreover, we obtain

$$\begin{aligned} \mathbf{T}^{*r}_X(A) &= \bigvee_{x \in X}(\Phi_{T^*_Y}(A, r)(x) \to A(x)) \\ &= (A(x) \vee (A(z) - r)) \to A(x) \\ &= (1 + r - A(z) + A(x)) \wedge 1. \end{aligned}$$

For $B(x)=0.9, B(y)=0.3, B(z)=0.2$, $\mathbf{T}^{*0.5}_{Y}(B)=(1+0.5-B(z)+B(x))\vee 0=1$.

$$\begin{aligned}\mathbf{T}_{T^*_Y}(A) &= \bigvee\{1-r\in L \mid \mathbf{T}^{*r}(A)=1\}\\ &=(1-A(z)+A(x))\wedge 1.\end{aligned}$$

Hence $\mathbf{T}_{T^*_Y}=\mathbf{T}_{H_{T^*_Y}}=\mathbf{T}^*_Y$. Since $R^r_{T^*_Y}(x,y)=\Phi_{T^*_Y}(1_x,r)(y)$, then $\Phi_{T^*_Y}(A,r)(y)=\bigvee_{x\in X}(A(x)\odot R^r_{T^*_Y}(x,y))$ with

$$R^r_{T^*_Y}=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 1-r & 0 & 1\end{pmatrix}.$$

(5) Let $X=\{a,b,c,d\}, Y=\{x,y,z\}$ be a set and $f:X\to Y$ be a map as follows:

$$f(a)=f(b)=x, f(c)=y, f(d)=z.$$

We obtain a map $\mathbf{T}^*_X:[0,1]^X\to[0,1]$ as

$$\mathbf{T}^*_X(A)=\begin{cases}(1-B(f(d))+B(f(a)))\wedge 1, & \text{if } A=f^{-1}(B),\\ 0, & \text{if } A\neq f^{-1}(B).\end{cases}$$

For $A_1(a)=A_1(b)=0.3, A_1(c)=0.8, A_1(d)=0.4$, we have $B_1(x)=0.3, B_1(y)=0.8, B_1(z)=0.4$ such that $A_1=f^{-1}(B_1)$. Hence $\mathbf{T}^*_X(A_1)=(1-B_1(z)+B_1(x))\wedge 1=0.9$. For $A_2(a)=0.6, A_2(b)=0.3, A_2(b)=0.8, A_2(c)=0.4$, $A_2\neq f^{-1}(B)$. Hence $\mathbf{T}^*_X(A_2)=0$. Since f is surjective and $f^{-1}(B_1)=f^{-1}(B_2)$, then $B_1=B_2$. Hence $\mathbf{T}^*_X(f^{-1}(B))=\mathbf{T}^*_Y(B)$.

We obtain an upper approximation operator $\Phi_{T^*_X}(-,r):L^X\to L^X$ as follows:

$$\Phi_{T^*_X}(1_d,r)(a)=\bigwedge\{f^{-1}(B)(a)\mid f^{-1}(B)\geq 1_d,\ \mathbf{T}^*_Y(B)\geq r^*\}.$$

Since $B(f(d))=B(z)=1$ and $\mathbf{T}^*_Y(B)=1-1+B(f(a))\geq 1-r$, then $B(f(a))\geq 1-r$. We have $\Phi_{T^*_X}(1_d,r)(a)=1-r$. Similarly, $\Phi_{T^*_X}(1_d,r)(b)=1-r$, $\Phi_{T^*_X}(1_d,r)(c)=0$, $\Phi_{T^*_X}(1_d,r)(d)=1$,

$$\Phi_{T^*_X}(1_a,r)(a)=\Phi_{T^*_X}(1_a,r)(b)=1, \Phi_{T^*_X}(1_a,r)(c)=\Phi_{T^*_X}(1_a,r)(d)=0,$$

$$\Phi_{T^*_X}(1_b,r)(a)=\Phi_{T^*_X}(1_b,r)(b)=1, \Phi_{T^*_X}(1_b,r)(c)=\Phi_{T^*_X}(1_b,r)(d)=0,$$

$$\Phi_{T^*_X}(1_c,r)(a)=\Phi_{T^*_X}(1_c,r)(b)=\Phi_{T^*_X}(1_c,r)(d)=0, \Phi_{T^*_X}(1_c,r)(c)=1.$$

Since $R^r_{T^*_X}(a,b) = \Phi_{T^*_X}(1_a, r)(b)$, then $\Phi_{T^*_X}(A,r)(b) = \bigvee_{x\in X}(A(x)\odot R^r_{T^*_X}(x,b))$ with

$$R^r_{T^*_X} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1-r & 1-r & 0 & 1 \end{pmatrix}.$$

For $A = \bigvee_{a\in X}(A(a)\odot \top_a)$, we have

$$\Phi_{T^*_X}(A,r)(b) = \bigvee_{a\in X}(A(a)\odot \Phi_{T^*_X}(\top_a, r)(b)),$$

$$\begin{aligned} \Phi_{T^*_X}(A,r) &= (A(a)\vee A(b)\vee(A(d)-r), \\ &\quad A(a)\vee A(b)\vee(A(d)-r), A(c), A(d)). \end{aligned}$$

If $A(a) = A(b)$ and $A(d)-r \le A(a)$, then $\Phi_{T^*_X}(A,r) = A$. Thus

$$\begin{aligned} \mathbf{T}_{H_{T^*_X}}(A) &= \bigvee\{r^* \in L \mid \Phi_{T^*_X}(A,r) = A\} \\ &= (1 - A(d) + A(a))\wedge 1 = \mathbf{T}_X(A). \end{aligned}$$

Moreover, we obtain

$$\begin{aligned} \mathbf{T}^r_{T^*_X}(A) &= \bigwedge_{x\in X}(\Phi_{T^*_X}(A,r)(x)\to A(x)) \\ &= (A(b)\to A(a))\wedge(A(a)\to A(b)) \\ &\wedge\Big(((A(d)-r)\to A(a))\wedge((A(d)-r)\to A(b))\Big), \end{aligned}$$

$$\begin{aligned} \mathbf{T}^r_{T^*_X}(A^*) &= (A^*(b)\to A^*(a))\wedge(A^*(a)\to A^*(b)) \\ &\wedge\Big(((A^*(d)-r)\to A^*(a))\wedge((A^*(d)-r)\to A^*(b))\Big) \\ &= (A(b)\to A(a))\wedge(A(a)\to A(b)) \\ &\wedge\Big(((A(a)-r)\to A(d))\wedge((A(b)-r)\to A(d))\Big) \\ &= \mathbf{T}^r_{T_X}(A). \end{aligned}$$

By Theorem 3.1 (11), $\mathbf{T}^r_{T_X}(A) = \mathbf{T}^r_{T^*_X}(A^*)$ for all $A \in L^X$ and $r \in L$ iff $\Phi_{T_X}(\top_a, r)(b) = \Phi_{T^*_X}(\top_b, r)(a)$ for all $x, y \in Y$ iff $R^r_{T^*_X}(a,b) = R^r_{T_X}(b,a)$ for all $a, b \in X$. For $B(x) = 0.8, B(y) = 0.6, B(z) = 0.3,$

$f^{-1}(B)(a) = f^{-1}(B)(b) = 0.8, f^{-1}(B)(c) = 0.6, f^{-1}(B)(d) = 0.3$. Then

$$\begin{aligned}
\mathbf{T}^{0.4}_{T_Y^*}(B^*) &= (1 + r - B^*(z) + B^*(x)) \vee 0 = 0.9 \\
&= (1 + r - B(x) + B(z)) \vee 0 = \mathbf{T}^{0.4}_{T_Y}(B) \\
\mathbf{T}^{0.5}_{T_X^*}(f^{-1}(B)^*) &= (((f^{-1}(B)^*(d) - 0.4) \to f^{-1}(B)^*(a)) \\
&\wedge ((f^{-1}(B)^*(d) - 0.4) \to f^{-1}(B)^*(b)) = 0.9 \\
&= (((f^{-1}(B)(a) - 0.4) \to f^{-1}(B)(d)) \\
&\wedge ((f^{-1}(B)(b) - 0.4) \to f^{-1}(B)(d)) = \mathbf{T}^{0.4}_{T_X}(f^{-1}(B)).
\end{aligned}$$

If $\mathbf{T}^r_{T_X^*}(A) = 1$, then $A(a) = A(b), A(d) - r \le A(a)$. So, $1 - r \le 1 - A(d) + A(a)$. Thus,

$$\begin{aligned}
\mathbf{T}_{T_X^*}(A) &= \bigvee \{1 - r \in L \mid \mathbf{T}^r_{T_X^*}(A) = 1\} \\
&= (1 - A(d) + A(a)) \wedge 1.
\end{aligned}$$

Hence $\mathbf{T}_{T_X^*} = \mathbf{T}_{H_{T_X^*}} = \mathbf{T}^*_X$.

(6) Let $X = \{a,b,c,d\}, Y = \{x,y,z\}$ be a set and $g : X \to Y$ be a map as follows:

$$g(a) = g(b) = x, g(c) = g(d) = y.$$

We obtain a map $\mathbf{T}^*_X : [0,1]^X \to [0,1]$ as

$$\mathbf{T}^*_X(A) = \begin{cases} 1, & \text{if } A = g^{-1}(B), \\ 0, & \text{if } A \neq g^{-1}(B). \end{cases}$$

For $A_1(a) = A_1(b) = 0.3, A_1(c) = A_1(d) = 0.5$, we have $B_1(x) = 0.3, B_1(y) = 0.5, B_1(z) = 0$ such that $A_1 = g^{-1}(B_1)$. Hence $\mathbf{T}^*_X(A_1) = (1 - B_1(z) + B_1(x)) \wedge 1 = 1$. For $A_2(a) = A_2(b) = 0.3, A_1(b) = 0.8, A_1(c) = 0.5, A_2 \neq g^{-1}(B)$. Hence $\mathbf{T}^*_X(A_2) = 0$. Since $R^r_{T_X^*}(a,b) = \Phi_{T_X^*}(1_a, r)(b) = \Phi_{T_X}(1_a, r)(b) = R^r_{T_X}(a,b)$, then $\Phi_{T_X^*}(A,r)(b) = \bigvee_{x\in X}(A(x) \odot R^r_{T_X^*}(x,b)) = \bigvee_{x\in X}(A(x) \odot R^r_{T_X}(x,b)) = \Phi_{T_X}(A,r)(b)$ with

$$R^r_{T_X^*} = R^r_{T_X} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

For $A = \bigvee_{a\in X}(A(a) \odot \top_a)$, we have

$$\Phi_{T_X^*}(A,r)(b) = \bigvee_{a\in X} (A(a) \odot \Phi_{T_X^*}(\top_a, r)(b)),$$

$$\begin{aligned}\Phi_{T_X^*}(A,r) &= (A(a)\vee A(b), A(a)\vee A(b),\\ &A(c)\vee A(d), A(c)\vee A(d)) = \Phi_{T_X}(A,r).\end{aligned}$$

If $A(a)=A(b)$ and $A(c)=A(d)$, then $\Phi_{T_X^*}(A,r)=A$. Thus

$$\begin{aligned}\mathbf{T}_{H_{T_X^*}}(A) &= \bigvee\{r^*\in L \mid \Phi_{T_X^*}(A,r)=A\}\\ &= 1 = \mathbf{T}_X(A^*) = \mathbf{T}_X^*(A).\end{aligned}$$

Moreover, we obtain

$$\begin{aligned}\mathbf{T}_X^{*r}(A) &= \bigwedge_{x\in X}(\Phi_{T_X^*}(A,r)(x)\to A(x))\\ &= (A(b)\to A(a))\wedge(A(b)\to A(a))\\ &\wedge(A(c)\to A(d))\wedge(A(d)\to A(c)).\end{aligned}$$

Moreover, $\mathbf{T}_X^r=\mathbf{T}_X^{*r}$. For $B(x)=0.8, B(y)=0.6, B(z)=0.3$, $g^{-1}(B)(a)=g^{-1}(B)(b)=0.8, g^{-1}(c)=0.6=g^{-1}(d)$. Then

$$\begin{aligned}\mathbf{T}_Y^{0.4}(B) &= (1+r-B(x)+B(z))\vee 0 = 0.9\\ \mathbf{T}_X^{0.5}(g^{-1}(B)) &= 1.\end{aligned}$$

If $\mathbf{T}_X^r(A)=1$, then $A(a)=A(b), A(c)=A(d)$. Thus,

$$\mathbf{T}_{T_X^*}(A)=\bigvee\{1-r\in L \mid \mathbf{T}^r(A)=1\}=1.$$

Hence $\mathbf{T}_{T_X^*}=\mathbf{T}_{H_{T_X^*}}=\mathbf{T}_X^*$.

Conflict of Interests

The author declares that there is no conflict of interests.

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