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ON PARAMETER DEPENDENT REFINEMENT OF DISCRETE JENSEN'S INEQUALITY FOR OPERATOR CONVEX FUNCTIONS

L. HORVÁTH[1], KHURAM ALI KHAN[2,3,*], AND J. PEČARIĆ[2,4]

[1]Department of Mathematics, University of Pannonia, Egyetem u.10, 8200 Veszprém, Hungary

[2]Abdus Salam School of Mathematical Sciences, GC University, 68-B, New Muslim Town, Lahore 54600, Pakistan

[3]Department of mathematics, University of Sargodha, Sargodha 40100, Pakistan

[4]Faculty of Textile Technology, University of Zagreb, 10000 Zagreb, Croatia

Abstract. In this paper, we consider the class of self-adjoint operators defined on a Hilbert space, whose spectra are contained in an interval. We give parameter dependent refinement of the well known discrete Jensen's inequality in this class. The parameter dependent mixed symmetric means are defined for a subclass of positive self-adjoint operators which insure the refinements of inequality between power means of strictly positive operators.

Keywords: self-adjoint operators, operator convex functions, operator means, symbolic calculus.

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*Corresponding author

E-mail addresses: lhorvath@almos.uni-pannon.hu (L. Horváth), khuramsms@gmail.com (K. A. Khan), pecaric@mahazu.hazu.hr (J. Pečarić)

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1. **Introduction and Preliminary Results**

Initially a complex Hilbert space H is given. The Banach algebra of all bounded linear operators on H is denoted by $B(H)$. $Sp(A)$ means the spectrum of the operator $A \in B(H)$. Let $S(I)$ be the class of all self-adjoint bounded operators on H whose spectra are contained in an interval $I \subset \mathbb{R}$. A function $f : D_f(\subset \mathbb{R}) \to \mathbb{R}$ is operator monotone on the interval I, if f is continuous on I and $f(A) \leq f(B)$ for all A, $B \in S(I)$ satisfying $A \leq B$ (i.e $A - B$ is a positive operator). The function f is operator convex on I, if f is continuous on I and

$$f(sA + tB) \leq sf(A) + tf(B)$$

for all A, $B \in S(I)$ and for all positive numbers s and t. The function f is called operator concave on I if $-f$ is operator convex on I.

If f is an operator convex function on the interval I, $T_i \in S(I)$, and $w_i > 0$ $(i = 1, ..., n)$ such that $\sum_{i=1}^{n} w_i = 1$, then the discrete Jensen's inequality is given by

$$f\left(\sum_{i=1}^{n} w_i T_i\right) \leq \sum_{i=1}^{n} w_i f(T_i). \tag{1}$$

If f is an operator concave function on I , then inequality in (1) is reversed.

Some interpolations of (1) are given in [7]. The power means for strictly positive operators with positive weights are also defined in [7] and their monotonicity is discussed. In [5], the class $S(I)$ is considered to give some refinements of the discrete Jensen's inequality, and the monotonicity property of the corresponding mixed symmetric means is studied. The interpolations given in [7] are special cases of some results in [5].

We start with a result from [5]. To formulate this result we need some notations and some hypotheses which will also give the basic context of our main results.

The power set of a set X is denoted by $P(X)$. $|X|$ means the number of elements in X.

The usual symbol $\mathbb{N}$ is used for the set of natural numbers (including 0), while $\mathbb{N}_+$ means $\mathbb{N} \setminus \{0\}$.

(H_1) Let $I \subset \mathbb{R}$ be an interval, and let $T_i \in S(I)$ $(1 \leq i \leq n)$.

(H$_2$) Let $w_1, \ldots, w_n$ be positive numbers such that $\sum_{j=1}^{n} w_j = 1$.

(H$_3$) Let the function $f : I \to \mathbb{R}$ be operator convex.

(H$_4$) Let h, $g : I \to \mathbb{R}$ be continuous and strictly operator monotone functions.

We do not apply Theorem 1.1 in this paper, and therefore on the score of the exact meaning of the following expresions $A_{k,l}$ $(k \geq l \geq 1)$ see [5] or [6]. Let

$$A_{k,k} := \sum_{(i_1,\ldots,i_k)\in I_k} \left(\sum_{s=1}^{k} \frac{w_{i_s}}{\alpha_{I_k,i_s}} \right) f \left(\frac{\sum_{s=1}^{k} \frac{w_{i_s}}{\alpha_{I_k,i_s}} T_{i_s}}{\sum_{s=1}^{k} \frac{w_{i_s}}{\alpha_{I_k,i_s}}} \right),$$

and for each $k-1 \geq l \geq 1$ let

$$A_{k,l} := \frac{1}{(k-1)\ldots l} \sum_{(i_1,\ldots,i_l)\in I_l} t_{I_k,l}\left(i_1,\ldots,i_l\right) \left(\sum_{s=1}^{l} \frac{w_{i_s}}{\alpha_{I_k,i_s}} \right) f \left(\frac{\sum_{s=1}^{l} \frac{w_{i_s}}{\alpha_{I_k,i_s}} T_{i_s}}{\sum_{s=1}^{l} \frac{w_{i_s}}{\alpha_{I_k,i_s}}} \right).$$

Now we are in a position to formulate one of the main results in [5]:

Theorem 1.1. *Assume (H$_1$)-(H$_3$) are satisfied. Then*

$$f\left(\sum_{r=1}^{n} w_r T_r\right) \leq A_{k,k} \leq A_{k,k-1} \leq \ldots \leq A_{k,2} \leq A_{k,1} = \sum_{r=1}^{n} w_r f(T_r). \tag{2}$$

In this paper, we first use the method of Horváth adopted in [3] to construct a new refinement of Jensen's inequality for operator convex functions. In this way we are able to generalize the refinement results given in [5] as well as the results of Mond and Pečarić in [7].

Secondly, we introduce a parameter dependent refinement of (1) by using the method given in [4]. With the help of this new refinement, we construct the parameter dependent mixed symmetric means for a subclass of $S(I)$ and also give the monotonicity property of these operator means.

2. Generalizations

To give the generalization of Theorem 1.1, we start with the following notations introduced in [3]:

Let X be a set. For every nonnegative integer m, define

$$P_m(X) := \{Y \subset X \mid |Y| = m\}.$$

We introduce two further hypotheses:

(H_5) Let $S_1, \ldots, S_n$ be finite, pairwise disjoint and nonempty sets, let

$$S := \bigcup_{j=1}^{n} S_j,$$

and let c be a function from S into $\mathbb{R}$ such that

$$c(s) > 0, \quad s \in S, \quad \text{and} \quad \sum_{s \in S_j} c(s) = 1, \quad j = 1, \ldots, n.$$

Let the function $\tau : S \to \{1, \ldots, n\}$ be defined by

$$\tau(s) := j, \quad \text{if} \quad s \in S_j.$$

(H_6) Suppose $\mathcal{A} \subset P(S)$ is a partition of S into pairwise disjoint and nonempty sets. Let

$$k := \max\{|A| \mid A \in \mathcal{A}\},$$

and let

$$\mathcal{A}_l := \{A \in \mathcal{A} \mid |A| = l\}, \quad l = 1, \ldots, k.$$

(We note that $\mathcal{A}_l$ ($l = 1, \ldots, k-1$) may be the empty set, and of course, $|S| = \sum_{l=1}^{k} l\,|\mathcal{A}_l|$.)

Now, we give a refinement of (1). The empty sum of numbers or vectors is taken to be zero.

Theorem 2.1. *If (H_1)-(H_3) and (H_5)-(H_6) are satisfied, then*

$$f\left(\sum_{j=1}^{n} w_j T_j\right) \le N_k \le N_{k-1} \le \ldots \le N_2 \le N_1 = \sum_{j=1}^{n} w_j f(T_j),$$

where

$$N_k := \sum_{l=1}^{k}\left(\sum_{A\in\mathcal{A}_l}\left(\left(\sum_{s\in A}c(s)w_{\tau(s)}\right)f\left(\frac{\sum_{s\in A}c(s)w_{\tau(s)}T_{\tau(s)}}{\sum_{s\in A}c(s)w_{\tau(s)}}\right)\right)\right),$$

and for every $1\le m\le k-1$ *the operator* N_{k-m} *is given by*

$$N_{k-m} := \sum_{l=1}^{m}\left(\sum_{A\in\mathcal{A}_l}\left(\sum_{s\in A}c(s)w_{\tau(s)}f(T_{\tau(s)})\right)\right)+\sum_{l=m+1}^{k}\left(\frac{m!}{(l-1)\dots(l-m)}\right.$$

$$\left.\cdot\sum_{A\in\mathcal{A}_l}\left(\sum_{B\in P_{l-m}(A)}\left(\left(\sum_{s\in B}c(s)w_{\tau(s)}\right)f\left(\frac{\sum_{s\in B}c(s)w_{\tau(s)}T_{\tau(s)}}{\sum_{s\in B}c(s)w_{\tau(s)}}\right)\right)\right)\right).$$

Proof. The proof is entirely similar to the proof of Theorem 1 in [3], so we omit it.. □

The first application of Theorem 2.1 leads to a generalization of Theorem 1.1.

Theorem 2.2. *Assume that* (H_1)*-*(H_3) *are satisfied, let* $k\ge 1$ *be a fixed integer, and let* $I_k\subset\{1,\dots,n\}^k$. *For* $j=1,\dots,n$ *we consider the sets*

$$S_j := \{((i_1,\dots,i_k),l)\mid (i_1,\dots,i_k)\in I_k,\quad 1\le l\le k,\quad i_l=j\}.$$

Let c *be a positive function on* $S:=\bigcup_{j=1}^{n}S_j$ *such that*

$$\sum_{((i_1,\dots,i_k),l)\in S_j}c((i_1,\dots,i_k),l)=1,\quad j=1,\dots,n.$$

Then

$$f\left(\sum_{j=1}^{n}w_jT_j\right)\le N_k\le N_{k-1}\le\dots\le N_2\le N_1=\sum_{j=1}^{n}w_jf(T_j), \tag{3}$$

where

$$N_k := \sum_{(i_1,\dots,i_k)\in I_k}\left(\left(\sum_{l=1}^{k}c((i_1,\dots,i_k),l)\,w_{i_l}\right)f\left(\frac{\sum_{l=1}^{k}c((i_1,\dots,i_k),l)\,w_{i_l}T_{i_l}}{\sum_{l=1}^{k}c((i_1,\dots,i_k),l)\,w_{i_l}}\right)\right),$$

and for every $1\le m\le k-1$

$$N_{k-m} := \frac{m!}{(k-1)\dots(k-m)}\sum_{(i_1,\dots,i_k)\in I_k}\left(\sum_{1\le l_1<\dots<l_{k-m}\le k}\right.$$

$$\left(\left(\sum_{j=1}^{k-m} c\left(\left(i_1, \ldots, i_k\right), l_j\right) w_{i_{l_j}}\right) f\left(\frac{\sum\limits_{l=1}^{k-m} c\left(\left(i_1, \ldots, i_k\right), l_j\right) w_{i_{l_j}} T_{i_{l_j}}}{\sum\limits_{l=1}^{k-m} c\left(\left(i_1, \ldots, i_k\right), l_j\right) w_{i_{l_j}}}\right)\right)\right)\right).$$

An immediate consequence of the previous result is Theorem 1.1: choosing

$$c\left(\left(i_1, \ldots, i_k\right), l\right) = \frac{1}{|S_j|} = \frac{1}{\alpha_{I_k,j}} \quad \text{if} \quad \left(\left(i_1, \ldots, i_k\right), l\right) \in S_j,$$

it can be checked easily that the inequality (3) corresponds to the inequality (2).

Theorem 1.1 has some interesting special cases (see [5]). Theorem 2.2 generalizes these results: apply it to either

$$I_k := \left\{(i_1, \ldots, i_k) \in \{1, \ldots, n\}^k \mid i_1 < \ldots < i_k\right\}, \quad 1 \leq k \leq n,$$

or

$$I_k := \left\{(i_1, \ldots, i_k) \in \{1, \ldots, n\}^k \mid i_1 \leq \ldots \leq i_k\right\}, \quad 1 \leq k.$$

Now we apply Theorem 2.1 to some special situations which correspond to some results about operator convexity.The next examples based on examples in [3].

Example 2.3. *Let n, m, r be fixed integers, where $n \geq 3$, $m \geq 2$ and $1 \leq r \leq n-2$. In this example, for every $i = 1, 2, \ldots, n$ and for every $l = 0, 1, \ldots, r$ the integer $i + l$ will be identified with the uniquely determined integer j from $\{1, \ldots, n\}$ for which*

$$l + i \equiv j \quad (mod\, n). \tag{4}$$

Introducing the notation

$$D := \{1, \ldots, n\} \times \{0, \ldots, r\},$$

let for every $j \in \{1, \ldots, n\}$

$$S_j := \{(i, l) \in D \mid i + l \equiv j \quad (mod\, n)\} \bigcup \{j\},$$

and let $\mathcal{A} \subset P(S)$ ($S := \bigcup\limits_{j=1}^{n} S_j$) contain the following sets:

$$A_i := \{(i, l) \in D \mid l = 0, \ldots, r\}, \quad i = 1, \ldots, n$$

and

$$A := \{1, \ldots, n\}.$$

Let c be a positive function on S such that

$$\sum_{(i,l)\in S_j} c(i,l) + c(j) = 1, \quad j = 1, \ldots, n.$$

A careful verification shows that the sets $S_1, \ldots, S_n$, the partition $\mathcal{A}$ and the function c defined above satisfy the conditions (H_5) and (H_6),

$$\tau(i,l) = i + l, \quad (i,l) \in D,$$

(by the agreement (see (4)), $i + l$ is identified with j)

$$\tau(j) = j, \quad j = 1, \ldots, n,$$

$$|S_j| = r + 2, \quad j = 1, \ldots, n,$$

and

$$|A_i| = r + 1, \quad i = 1, \ldots, n, \quad |A| = n.$$

Now we suppose (H_1)-(H_3) are satisfied. Then by Theorem 2.1

$$f\left(\sum_{j=1}^{n} w_j T_j\right) \leq N_k = \sum_{i=1}^{n}\left(\left(\sum_{l=0}^{r} c(i,l)\, w_{i+l}\right) f\left(\frac{\sum\limits_{l=0}^{r} c(i,l)\, w_{i+l} T_{i+l}}{\sum\limits_{l=0}^{r} c(i,l)\, w_{i+l}}\right)\right)$$

$$+\left(\sum_{j=1}^{n} c(j) w_j\right) f\left(\frac{\sum\limits_{j=1}^{n} c(j) w_j T_j}{\sum\limits_{j=1}^{n} c(j) w_j}\right) \leq \sum_{j=1}^{n} w_j f(T_j). \tag{5}$$

In case

$$w_j := \frac{1}{n}, \quad j = 1, \ldots, n,$$

$$c(i,l) := \frac{1}{m(r+1)}, \quad (i,l) \in D, \quad c(j) := \frac{m-1}{m} \quad j = 1, \ldots, n,$$

it follows from (5) that

$$f\left(\frac{1}{n}\sum_{j=1}^{n} T_j\right) \leq \frac{1}{mn}\sum_{i=1}^{n} f\left(\frac{T_i + T_{i+1} + \ldots + T_{i+r}}{r+1}\right)$$

$$+\frac{m-1}{m}f\left(\frac{1}{n}\sum_{j=1}^{n}T_j\right)\leq\frac{1}{n}\sum_{j=1}^{n}f(T_j).$$

Example 2.4. *Let n and k be fixed positive integers. Let*

$$D:=\left\{(i_1,\ldots,i_n)\in\{1,\ldots,k\}^n\mid i_1+\ldots+i_n=n+k-1\right\},$$

and for each $j=1,\ldots,n$, denote S_j the set

$$S_j:=D\times\{j\}.$$

For every $(i_1,\ldots,i_n)\in D$ designate by $A_{(i_1,\ldots,i_n)}$ the set

$$A_{(i_1,\ldots,i_n)}:=\left\{\left((i_1,\ldots,i_n),l\right)\mid l=1,\ldots,n\right\}.$$

It is obvious that S_j $(j=1,\ldots,n)$ and $A_{(i_1,\ldots,i_n)}$ $((i_1,\ldots,i_n)\in D)$ are decompositions of $S:=\bigcup_{j=1}^{n}S_j$ into pairwise disjoint and nonempty sets, respectively. Let c be a function on S such that

$$c\left((i_1,\ldots,i_n),j\right)>0,\quad\left((i_1,\ldots,i_n),j\right)\in S$$

and

$$\sum_{(i_1,\ldots,i_n)\in D}c\left((i_1,\ldots,i_n),j\right)=1,\quad j=1,\ldots,n. \tag{6}$$

In summary we have that the conditions (H_5) and (H_6) are valid, and

$$\tau\left((i_1,\ldots,i_n),j\right)=j,\quad\left((i_1,\ldots,i_n),j\right)\in S.$$

Suppose (H_1)-(H_3) are satisfied. Then by Theorem 2.1

$$f\left(\sum_{j=1}^{n}w_jT_j\right)\leq N_k=\sum_{(i_1,\ldots,i_n)\in D}\left(\left(\sum_{l=1}^{n}c\left((i_1,\ldots,i_n),l\right)w_l\right)\right.$$

$$\left.f\left(\frac{\sum_{l=1}^{n}c\left((i_1,\ldots,i_n),l\right)w_lT_l}{\sum_{l=1}^{n}c\left((i_1,\ldots,i_n),l\right)w_l}\right)\right)\leq\sum_{j=1}^{n}w_jf(T_j). \tag{7}$$

If we set

$$w_j:=\frac{1}{n},\quad j=1,\ldots,n,$$

and

$$c\left(\left(i_1,\ldots,i_n\right),j\right) := \frac{i_j}{\binom{n+k-1}{k-1}},$$

then (6) holds, since by some combinatorial considerations

$$|D| = \binom{n+k-2}{n-1},$$

and

$$\sum_{(i_1,\ldots,i_n)\in D} i_j = \frac{n+k-1}{n}\binom{n+k-2}{n-1} = \binom{n+k-1}{k-1}, \quad j=1,\ldots,n.$$

In this situation (7) can therefore be expressed as

$$f\left(\frac{1}{n}\sum_{j=1}^{n} T_j\right) \leq \frac{1}{\binom{n+k-2}{k-1}} \sum_{(i_1,\ldots,i_n)\in D} f\left(\frac{1}{n+k-1}\sum_{l=1}^{n} i_l T_l\right) \leq \frac{1}{n}\sum_{j=1}^{n} f(T_j).$$

Let us close this section by deriving a sharpened version of the arithmetic mean - geometric mean inequality.

Example 2.5. *Let $n \geq 2$ be a fixed positive integer, let*

$$S_j := \left\{(i,j) \in \{1,\ldots,n\}^2 \mid i = 1,\ldots,j\right\}, \quad j = 1,\ldots,n,$$

and let

$$A_i := \left\{(i,j) \in \{1,\ldots,n\}^2 \mid j = i,\ldots,n\right\}, \quad i = 1,\ldots,n.$$

If $T_1,\ldots,T_n$ are strictly positive operators, then it follows from Theorem 2.1 that

$$-\ln\left(\frac{T_1+\ldots+T_n}{n}\right) \leq \sum_{i=1}^{n}\left(-\left(\frac{1}{n}\sum_{j=i}^{n}\frac{1}{j}\right)\ln\left(\frac{\sum\limits_{j=i}^{n}\frac{T_j}{j}}{\sum\limits_{j=i}^{n}\frac{1}{j}}\right)\right)$$

$$\leq -\frac{\ln(T_1)+\ldots+\ln(T_n)}{n},$$

and therefore

$$(T_1\ldots T_n)^{\frac{1}{n}} \leq \prod_{i=1}^{n}\left(\frac{\sum\limits_{j=i}^{n}\frac{T_j}{j}}{\sum\limits_{j=i}^{n}\frac{1}{j}}\right)^{\frac{1}{n}\sum\limits_{j=i}^{n}\frac{1}{j}} \leq \frac{T_1+\ldots+T_n}{n}.$$

3. Parameter Dependent Refinement

In this part of the paper we use the following hypothesis:

(H_7) Consider a real number λ such that $\lambda \geq 1$.

Now we give a parameter dependent refinement of the discrete Jensen's inequality (1).

Theorem 3.1. *Suppose (H_1)-(H_3) and (H_7). For $k \in \mathbb{N}$, we introduce the sets*

$$S_k := \left\{ (i_1, \dots, i_n) \in \mathbb{N}^n \mid \sum_{j=1}^{n} i_j = k \right\}, \quad k \in \mathbb{N},$$

and define the operators

$$C_k(\lambda) = C_k(T_1, \dots, T_n; w_1, \dots, w_n; \lambda)$$

$$:= \frac{1}{(n+\lambda-1)^k} \sum_{(i_1,\dots,i_n)\in S_k} \frac{k!}{i_1! \dots i_n!} \left(\sum_{j=1}^{n} \lambda^{i_j} w_j \right) f \left(\frac{\sum_{j=1}^{n} \lambda^{i_j} w_j T_j}{\sum_{j=1}^{n} \lambda^{i_j} w_j} \right). \tag{8}$$

Then

(a)

$$f\left(\sum_{j=1}^{n} w_j T_j\right) = C_0(\lambda) \leq C_1(\lambda) \leq \dots \leq C_k(\lambda) \leq \dots \leq \sum_{j=1}^{n} w_j f(T_j), \quad k \in \mathbb{N}.$$

(b) For every fixed $\lambda > 1$

$$\lim_{k\to\infty} C_k(\lambda) = \sum_{j=1}^{n} w_j f(T_j).$$

It follows from the definition of S_k that $S_k \subset \{0, \dots, k\}^n$ ($k \in \mathbb{N}$), and it is obvious that

$$C_k(1) = f\left(\sum_{j=1}^{n} w_j T_j\right), \quad k \in \mathbb{N}.$$

The proof of Theorem 3.1 is essentially the same as the proofs of the similar results in [4], so it is omitted. But to prove the second part of the theorem we need the following two results. First, we generalize Lemma 15 in [4].

Lemma 3.2. *Let $(X, \|\cdot\|)$ be a normed space. Let $p_1, \ldots, p_n$ be a discrete distribution with $n \geq 2$, and let $\lambda > 1$. Let $l \in \{1, \ldots, n\}$ be fixed. e_l denotes the vector in $\mathbb{R}^n$ that has 0s in all coordinate positions except the lth, where it has a 1. Let $q_1, \ldots, q_n$ be also a discrete distribution such that $q_j > 0$ $(1 \leq j \leq n)$ and*

$$q_l > \max(q_1, \ldots q_{l-1}, q_{l+1}, \ldots, q_n).$$

If

$$g : \left\{ (t_1, \ldots, t_n) \in \mathbb{R}^n \mid t_j > 0 \ (1 \leq j \leq n), \ \sum_{j=1}^{n} t_j = 1 \right\} \to X$$

is a bounded function for which

$$\tau_l := \lim_{e_l} g$$

exists, and $p_l > 0$, then

$$\lim_{k \to \infty} \sum_{(i_1, \ldots, i_n) \in S_k} \frac{k!}{i_1! \ldots i_n!} q_1^{i_1} \ldots q_n^{i_n} g\left(\frac{\lambda^{i_1} p_1}{\sum_{j=1}^{n} \lambda^{i_j} p_j}, \ldots, \frac{\lambda^{i_n} p_n}{\sum_{j=1}^{n} \lambda^{i_j} p_j} \right) = \tau_l.$$

Proof. We have to modify just the final part of the proof of Lemma 15 in [4]. We can suppose that $l = 1$.

Choose $0 < \varepsilon < 1$. Since the distribution function F_{n-1} of the Chi-square distribution (χ^2-distribution) with $n-1$ degrees of freedom is continuous, and strictly increasing on $]0, \infty[$, there exists a unique $t_\varepsilon > 0$ such that

$$F_{n-1}(t_\varepsilon) = 1 - \varepsilon.$$

Define

$$S_k^1 := \left\{ (i_{1k}, \ldots, i_{nk}) \in S_k \mid \sum_{j=1}^{n} k \frac{\left(\frac{i_{jk}}{k} - q_j \right)^2}{q_j} < t_\varepsilon \right\},$$

let $S_k^2 := S_k \setminus S_k^1$ $(k \in \mathbb{N}_+)$, and consider the sequences

$$a_k^1 := \sum_{(i_{1k}, \ldots, i_{nk}) \in S_k^1} \frac{k!}{i_{1k}! \ldots i_{nk}!} q_1^{i_{1k}} \ldots q_n^{i_{nk}} g\left(\frac{\lambda^{i_{1k}} p_1}{\sum_{j=1}^{n} \lambda^{i_{jk}} p_j}, \ldots, \frac{\lambda^{i_{nk}} p_n}{\sum_{j=1}^{n} \lambda^{i_{jk}} p_j} \right),$$

and

$$a_k^2 := \sum_{(i_{1k},\ldots,i_{nk})\in S_k^2} \frac{k!}{i_{1k}!\ldots i_{nk}!} q_1^{i_{1k}} \ldots q_n^{i_{nk}} g\left(\frac{\lambda^{i_{1k}} p_1}{\sum_{j=1}^{n} \lambda^{i_{jk}} p_j}, \ldots, \frac{\lambda^{i_{nk}} p_n}{\sum_{j=1}^{n} \lambda^{i_{jk}} p_j}\right),$$

where $k \in \mathbb{N}_+$.

By using the first part of the proof of Lemma 15 in [4], we have that

(i)

$$\sum_{(i_{1k},\ldots,i_{nk})\in S_k^1} \frac{k!}{i_{1k}!\ldots i_{nk}!} q_1^{i_{1k}} \ldots q_n^{i_{nk}} = 1 - \varepsilon + \delta_\varepsilon(k), \quad k \in \mathbb{N}_+,$$

where $\lim\limits_{k\to\infty} \delta_\varepsilon(k) = 0$ (let $k_\varepsilon \in \mathbb{N}_+$ such that $\delta_\varepsilon(k) < \varepsilon$ for all $k > k_\varepsilon$),

(ii) for every $\varepsilon_1 > 0$ we can find an integer $k_{\varepsilon_1} > k_\varepsilon$ such that for all $k > k_{\varepsilon_1}$

$$\left\| g\left(\frac{\lambda^{i_{1k}} p_1}{\sum_{j=1}^{n} \lambda^{i_{jk}} p_j}, \ldots, \frac{\lambda^{i_{nk}} p_n}{\sum_{j=1}^{n} \lambda^{i_{jk}} p_j}\right) - \tau_1 \right\| < \varepsilon_1, \quad (i_{1k}, \ldots, i_{nk}) \in S_k^1.$$

Since g bounded on its domain ($\|g - \tau_1\| \leq m$), it follows from (i) and (ii) that

$$\left\| \sum_{(i_1,\ldots,i_n)\in S_k} \frac{k!}{i_1!\ldots i_n!} q_1^{i_1} \ldots q_n^{i_n} g\left(\frac{\lambda^{i_1} p_1}{\sum_{j=1}^{n} \lambda^{i_j} p_j}, \ldots, \frac{\lambda^{i_n} p_n}{\sum_{j=1}^{n} \lambda^{i_j} p_j}\right) - \tau_1 \right\|$$

$$\leq \sum_{(i_1,\ldots,i_n)\in S_k^1} \frac{k!}{i_1!\ldots i_n!} q_1^{i_1} \ldots q_n^{i_n} \left\| g\left(\frac{\lambda^{i_1} p_1}{\sum_{j=1}^{n} \lambda^{i_j} p_j}, \ldots, \frac{\lambda^{i_n} p_n}{\sum_{j=1}^{n} \lambda^{i_j} p_j}\right) - \tau_1 \right\|$$

$$+ \sum_{(i_1,\ldots,i_n)\in S_k^2} \frac{k!}{i_1!\ldots i_n!} q_1^{i_1} \ldots q_n^{i_n} \left\| g\left(\frac{\lambda^{i_1} p_1}{\sum_{j=1}^{n} \lambda^{i_j} p_j}, \ldots, \frac{\lambda^{i_n} p_n}{\sum_{j=1}^{n} \lambda^{i_j} p_j}\right) - \tau_1 \right\|$$

$$\leq \varepsilon_1 (1 - \varepsilon + \delta_\varepsilon(k)) + m(\varepsilon - \delta_\varepsilon(k)), \quad k > k_{\varepsilon_1},$$

and this gives the result. □

The second lemma corresponds to the symbolic calculus for self-adjoint operators.

Lemma 3.3. *Assume (H_1) and let $f : I \to \mathbb{R}$ be continuous. Let the function*

$$g : \left\{ (t_1, \ldots, t_n) \in \mathbb{R}^n \mid t_j > 0 \ (1 \le j \le n), \ \sum_{j=1}^{n} t_j = 1 \right\} \to B(H)$$

defined by

$$g\left(t_1, \ldots, t_n\right) := f\left(\sum_{j=1}^{n} t_j T_j\right).$$

Then

$$\lim_{e_l} g = f(T_l), \quad 1 \le l \le n.$$

Proof. Let

$$\alpha := \min_{1 \le j \le n} \left(\min Sp(T_j)\right) \quad \text{and} \quad \beta := \max_{1 \le j \le n} \left(\max Sp(T_j)\right),$$

where $Sp(T)$ denotes the spectrum of T. Then

$$Sp\left(\sum_{j=1}^{n} t_j T_j\right) \subset [\alpha, \beta] \subset I$$

for all $t_j \ge 0$ $(1 \le j \le n)$ with $\sum_{j=1}^{n} t_j = 1$.

It is enough to prove that f is continuous on $S([\alpha, \beta])$.

To prove this let $\varepsilon > 0$ be fixed, and let $\left(A_n\right)_{n \in \mathbb{N}}$ be a sequence in $S([\alpha, \beta])$ such that $A_n \to A \in S([\alpha, \beta])$.

Since f is continuous on $[\alpha, \beta]$, the Stone-Weierstrass theorem implies the existence of a sequence of real polynomial functions $\left(f_k\right)_{k \in \mathbb{N}}$ which converges uniformly on $[\alpha, \beta]$ to f. It follows that there exists $k_0 \in \mathbb{N}$ such that

$$\left|f_{k_0}(t) - f(t)\right| < \frac{\varepsilon}{3}, \quad t \in [\alpha, \beta].$$

The fundamental result for continuous functional calculus (see for example [2]) yields that

$$\left\|f(A_n) - f_{k_0}(A_n)\right\| = \left\|\left(f - f_{k_0}\right)(A_n)\right\| = \sup_{t \in Sp(A_n)} \left|f(t) - f_{k_0}(t)\right| \tag{9}$$

$$\le \sup_{t \in [\alpha, \beta]} \left|f(t) - f_{k_0}(t)\right| < \frac{\varepsilon}{3}, \quad n \in \mathbb{N},$$

where $\|\cdot\|$ means the norm on H. Similarly, we have

$$\left\|f_{k_0}(A) - f(A)\right\| < \frac{\varepsilon}{3}. \tag{10}$$

Since $A_n \to A$, we obtain $A_n^i \to A^i$ for every $i \in \mathbb{N}$, and therefore there is $n_0 \in \mathbb{N}$ such that

$$\|f_{k_0}(A_n) - f_{k_0}(A)\| < \frac{\varepsilon}{3} \tag{11}$$

for all $n > n_0$.

Now the inequalities (9-11) give that

$$\|f(A_n) - f(A)\| \leq \|f(A_n) - f_{k_0}(A_n)\| + \|f_{k_0}(A_n) - f_{k_0}(A)\|$$
$$+ \|f_{k_0}(A) - f(A)\| < \varepsilon$$

for all $n > n_0$, and hence $f(A_n) \to f(A)$.

The proof is complete. □

Suppose (H$_1$)-(H$_3$) and (H$_7$). We consider three special cases of (8).

(a) $k = 1$, $n \in \mathbb{N}_+$:

$$C_1(\lambda) = \frac{1}{n+\lambda-1} \sum_{i=1}^{n} (1 + (\lambda-1) w_i) f \left(\frac{\sum\limits_{j=1}^{n} w_j T_j + (\lambda-1) w_i T_i}{1 + (\lambda-1) w_i} \right).$$

(b) $k \in \mathbb{N}$, $n = 2$:

$$C_k(\lambda) = \frac{1}{(\lambda+1)^k} \sum_{i=0}^{k} \binom{k}{i} \left(\lambda^i w_1 + \lambda^{k-i} w_2\right) f \left(\frac{\lambda^i w_1 T_1 + \lambda^{k-i} w_2 T_2}{\lambda^i w_1 + \lambda^{k-i} w_2} \right).$$

(c) $w_1 = \ldots = w_n := \frac{1}{n}$:

$$C_k(\lambda) = \frac{1}{n(n+\lambda-1)^k} \sum_{(i_1,\ldots,i_n)\in S_k} \frac{k!}{i_1!\ldots i_n!} \left(\sum_{j=1}^{n} \lambda^{i_j} \right) f \left(\frac{\sum\limits_{j=1}^{n} \lambda^{i_j} T_j}{\sum\limits_{j=1}^{n} \lambda^{i_j}} \right).$$

Next, we define some further operator means and study their monotonicity and convergence.

Definition 3.4. *We assume that (H$_1$), (H$_2$) and (H$_4$) are satisfied and $\lambda \geq 1$. Then we define the operator means with respect to (8) by*

$$M_{h,g}(k,\lambda) := h^{-1} \left(\frac{1}{(n+\lambda-1)^k} \sum_{(i_1,\ldots,i_n)\in S_k} \frac{k!}{i_1!\ldots i_n!} \left(\sum_{j=1}^{n} \lambda^{i_j} w_j \right) \right. \tag{12}$$

$$\cdot(h\circ g^{-1})\left(\frac{\sum_{j=1}^{n}\lambda^{i_j}w_j g(T_j)}{\sum_{j=1}^{n}\lambda^{i_j}w_j}\right)\Bigg), \quad k\in\mathbb{N}.$$

We now give the monotonicity of the means (12) by the virtue of Theorem 3.1.

Proposition 3.5. *For $\lambda \geq 1$, we assume (H_1), (H_2) and (H_4). Then*

(a)

$$M_g = M_{h,g}(0,\lambda) \leq \ldots \leq M_{h,g}(k,\lambda) \leq \ldots \leq M_h, \quad k\in\mathbb{N},$$

if either $h\circ g^{-1}$ is operator convex and h^{-1} is operator monotone or $h\circ g^{-1}$ is operator concave and $-h^{-1}$ is operator monotone.

(b)

$$M_g = M_{h,g}(0,\lambda) \geq \ldots \geq M_{h,g}(k,\lambda) \geq \ldots \geq M_h, \quad k\in\mathbb{N},$$

if either $h\circ g^{-1}$ is operator convex and $-h^{-1}$ is operator monotone or $h\circ g^{-1}$ is operator concave and h^{-1} is operator monotone.

(c) In both cases

$$\lim_{k\to\infty} M_{h,g}(k,\lambda) = M_h.$$

Proof. The idea of the proof is the same as given in [5]. □

As a special case we consider the following example.

Example 3.6. *If $I :=]0,\infty[$, $h := \ln$ and $g(x) := x$ ($x\in]0,\infty[$), then by Proposition 3.5 (b), we have the following inequality: for every $T_j > 0$ ($1\leq j\leq n$), $\lambda\geq 1$, and $k\in\mathbb{N}_+$*

$$\prod_{j=1}^{n} T_j^{w_j} \leq \prod_{(i_1,\ldots,i_n)\in S_k}\left(\frac{\sum_{j=1}^{n}\lambda^{i_j}w_jT_j}{\sum_{j=1}^{n}\lambda^{i_j}w_j}\right)^{\frac{1}{(n+\lambda-1)^k}\frac{k!}{i_1!\ldots i_n!}\sum_{j=1}^{n}\lambda^{i_j}w_j} \leq \sum_{j=1}^{n} w_jT_j,$$

which gives a sharpened version of the arithmetic mean - geometric mean inequality

$$\prod_{j=1}^{n} T_j^{\frac{1}{n}} \leq \prod_{(i_1,\ldots,i_n)\in S_k}\left(\frac{\sum_{j=1}^{n}\lambda^{i_j}T_j}{\sum_{j=1}^{n}\lambda^{i_j}}\right)^{\frac{1}{n(n+\lambda-1)^k}\frac{k!}{i_1!\ldots i_n!}\sum_{j=1}^{n}\lambda^{i_j}} \leq \frac{1}{n}\sum_{j=1}^{n} T_j.$$

Supported by the power means we can introduce mixed symmetric operator means corresponding to (8):

Definition 3.7. *Assume (H_1) with $I :=]0, \infty[$ and (H_2). We define the mixed symmetric means with respect to (8) by*

$$M_{s,r}(k,\lambda)$$
$$:= \left(\frac{1}{(n+\lambda-1)^k} \sum_{(i_1,\dots,i_n)\in S_k} \frac{k!}{i_1!\dots i_n!} \left(\sum_{j=1}^{n} \lambda^{i_j} w_j \right) \cdot M_r^s \left(T_1,\dots,T_n; \frac{\lambda^{i_1} w_1}{\sum\limits_{j=1}^{n} \lambda^{i_j} w_j}, \dots, \frac{\lambda^{i_n} w_n}{\sum\limits_{j=1}^{n} \lambda^{i_j} w_j} \right) \right)^{\frac{1}{s}},$$

if s, $r \in \mathbb{R}$ and $s \neq 0$.

The monotonicity and the convergence of the previous means is studied in the next result.

Proposition 3.8. *Assume (H_1) with $I :=]0, \infty[$ and (H_2). Then*

(a)

$$M_s \leq \ldots \leq M_{s,r}(k,\lambda) \leq \ldots \leq M_{s,r}(0,\lambda) = M_r, \tag{13}$$

if either

(i) $1 \leq s \leq r$ or

(ii) $-r \leq s \leq -1$ or

(iii) $s \leq -1$, $r \geq s \geq 2r$;

while the reverse inequalities hold in (13) if either

(iv) $r \leq s \leq -1$ or

(v) $1 \leq s \leq -r$ or

(vi) $s \geq 1$, $r \leq s \leq 2r$.

(b) All of these cases

$$\lim_{k\to\infty} M_{s,r}(k,\lambda) = M_s$$

for each fixed $\lambda > 1$.

Proof. We apply Proposition 3.5 (b). □

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