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SYMMETRIC SKEW 4-DERIVATIONS ON PRIME RINGS

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Abstract. For a ring R with an automorphism α a 4-additive mapping $D : R^4 \longrightarrow R$ is called a skew 4-derivation w.r.t. α if it is a α-derivation of R for each argument. Namely it is always an α-derivation of R for the argument being left once (3) arguments are fixed by (3) elements in R. In the present note, begin with a result of Jung and Park [5], we prove that if a skew 4-derivation D associated with an automorphism α with trace f of a noncommutative prime ring R under suitable torsion condition satisfying $[f(x), \alpha(x)] = 0$ for all $x \in I$, a nonzero ideal of R, then $D = 0$.

Keywords: prime (semiprime) ring; skew derivation; commuting mappings.

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1. Introduction

Throughout the paper R will denote a ring with centre $Z(R)$. A ring R is said to be prime (resp. semiprime) if $aRb = (0)$ implies that either $a = 0$ or $b = 0$ (resp. $aRa = (0)$ implies that $a = 0$). We shall write $[x, y]$ the commutator $xy - yx$. We make extensive use of basic commutator identities $[xy, z] = [x, z]y + x[y, z]$ and $[x, yz] = [x, y]z + y[x, z]$. An additive mapping $d : R \longrightarrow R$ is called a derivation if $d(xy) = d(x)y + xd(y)$, for all $x, y \in R$. A derivation d is inner if there exists

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an element $a \in R$ such that $d(x) = [a,x]$ for all $x \in R$. A mapping $D(.,.) : R \times R \longrightarrow R$ is said to be symmetric if $D(x,y) = D(y,x)$, for all $x,y \in R$. A mapping $f : R \longrightarrow R$ defined by $f(x) = D(x,x)$, where $D(.,.) : R \times R \longrightarrow R$ is a symmetric mapping, is called the trace of D. It is obvious that in the case $D(.,.) : R \times R \longrightarrow R$ is a symmetric mapping which is also biadditive (i.e. additive in both arguments), the trace f of D satisfies the relation $f(x+y) = f(x) + f(y) + 2D(x,y)$, for all $x,y \in R$. A biadditive mapping $D : R \times R \longrightarrow R$ is said to be a biderivation if for every $x \in R$, the map $y \mapsto D(x,y)$ as well as if for every $y \in R$, the map $x \mapsto D(x,y)$ are derivations of R. G. Maksa [6] introduced the concept of a symmetric biderivation (see also [7], where an example can be found). It was shown in [6] that symmetric biderivations are related to general solution of some functional equations. Some results on symmetric biderivation in prime and semiprime rings can be found in [1, 2, 8, 9]. The notion of additive commuting mappings is closely connected with the notion of biderivations. Namely linearizing $[x, f(x)] = 0$ for all $x,y \in R$, $(x,y) \mapsto [f(x),y]$ is a biderivation (moreover, all derivations appearing are inner). There has been ongoing interest concerning the relationship between the commutativity of a ring and the existence of certain specific types of derivations.

An additive mapping $d : R \longrightarrow R$ is called a skew derivation (α-derivation) of R associated with an automorphism α if $d(xy) = d(x)y + \alpha(x)d(y)$, for all $x,y \in R$. Skew derivations are one of the natural generalization of usual derivations, when $\alpha = I$, the identity map on R. A mapping $D : R^4 \longrightarrow R$ is said to be 4-additive if its additive in each argument and it is called symmetric if $D(x_1,x_2,x_3,x_4) = D(x_{\pi(1)},x_{\pi(2)},x_{\pi(3)},x_{\pi(4)})$ for all $x_1,...,x_4 \in R$ and every permutation $\pi \in S_4$. A 4-additive map $D : R^4 \longrightarrow R$ is called a skew $4-$derivation associated with an automorphism α if for every $x_1,x_2,x_3 \in R$, the map $x \longmapsto D(x_1,x_2,x_3,x)$ is a skew derivation of R associated with an automorphism α.

Example Let R be a commutative ring, α be an automorphism of R. Suppose $d : R \longrightarrow R$ is a skew derivation of R with an automorphism α. Then a map $\delta : R^4 \longrightarrow R$ defined as $\delta(w,x,y,z) = d(w)d(x)d(y)d(z)$ for all $w,x,y,z \in R$ is a symmetric skew 4-derivation on R associated with automorphism α.

A trivial generalization of skew n-derivation for $n \geq 1$ is defined as follows: A mapping $D : R^n \longrightarrow R$ is said to be n additive if it is additive in each argument and it is called symmetric if $D(x_1,x_2,....,x_n) = D(x_{\pi(1)},x_{\pi(2)},.....x_{\pi(n)})$ for all $x_1,x_2,...,x_n \in R$ and every permutation $\pi \in S_n$. An n-additive map $D : R^n \longrightarrow R$ is called a skew $n-$derivation associated with automorphism

α if for every $k = 1,2,.,n$ and all $x_1, x_2, ...x_n \in R$, the map $x \longmapsto D(x_1, x_{k-1}, x, x_{k+1}, ..., x_n)$ is a skew derivation of R associated with automorphism α. This definition covers both the notion of skew derivations as well as the notion of skew biderivation. Namely, a skew 1-derivation is a skew derivation and skew 2-derivation is a skew biderivation.

In 1957, Posner [10] proved a very striking theorem, which states that the existence of a nonzero centralizing derivation on a prime ring forces the ring to be commutative. This theorem has been extremely influential and it initiated the study of centralizing mappings. Further Vukman [9] extend above result for biderivations. Recently Jung and Park [5] considered permuting 3-derivations on prime and semiprime rings and obtained the following: Let R be a noncommutative 3-torsion free semiprime ring and let I be a nonzero two sided ideal of R. Suppose that there exists a permuting 3-derivation $D : R^3 \longrightarrow R$ such that f is centralizing on I. Then f is commuting on I. Very recently above mentioned results extend by Fosner, A. in [3]. Motivated by all these observations, we prove the following theorems. Moreover, at the end we present some corollaries and open problems.

2. Main Results

Theorem 2.1 *Let R be a 2, 3-torsion free noncommutative prime ring and I be a nonzero ideal of R. Suppose α is an automorphism of R and $D : R^4 \longrightarrow R$ is a symmetric skew 4-derivation associated with α. If f is a trace of D such that $[f(x), \alpha(x)] = 0$ for all $x \in I$, then $D = 0$.*

Proof. Let

$$[f(x), \alpha(x)] = 0 \text{ for all } x \in I. \tag{1}$$

Linearization of (1) yields that

$$\begin{aligned} &[f(x), \alpha(x)] + 4[D(x,x,x,y), \alpha(x)] + 6[D(x,x,y,y), \alpha(x)] + 4[D(x,y,y,y), \alpha(x)] \\ &+[f(y), \alpha(x)] + [f(x), \alpha(y)] + 4[D(x,x,x,y), \alpha(y)] + 6[D(x,x,y,y), \alpha(y)] \\ &+4[D(x,y,y,y), \alpha(y)] + [f(y), \alpha(y)] = 0 \text{ for all } x, y \in I. \end{aligned} \tag{2}$$

In view of (1), (2) yields that

$$\begin{aligned} &4[D(x,x,x,y), \alpha(x)] + 6[D(x,x,y,y), \alpha(x)] + 4[D(x,y,y,y), \alpha(x)] \\ &+[f(y), \alpha(x)] + [f(x), \alpha(y)] + 4[D(x,x,x,y), \alpha(y)] \\ &+6[D(x,x,y,y), \alpha(y)] + 4[D(x,y,y,y), \alpha(y)] = 0 \text{ for all } x, y \in I. \end{aligned} \tag{3}$$

Replacing y by $-y$ in (3) we find

$$\begin{aligned} &-4[D(x,x,x,y),\alpha(x)]+6[D(x,x,y,y),\alpha(x)]-4[D(x,y,y,y),\alpha(x)] \\ &+[f(y),\alpha(x)]-[f(x),\alpha(y)]+4[D(x,x,x,y),\alpha(y)] \\ &-6[D(x,x,y,y),\alpha(y)]+4[D(x,y,y,y),\alpha(y)]=0 \text{ for all } x,y\in I. \end{aligned} \tag{4}$$

Comparing (3) and (4) and using 2-torsion freeness of R we get

$$\begin{aligned} &4[D(x,x,x,y),\alpha(x)]+4[D(x,y,y,y),\alpha(x)] \\ &+[f(x),\alpha(y)]+6[D(x,x,y,y),\alpha(y)]=0 \text{ for all } x,y\in I. \end{aligned} \tag{5}$$

Substitute $y+z$ for y in (5) and use (5) to get

$$\begin{aligned} &12[D(x,y,z,z),\alpha(x)]+12[D(x,z,y,y),\alpha(x)]+[D(x,x,y,z),\alpha(y)]+6[D(x,x,z,z),\alpha(y)] \\ &+6[D(x,x,y,y),\alpha(z)]+12[D(x,x,y,z),\alpha(z)]=0 \text{ for all } x,y,z\in I. \end{aligned} \tag{6}$$

Replacing z by $-z$ in (6) and compare with (6) we obtain

$$\begin{aligned} &12[D(x,z,y,y),\alpha(x)]+12[D(x,x,y,z),\alpha(y)] \\ &+6[D(x,x,y,y),\alpha(z)]=0 \text{ for all } x,y,z\in I. \end{aligned} \tag{7}$$

Substitute $y+u$ for y in (7) and use (7) we get

$$\begin{aligned} &24[D(x,z,y,u),\alpha(x)]+12[D(x,x,y,z),\alpha(u)]+12[D(x,x,u,z),\alpha(y)] \\ &+12[D(x,x,y,u),\alpha(z)]=0 \text{ for all } u,x,y,z\in I. \end{aligned} \tag{8}$$

Since R is 2 and 3 -torsion free and replacing y,u by x in (8), we have

$$4[D(x,x,x,z),\alpha(x)]+[f(x),\alpha(z)]=0 \text{ for all } x,z\in I. \tag{9}$$

Again replace z by zy in (9) and using (9) we obtain

$$\begin{aligned} &4[D(x,x,x,z),\alpha(x)]y+4D(x,x,x,z)[y,\alpha(x)] \\ &+4[\alpha(z),\alpha(x)]D(x,x,x,y)+[f(x),\alpha(z)]\alpha(y)=0 \text{ for all } x,y,z\in I. \end{aligned} \tag{10}$$

Substitute x for z in (10) and in view of (1) we find

$$4f(x)[y,\alpha(x)]=0 \text{ for all } x,y\in I. \tag{11}$$

Using 2-torsion freeness of R we obtain

$$f(x)[y,\alpha(x)]=0 \text{ for all } x,y\in I. \tag{12}$$

Substitute yz for y to get

$$f(x)y[z,\alpha(x)]=0 \text{ for all } x,y,z\in I. \tag{13}$$

Primeness of R yields that either $f(x) = 0$ or $[z, \alpha(x)] = 0$ for all $x \in I \setminus Z(R),\ z \in I$.

Next we will show that $f(x) = 0$ for all $x \in I$. Let $z \in I \cap Z(R)$ and $x \in I \setminus Z(R)$. Then

$$x+z, x-z \in I \setminus Z(R)$$

and we have

$$0 = f(x+z) = f(z) + 4D(x,x,x,z) + 4D(x,z,z,z) + 6D(x,x,z,z) \tag{14}$$

and

$$0 = f(x-z) = f(z) - 4D(x,x,x,z) - 4D(x,z,z,z) + 6D(x,x,z,z) \tag{15}$$

Comparing the last two relation and using torsion condition, we get

$$f(z) + 6D(x,x,z,z) = 0. \tag{16}$$

On suitable linearization and using (16) we arrive at $f(x) = 0$ for all $x \in I$. Hence we have $D(x,y,z,w) = 0$ for all $x,y,z,w \in I$. Substitute rx for x for all $x \in I,\ r \in R$ to get

$$0 = D(rx,y,z,w) = D(r,y,z,w)x + \alpha(r)D(x,y,z,w) = D(r,y,z,w)x. \tag{17}$$

This implies that $D(r,y,z,w)I = 0$ for all $y,z,w \in I,\ r \in R$. Since R is prime we obtain $D(r,y,z,w) = 0$ for all $y,z,w \in I,\ r \in R$. Repeating this process untill we get $D(r,s,t,p) = 0$ for all $r,s,t,p \in R$. Hence $D = 0$.

In [8], author proved that: let R be a 2-torsion free semiprime ring. Suppose that there exists a symmetric biderivation $D : R^2 \longrightarrow R$ such that $D(f(x),x) = 0$ for all $x \in R$, where f denotes the trace of D. Then we have $D = 0$. We consider the case when the ring is semiprime and replace symmetric biderivation with symmetric skew 3-derivation. In this sense we obtain the following:

Theorem 2.2. *Let R be a 2, 3-torsion free semiprime ring and $D : R^3 \longrightarrow R$ be a symmetric skew 3-derivation of R with trace f such that $D(f(x),x,x) = 0$ for all $x \in R$. Then $D = 0$.*

Proof. Let

$$D(f(x),x,x) = 0 \text{ for all } x \in R. \tag{18}$$

Linearization yields that

$$\begin{aligned}
&D(f(x),x,x)+3D(D(x,x,y),x,x)+3D(D(y,y,x),x,x)\\
&+D(f(y),x,x)+2D(f(x),x,y)+6D(D(x,x,y),x,y)\\
&+6D(D(y,y,x),x,y)+2D(f(y),x,y)+D(f(x),y,y)\\
&+3D(D(x,x,y),y,y)+3D(D(y,y,x),y,y)+D(f(y),y,y)=0 \text{ for all } x,y\in R.
\end{aligned}\tag{19}$$

Comparing (18) and (19) we have

$$\begin{aligned}
&3D(D(x,x,y),x,x)+3D(D(y,y,x),x,x)+D(f(y),x,x)+2D(f(x),x,y)\\
&+6D(D(x,x,y),x,y)+6D(D(y,y,x),x,y)+2D(f(y),x,y)+D(f(x),y,y)\\
&+3D(D(x,x,y),y,y)+3D(D(y,y,x),y,y)=0 \text{ for all } x,y\in R.
\end{aligned}\tag{20}$$

Replace y by $-y$ in (20) to get

$$\begin{aligned}
&-3D(D(x,x,y),x,x)+3D(D(y,y,x),x,x)+D(f(y),x,x)-2D(f(x),x,y)\\
&+6D(D(x,x,y),x,y)-6D(D(y,y,x),x,y)-2D(f(y),x,y)+D(f(x),y,y)\\
&-3D(D(x,x,y),y,y)+3D(D(y,y,x),y,y)=0 \text{ for all } x,y\in R.
\end{aligned}\tag{21}$$

Subtracting (21) and (20) we obtain

$$\begin{aligned}
&6D(D(x,x,y),x,x)+2D(f(y),x,x)+4D(f(x),x,y)\\
&+12D(D(y,y,x),x,y)+6D(D(x,x,y),y,y)=0 \text{ for all } x,y\in R.
\end{aligned}\tag{22}$$

Substitute $y+z$ for y in (22) and use (22) we find

$$\begin{aligned}
&6D(D(z,z,y),x,x)+6D(D(y,y,z),x,x)+12D(D(y,y,x),x,z)\\
&+12D(D(z,z,x),x,y)+24D(D(y,z,x),x,y)+24D(D(y,z,x),x,z)\\
&+6D(D(x,x,z),y,y)+12D(D(x,x,y),y,z)\\
&+12D(D(x,x,z),y,z)+6D(D(x,x,y),z,z)=0 \text{ for all } x,y,z\in R.
\end{aligned}\tag{23}$$

Replacing y by $-y$ in (23) we have

$$\begin{aligned}
&-6D(D(z,z,y),x,x)+6D(D(y,y,z),x,x)+12D(D(y,y,x),x,z)\\
&-12D(D(z,z,x),x,y)+24D(D(y,z,x),x,y)-24D(D(y,z,x),x,z)\\
&+6D(D(x,x,z),y,y)+12D(D(x,x,y),y,z)\\
&-12D(D(x,x,z),y,z)-6D(D(x,x,y),z,z)=0 \text{ for all } x,y,z\in R.
\end{aligned}\tag{24}$$

Adding (23)and (24) and using 2-torsion freeness of R we get

$$\begin{aligned}
&6D(D(y,y,z),x,x)+12D(D(y,y,x),x,z)+24D(D(y,z,x),x,y)\\
&+6D(D(x,x,z),y,y)+12D(D(x,x,y),y,z)=0 \text{ for all } x,y,z\in R.
\end{aligned}\tag{25}$$

Again replacing z by zw in (25) and using (25) we obtain

$$\begin{aligned} &6D(y,y,z)D(w,x,x)+6D(\alpha(z),x,x)D(y,y,w)\\ &+24D(y,z,x)D(w,x,y)+24D(\alpha(z),x,y)D(w,x,y)\\ &+6D(x,z,x)D(w,y,y)+6D(\alpha(z),y,y)D(x,x,w)=0 \text{ for all } w,x,y,z\in R. \end{aligned} \tag{26}$$

Substitute x for y and use symmetry of D and apply torsion condition to get

$$D(x,z,x)D(w,x,x)+D(\alpha(z),x,x)D(x,x,w)=0 \text{ for all } w,x,z\in R. \tag{27}$$

Since α is an automorphism of R and using torsion freeness of R, we have $D(x,z,x)D(w,x,x)=0$ for all $w,x,z\in R$. Using the symmetry of D we get

$$D(x,x,z)D(w,x,x)=0 \text{ for all } w,x,z\in R. \tag{28}$$

Replacing z by zu in (28) and using (28) we have

$$D(x,x,z)uD(w,x,x)=0 \text{ for all } u,w,x,z\in R. \tag{29}$$

Semiprimeness of R yields that $D(w,x,x)=0$ for all $w,x\in R$. A suitable linearization implies that $D(w,x,y)=0$ for all $w,x,y\in R$. Hence $D=0$.

Theorem 2.3. *Let R be a 2, 3-torsion free semiprime ring and I a nonzero ideal of R. If D is a symmetric skew 3-derivation of R with trace f such that $D(I,I,I)\subseteq I$ and $D(f(x),x,x)=0$ for all $x\in I$. Then $D=0$.*

To prove above theorem we require the following lemma:

Lemma 2.1 [4] If R is a semiprime ring and I is an ideal of R, then $I\cap ann(I)=(0)$, where ann(I) denotes the annihilator of I.

Proof of theorem 2.3 Application of Lemma 2.1 and Theorem 2.2 yields the required result.

Corollary 2.1. *Let R be a 2, 3-torsion free prime ring and I be a nonzero ideal of R. If $D:R^3\longrightarrow R$ is a symmetric skew 3-derivation of R with trace f such that $D(f(x),x,x)=0$ for all $x\in I$. Then $D=0$.*

Corollary 2.2. *Let R be a 2, 3-torsion free prime ring and I be a nonzero ideal of R. If $D:R^3\longrightarrow R$ is a symmetric 3-derivation of R with trace f such that $D(f(x),x,x)=0$ for all $x\in I$. Then $D=0$.*

Conjecture 2.1. *Let R be a noncommutative prime ring under suitable torsion restriction and I be a nonzero ideal of R. Suppose α is automorphism of R and $D : R^n \longrightarrow R$ is a symmetric skew n-derivation associated with α. If f is the trace of D such that $[f(x), \alpha(x)] = 0$ for all $x \in I$, then $D = 0$.*

Conjecture 2.2. *Let R be a semiprime ring with suitable torsion restriction and $D : R^n \longrightarrow R$ be a symmetric skew n-derivation of R with trace f such that $D(f(x), \underbrace{x, x, ..., x}_{(n-1)-times}) = 0$ for all $x \in R$. Then $D = 0$.*

Conflict of Interests

The authors declare that there is no conflict of interests.

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