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ON THE k-METRIC DIMENSION OF GRAPHS

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Abstract. Let $G(V,E)$ be a connected graph. A subset S of V is said to be *k-resolving set* of G, if for every pair of distinct vertices $u, v \notin S$, there exists a vertex $w \in S$ such that $|d(u,w) - d(v,w)| \geq k$, for some $k \in Z^+$. Among all k-resolving sets of G, the set having minimum cardinality is called a *k-metric basis* of G and its cardinality is called the *k-metric dimension* of G and is denoted by $\beta_k(G)$. In this paper, we have discussed some characterizations of k-metric dimension in terms of some graphical parameters. We have mainly focused on 2-metric dimension of graphs and discussed few characterizations. Further 2-metric dimension of trees is determined and from this result 2- metric dimension of path, cycle and sharp bounds of unicyclic graphs are established.

Keywords: 2-metric dimension; metric dimension; resolving set.

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1. Introduction

All the graphs considered in this paper are simple, finite, undirected and connected. The *distance* between the vertices u and v, denoted by $d(u,v)$, is the length of the shortest path between

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them. The maximum distance from a vertex v to any vertex of G is called the *eccentricity* of the vertex v and is denoted by $e(v)$. The *radius* of a graph G, denoted by $rad(G)$, is the minimum eccentricity of its vertices and *diameter* of a graph G, denoted by $diam(G)$, is the maximum of eccentricity of its vertices. The minimum degree of a vertex in G is denoted by $\delta(G)$ and the maximum degree of a vertex in G is denoted by $\Delta(G)$. We use standard terminology, the terms not defined here are found in [1, 2].

In 1976, F.Harary and R.A. Melter [7] introduced the notion of metric dimension. A subset S of $V(G)$ is said to be a *resolving set* of G, if for every $u, v \in V(G)$ and $u \neq v$, there exists a vertex $w \in S$ with the property that $|d(u,w) - d(v,w)| \geq 1$. Among all the resolving sets of G, a set having minimum cardinality is called a *metric basis* of G, and its cardinality is called the *metric dimension* denoted by $\beta(G)$.

2. Some known results on Metric dimension

In this section we mention some of the known results of metric dimension due to various authors, which we use in the subsequent sections.

Theorem 2.1. [F.Harary and R.A Melter [7]] For a non-trivial graph G on n vertices, $\beta(G) = n-1$ if and only if $G \cong K_n$.

Theorem 2.2. [B. Shanmukha et al.[3]] For a wheel graph $W_{1,n}, n \geq 3$,

(1) $\beta(W_{1,3}) = \beta(W_{1,6}) = 3$

(2) $\beta(W_{1,4}) = \beta(W_{1,5}) = 2$

(3) $\beta(W_{1,x+5k}) = \begin{cases} 3+2k, & \text{when } x = 7 \text{ or } 8,\ k = 0,1,2\ldots \\ 4+2k, & \text{when } x = 9 \textit{ or } 10 \text{ or } 11,\ k = 0,1,2\ldots. \end{cases}$

The metric dimension of trees is studied by various authors in [6-8].

The following definitions are defined in [6] which we have used in obtaining our result. A vertex of degree at least 3 in a graph G is called a *major vertex* of G. Any end-vertex u of G is said to be a *terminal vertex* of a major vertex v of G if $d(u,v) < d(u,w)$ for every other major vertex w of G. The *terminal degree* $ter(v)$, of a major vertex v, is the number of terminal vertices of v. A major vertex v of G, is an *exterior major vertex* of G if it has positive terminal degree.

Theorem 2.3. [Gary Chartrand et al. [6]] If T is a tree that is not a path, then $\beta(T) = \sigma(T) - ex(T)$, where $\sigma(T)$ denotes the sum of the terminal degrees of the major vertices of G, and $ex(G)$ denotes the number of exterior major vertices of G.

Theorem 2.4. [6] If T is a tree of order at least 3 and e is an edge of $\overline{T}$, then

$$\beta(T) - 2 \leq \beta(T+e) \leq \beta(T) + 1.$$

3. k-Metric dimension

In this section we introduce the notion of k-metric dimension and study some of its bounds in terms of other graphical parameters.

Definition 3.1. Let $G(V,E)$ be a graph. A subset S of V is said to be a *k-resolving set* of G, if for every $u, v \in V(G) - S$ and $u \neq v$, there exists a vertex $w \in S$ with the property that $|d(u,w) - d(v,w)| \geq k$ for some $k \in Z^{+}$. Among all the k-resolving sets of G, a set having minimum cardinality is called a *k-metric basis* of G and its cardinality is called the *k-metric dimension of* G denoted by $\beta_k(G)$.

Remark 3.2. For any $k \in Z^{+}$ and a graph $G(V,E)$, the definition 3.1 implies that;

(1) $1 \leq \beta_k(G) \leq |V| - 1$.
(2) If $k = 1$, then $\beta_1(G) = \beta(G)$, the metric dimension of a graph dealt by various authors in [3-8].
(3) Every $k+1$-metric basis of a graph G is also a k resolvable set for G and hence $1 \leq \beta_k(G) \leq \beta_{k+1}(G) \leq |V(G)| - 1$. In particular $1 \leq \beta(G) \leq \beta_k(G) \leq |V| - 1$.
(4) By theorem 2.1, $\beta(K_n) = n - 1$ and by item (3) of same Remark 3.2, $n - 1 \leq \beta_k(K_n) \leq n - 1$ which implies $\beta_k(K_n) = n - 1$.
(5) If H is a connected subgraph of G, then $\beta_k(G) \geq \beta_k(H)$.

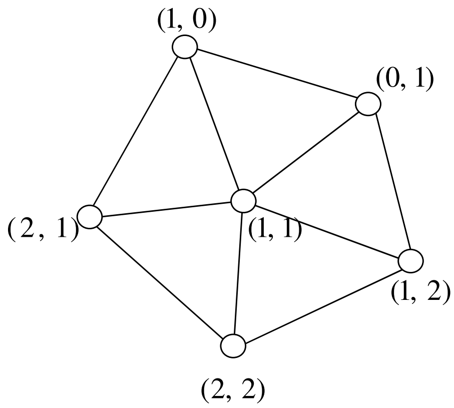

FIGURE 1. $\beta(W_6) = 3$

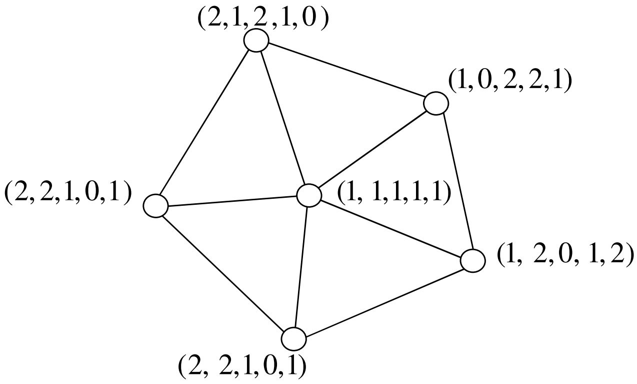

FIGURE 2. $\beta_2(W_6) = 5$

For any graph G on n vertices, by item 4 of Remark 3.2, if $\beta(G) = n-1$, then $\beta_k(G) = n-1$, for $k \geq 2$. But the converse of this need not be true in general. As a counter example it is verified that $\beta(W_6) = 3$ but $\beta_2(W_6) = 5$ (refer Figure 1 and Figure 2).

We now prove the following theorem as a generalization to theorem 2.1.

Theorem 3.3. For any non-trivial graph G on $n \geq 2$ vertices, $\beta_k(G) = n-1$ if and only if $diam(G) \leq k$, where $k \geq 1$ is any integer.

Proof: The case of $k = 1$, follows from the Theorem 2.1. Now we consider the case when $k \geq 2$. Suppose $diam(G) \leq k$. Then, for any three distinct vertices $u, v, w \in V(G)$, $d(u,w) \leq k$ and $d(v,w) \leq k$ and hence $0 \leq |d(u,w) - d(v,w)| < k$. Therefore, no vertex $w \in V(G)$ can k-resolve any pair of vertices $u, v \in V(G)$. Hence for any metric basis S of the graph G, the set $V(G) - S$ cannot have more than one element which implies $\beta_k(G) = n-1$.

We now prove the converse part. Suppose $diam(G) > k \geq 2$. The only connected graphs with 2 or 3 vertices are P_2, P_3, K_3 and the result can be easily verified in these three graphs.

We now consider the graphs G with at least 4 vertices. Let u and v be the end vertices of a longest path P in such a graph G. Let x be a vertex adjacent to u on the longest path P. Let $S = V(G) - \{x, v\}$ so that $V(G) - S = \{x, v\}$. Then $|d(u,v) - d(u,x)| = diam(G) - 1 > k - 1$. This implies $|d(u,v) - d(u,x)| \geq k$. Hence S is a k-resolving set and $\beta_k(G) \leq n - 2$, a contradiction to the fact that $\beta_k(G) = n - 1$.

Remark 3.4. From Theorem 3.3, it directly follows that, for $k \geq 1$ if $\beta_k(G) = 2$, then G cannot have K_5 or $K_{3,3}$ as a subgraph.

Lemma 3.5. For a graph G of order at least 2 and an integer $k \geq 2$, if S is a k-metric basis of G, then no two vertices $u, v \in V(G) - S$ are adjacent in G.

Proof: If not, for some $u, v \in V(G) - S$, $uv \in E(G)$ implies that $|d(u,w) - d(v,w)| \leq 1$ for any $w \in S$, a contradiction.

Lemma 3.6. Let G be a graph of order at least 2 and $k \geq 1$ be any integer. If S is a k-metric basis of G then for every two vertices $u, v \in V(G) - S$, $d(u,v) \geq k$.

Proof: Since S is a k-metric basis of G, there exists a vertex $w \in S$ such that for every $u, v \in V(G) - S$, $|d(u,w) - d(v,w)| \geq k$. As $k \geq 1$, $d(u,w) \neq d(v,w)$. So without loss of generality we assume, $d(u,w) > d(v,w)$. Hence,

$$\begin{aligned}
|d(u,w) - d(v,w)| &\geq k \\
d(u,w) - d(v,w) &\geq k \\
\Rightarrow k &\leq d(u,w) - d(v,w) \\
\Rightarrow k &\leq [d(u,v) + d(v,w)] - d(v,w) \text{ by triangular inequality} \\
\Rightarrow k &\leq d(u,v) \\
\Rightarrow d(u,v) &\geq k.
\end{aligned}$$

Remark 3.7. The converse of the Lemma 3.5 and Lemma 3.6 need not be true in general.

Lemma 3.8. For an integer $k \geq 1$, if S is a k-metric basis of a graph G of order at least 2, then at most one vertex in $V(G) - S$ is adjacent to every vertex in S.

Proof: Suppose two vertices $u, v \in V(G) - S$, are adjacent to every vertex in S, then $d(w,u) = d(w,v) = 1$ for every vertex $w \in S$, a contradiction to definition 3.1.

Remark 3.9. By Lemma 3.8, for every $v \in V(G) - S$, $deg(v) \leq |S|$. Further, there exists at most one vertex $v \in V(G) - S$ such that $deg(v) = |S|$.

4. 2-Metric dimension

In this section, we focus particularly on 2-metric dimension and have characterized graphs with 2-metric dimension.

Remark 4.1. By Theorem 3.3, it follows that 2-metric dimension of graphs on $n \geq 2$ vertices such as Complete graphs, Wheels, Stars, Complete bipartite graphs, fans is $n-1$. For any graph G, with $n \geq 2$ vertices, the graph $G + K_1$ obtained by taking the disjoint union of G with K_1 and joining every vertex of G to K_1 is a graph with diameter 2 and hence $\beta_2(G + K_1) = n$.

Lemma 4.2. Let G be a graph with $n \geq 3$ vertices with S as a 2-metric basis. Let $w \in V(G)$ be adjacent to p pendant vertices $v_1, v_2, \ldots v_p$, for $p \geq 2$. Then exactly p vertices from the set $\{w, v_1, v_2, \ldots v_p\}$ are in S.

Proof: Let S be a 2-metric basis of a graph G. By Lemma 3.5, no two vertices in $V(G) - S$ are adjacent. Since w is adjacent to every $v_i, 1 \leq i \leq p$, either $w \in S$ or $w \in V(G) - S$ and hence we have the following 2 cases.

Case 1: $w \in S$

For every $s \in S - \{w\}$, any path from s to v_i or v_j, for all $1 \leq i, j \leq p$, has to pass through w, and $d(s, v_i) = d(s, v_j) = d(s, w) + 1$. No two of the pendant vertices adjacent to w are resolved by any $s \in S$. Hence at most one of the pendant vertices can be in $V(G) - S$ and at least $p-1$ should be in S. Since S is a 2-metric basis, $p-1$ pendant vertices are in S. Since $w \in S$, S has exactly p vertices of the set $\{w, v_1, v_2, \ldots v_p\}$.

Case 2: $w \in V(G) - S$

Since all the p pendant vertices are adjacent to w, by Lemma 3.5, none of them can be in

$V(G) - S$. Hence all of them are in S. Then exactly p vertices from the set $\{w, v_1, v_2, \dots v_p\}$ are in S.

Lemma 4.3. Let G be a graph with $n \geq 4$ vertices and $S \subset V(G)$. Let $w \in S$ be adjacent to both $u, v \in V(G) - S$ of which at most one is a pendant vertex. If S is a 2-metric basis of G then there exists at least one vertex $w_1 \in S - \{w\}$, which is adjacent to either u or v and non adjacent to w.

Proof: As S is a 2-metric basis, $u, v \in V(G) - S$ are not adjacent to each other. Since both are adjacent to w and at most one among them is a pendant vertex, by Lemma 3.5, at least one of them should be adjacent to a vertex in S other than w.

Remark 4.4. The converse of the above Lemma 4.2 and Lemma 4.3 need not be true in general.

Remark 4.5. The generalization of the Lemma 4.3 is as follows:
Consider a graph G with $\beta_2(G) = m$. Suppose $w \in S$ is adjacent to m vertices $\{v_1, v_2, \dots v_m\} \in V(G) - S$ of which at most one is a pendant vertex, then at least $m-1$ vertices among these are adjacent to $m-1$ distinct, mutually non adjacent vertices $w_j \in S$, $1 \leq j \leq m-1$, where $w \neq w_j$ and w_j non adjacent to w.

Remark 4.6. For a graph G of order at least 2 and $S \subset V(G)$, by Lemma 3.5 it follows that if S is a 2-resolving set, then $\bigcup_{v \in V(G)-S} N(v) = A \subseteq S$. But $S \nsubseteq A$ always.

Lemma 4.7. Let S be a 2-metric basis of a graph G of order n. Then

$$\Delta(G) \leq |S| + 1.$$

Proof: Let G be a graph with 2-metric basis S. Let v be the vertex with maximum degree $\triangle(G) = m$. Then either $v \in S$ or $v \in V(G) - S$.

Case 1: Let $v \in S$ and v be adjacent to m vertices $u_1, u_2, \dots u_m$ in $V(G) - S$. Of these m vertices at most one vertex is adjacent to all $v \in S$ by Lemma 3.8. By Remark 4.5, of the remaining $m-1$ vertices in $V(G) - S$, at least $m-2$ vertices are adjacent to $m-2$ distinct, mutually non adjacent vertices of S that are all different from v and all are not adjacent to v. Hence $|S| \geq m - 2 + 1 = m - 1$. Hence $\Delta(G) = deg(v) = m \leq |S| + 1$.

Case 2: If $v \in S$ and v is adjacent to k vertices in S and $m-k$ vertices in $V(G) - S$, then $\Delta(G) = deg(v) \leq |S|$ since $k \leq m$.

Case 3: If $v \in V(G) - S$, then by Remark 3.9, $deg(v) = \triangle(G) \leq |S|$. Hence the proof.

Theorem 4.8. For a graph G of order n, if $\beta_2(G) = k$, then $k+1 \leq n \leq \frac{k^2+3k}{2}$.

Proof: From remark 3.2 item 1, $k = \beta_2(G) \leq |V| - 1 = n - 1$. Therefore, $k+1 \leq n$.

Let $s_1, s_2, \ldots, s_k$ be the vertices in a 2-metric basis S. Consider the vertex s_1. Let s_1 be adjacent to l_1 vertices in $V(G) - S$ (certainly $l_1 \leq k$, by Lemma 4.7). Then out of these l_1 vertices $l_1 - 1$ vertices should be adjacent to $l_1 - 1$ distinct vertices in S other than s_1. Let s_2 be one such vertex. Let us repeat the arguments for s_2. Let s_2 be adjacent to l_2 vertices in $V(G) - S$ (certainly exactly one of these vertices is counted in l_1 as in Figure 3). Then s_2 is adjacent to $l_2 - 1$ new vertices in $V(G) - S$ and out of these $l_2 - 1$ vertices $l_2 - 2$ vertices should be adjacent to $l_2 - 2$ distinct vertices in S other than s_1 and s_2. Let s_3 be one such vertex. Repeating the same argument for s_3, then to s_4, so that

$$\begin{aligned}
|V(G) - S| &= l_1 + [l_2 - 1] + [l_3 - 2] + \ldots + [l_k - (k-1)] \\
&\leq deg(s_1) + [deg(s_2) - 1] + [deg(s_3) - 2] + \ldots + [deg(s_k) - (k-1)], \\
&= |S| + [|S| - 1] + \ldots + 1 (Lemma 4.7) \\
&= |S|(|S| + 1)/2 \\
&= k(k+1)/2.
\end{aligned}$$

Thus, $n = |S| + |V(G) - S| \leq k + k(k+1)/2 = (k^2 + 3k)/2$.

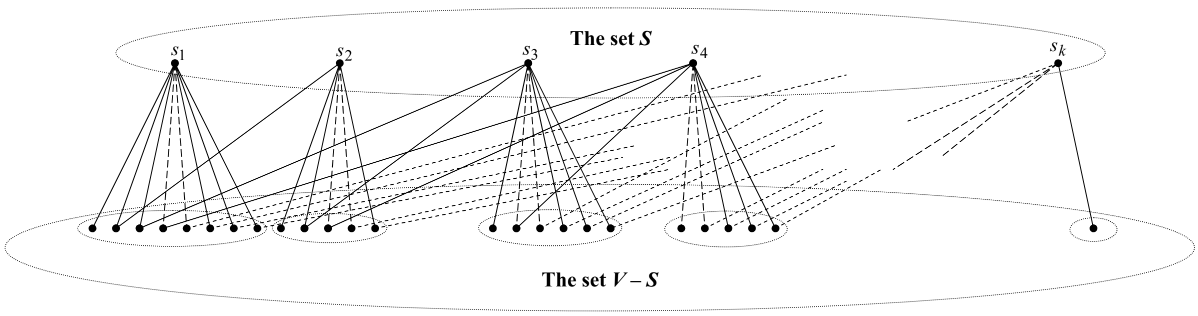

FIGURE 3. Illustration of proof of Lemma 4.8.

5. Graphs with 2-metric dimension 3

In this section, we determine the graphs with 2-metric dimension 1, 2 and characterize the graphs with $\beta_2(G) = 3$.

Theorem 5.1. For a graph G, $\beta_2(G) = 1$ if and only if $G \cong K_1$ or K_2.

Proof: Direct part follows by Lemma 4.8 with $m = 1$ and it is easy to verify converse part.

Theorem 5.2. For a graph G, $\beta_2(G) = 2$ if and only if $G \cong P_3$ or P_4 or P_5 or C_3.

Proof: Let G be a graph such that $\beta_2(G) = 2$ and $S = \{w_1, w_2\}$ be a 2-metric basis of G. Then, by Theorem 4.8, it follows that $3 \leq |V| \leq 5$. In case if $|V| = 3$, then G is one of P_3 or C_3 (since G is connected). In case if $|V| = 4$, then by Theorem 3.3, the only graph with $diam(G) > 2$ is P_4. When $|V| = 5$, let $V = \{v_1, v_2, v_3, w_1, w_2\}$, then $V(G) - S = \{v_1, v_2, v_3\}$. By Lemma 4.7, $\Delta(G) \leq 2$. Since there is no connected graph with $|V| = 5$ and $\Delta(G) = 1$, we consider the case when $\Delta(G) = 2$ and let us consider $\Delta(v_1) = 2$. By Lemma 3.5, $v_1v_2, v_1v_3 \notin E(G)$. Suppose $v_1w_1, v_1w_2 \in E(G)$. By Lemma 3.8, both v_2 and v_3 cannot be adjacent to w_1 or w_2 only. Thus v_2 is adjacent to w_1 and v_3 is adjacent to w_2 and the only possible graph that satisfies all the conditions is P_5.

Conversely, it is easy to verify that for the graphs P_3, P_4, P_5 and C_3, $\beta_2(G) = 2$. Hence the proof.

Theorem 5.3. A graph G of order at least 6 with $\beta_2(G) = 3$, cannot have K_5 as a subgraph.

Proof: Let G be a graph with $\beta_2(G) = 3$. If K_5 is a subgraph of G, then by Remark 3.2, item (5), $\beta_2(K_5) \leq \beta_2(G) = 3$. But by the item (4) of Remark 3.2, $\beta_2(K_5) = 4$, a contradiction. Hence the proof.

Remark 5.4. The proof of the above theorem can be extended to obtain the following result. For $m \geq 4$, a graph G of order at least $m+3$ with $\beta_2(G) = m$ cannot have K_{m+2} as a subgraph.

Theorem 5.5. A graph G of order at least 7 with $\beta_2(G) = 3$ cannot have $K_{3,3}$ as a subgraph.

Proof: Let G be a graph with $\beta_2(G) = 3$ having $K_{3,3}$ as a subgraph. Then by Remark 3.2, item 5 , $\beta_2(K_{3,3}) \leq \beta_2(G) = 3$. But by Remark 4.1 $\beta_2(K_{3,3}) = 5$, a contradiction. Hence the proof.

Remark 5.6. The proof of the above theorem can be extended to obtain the following result. A graph G of order at least $2m+1$ with $\beta_2(G) = m$ where $m \geq 1$ cannot have $K_{m,m}$ as a subgraph.

Theorem 5.7. A graph G on $n \geq 3$ vertices with $\beta_2(G) = m, m \geq 1$, cannot have a subgraph isomorphic to $K_{m+2} - me$, where all the m edges are incident on a vertex of K_{m+2}.

Proof: Let G be a graph on $n \geq 3$ vertices with S as a metric basis and $|S| = \beta_2(G) = m, m \geq 1$. If possible assume the graph G to contain $G_1 = K_{m+2} - me$ as a subgraph, with $V(G_1) = \{v_1, v_2, \ldots v_{m+2}\}$ and $E(G_1) = \{v_1 v_{m+2}, v_i v_j, 1 \leq i, j \leq m+1\}$ Then $< V(G_1) - v_{m+2} > \cong K_{m+1}$ and by Remark 3.2 item 4, $\beta_2(K_{m+1}) = m$. Further S should contain any m vertices of $V(K_{m+1})$. Also, for all $i, 2 \leq i \leq m+1$ we have, $|d(v_i, v_{m+2}) - d(v_i, v_1)| = 2 - 1 = 1 < 2$. Hence none of the m vertices in S, 2-resolves v_1 and v_{m+2}. This implies $\beta_2(G_1) > m$. But by the item (5) of Remark 3.2, $\beta_2(G_1) \leq \beta_2(G) = m$, a contradiction. Hence the proof.

Remark 5.8. A graph G on n vertices with $\beta_2(G) = k$, cannot have a subgraph isomorphic to $K_{k+2} - re$ where $0 \leq r \leq k$ and all the r edges are incident on a vertex of K_{k+2}.

6. 2-metric dimension of trees, paths and cycles

In this section, we have obtained a formula for metric dimension of trees. From this we have obtained the 2-metric dimension of path P_n and cycle C_n.

A tree $T(V,E)$ has one or two centers. Throughout this section, we consider all trees $T(V,E)$ as rooted trees with center c as root and denote it by (T,c). In case of trees with two centers any one of the center is considered as a root. Then for a vertex $v \in V(T,c)$, if the distance $d(c,v) = i$, $0 \leq i \leq e(c)$ then we say that v is in i^{th} level of (T,c). Let $L(i)$ be the set of all vertices that are at level i. Let $L(odd) = \bigcup_{i \text{ is odd}} L(i)$ and $L(even) = \bigcup_{i \text{ is even}} L(i) \bigcup L(0)$.

For example, consider the tree $T(V,E)$ in figure 4 with two centers at vertices 1 and 2. First, we view this tree as a rooted tree, rooted at 1 as shown in figure 5. Then $L(0) = \{1\}$, $L(1)$ = {2, 14, 15}, $L(2)$ = {3, 7, 18, 17, 16}, $L(3)$ = {4, 8, 11, 20, 19} and $L(4)$ = {5, 6, 9, 10, 12, 13, 21} (note that here 4 is also the eccentricity of the vertex 1). Also, the vertices 1, 2, 4, 7, 8, 11, 15, 18 are the major vertices of $T(V,E)$. The vertices 5, 6 are the terminal vertex of 4. Similarly, the vertices $\{9,10\}, \{12,13\}, \{21,19\}, \{17,16\}$ are the terminal vertices of the vertices 8, 11, 18, 15 respectively. Also, *ter*(4) = 2, *ter*(8) = 2, *ter*(11) = 2, *ter*(1) = 0, *ter*(15) = 2, *ter*(7) =

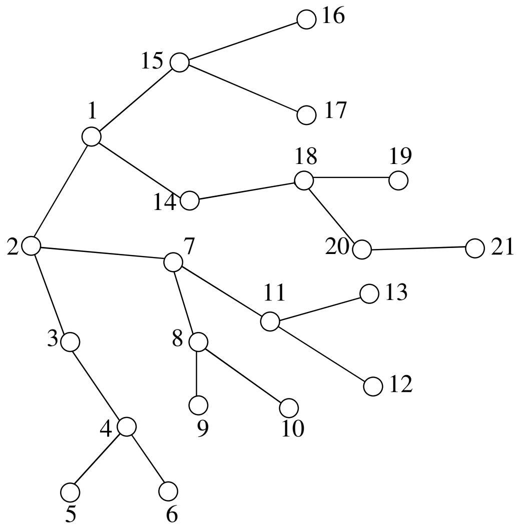

FIGURE 4. Tree $T(V,E)$ with two centers at vertices 1 and 2

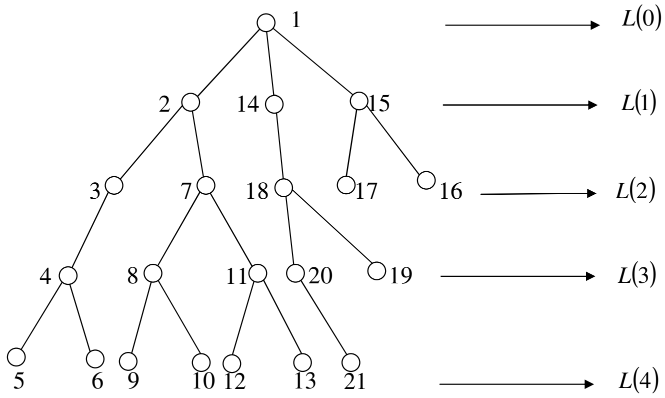

FIGURE 5. Tree $T(V,E)$ in figure 4 viewed as rooted tree $(T,1)$

0, $ter(18) = 2$. Since the terminal degree of the vertices 4, 8, 11, 15 and 18 is positive, these terminal vertices are also exterior major vertices.

Now, we denote the set of all the terminal vertices adjacent to an exterior major vertex v by $M(v)$. Thus, $M(4) = \{5,6\}$, $M(8) = \{9,10\}$, $M(11) = \{12,13\}$, $M(18) = \{21,19\}$ and $M(15) = \{17,16\}$.

Lemma 6.1. Let v be an exterior major vertex of a tree (T,c). Let m be an arbitrary element of $M(v)$. Then the sets

$$S_1 = L(odd) \bigcup [\bigcup_{v \in L(odd)} M(v) - \{m\}]$$

and

$$S_2 = L(even) \bigcup [\bigcup_{v \in L(even)} M(v) - \{m\}]$$

are 2-resolving sets of T.

Proof: We now prove that S_1 is a 2-resolving set.

Let $x, y \in V(T,c) - S_1$. By the definition of S_1, x and y are in some even level of (T,c) which implies $d(x,y) \geq 2$ since they are not adjacent by Lemma 3.5. Then we have the following two cases.

Case 1: $d(x,y) = 2$.

Since $x, y \in V(T,c) - S_1$, both x and y are adjacent to a vertex $w \in S_1$. By definition of S_1, at most one of the vertices x and y is a pendant vertex. We now consider two subcases:

Subcase 1.1: Only x is a pendant vertex at level say i where i is even. Then we have two possibilities depending on the level of y.

(1) x and y are in the same level i:

Now, w is in level $i-1$. Since y is not a pendant vertex, it must be adjacent to at least one vertex of (T,c) say v, which is in the odd level $i+1$ of (T,c) (refer figure 6) and hence $v \in S_1$. Then $|d(x,v) - d(y,v)| = 2$ which implies the vertex v, 2-resolves the vertices x and y.

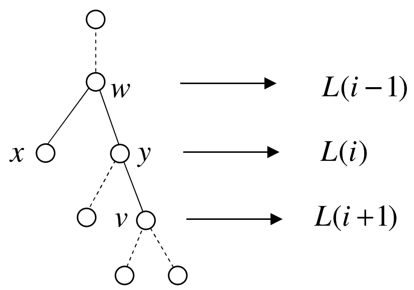

FIGURE 6. Illustration of proof of Lemma 6.1 (Case 1, Subcase 1.1, item 1)

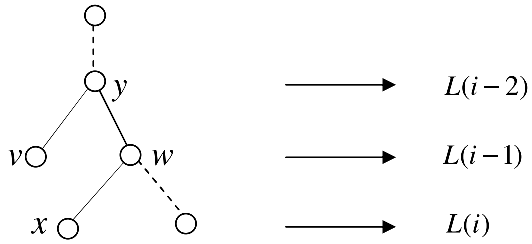

FIGURE 7. Illustration of proof of Lemma 6.1 (Case 1, Subcase 1.1, item 2)

(2) x and y are in different levels:

Since x is in level i, w is in level $i-1$ and y is in level $i-2$. Since y is not a pendant vertex, it must be adjacent to at least one vertex of (T,c), say v, which may be in one of the odd levels $i-1$ (refer figure 7) or $i-3$ (refer figure 8) of (T,c) and hence $v \in S_1$. Then $|d(x,v)-d(y,v)| = 2$ which implies the vertex v, 2-resolves the vertices x and y.

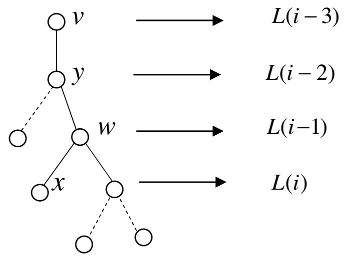

FIGURE 8. Illustration of proof of Lemma 6.1 (Case 1, Subcase 1.1, item 2)

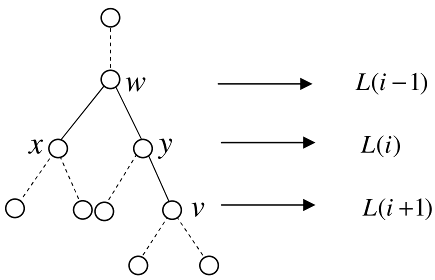

FIGURE 9. Illustration of proof of Lemma 6.1 (Case 1, Subcase 1.2, item 1)

Subcase 1.2: Both x and y are not terminal vertices.

Again we have two possibilities depending on the the level of y.

(1) x and y are in the same level i:

Now, w_1 is in level $i-1$. Since y is not a pendant vertex, it must be adjacent to at least one vertex of (T,c) say v, which is in the odd level $i+1$ of (T,c) (refer Figure 9) and hence $v \in S_1$. Then $|d(x,v)-d(y,v)| = 2$ which implies the vertex v, 2-resolves the vertices x and y.

(2) x and y are in different levels:

Since x is in level i, w is in level $i-1$ and y is in level $i-2$. Since y is not a pendant vertex, it must be adjacent to at least one vertex of (T,c) say v, which may be in one of the odd levels $i-1$ (refer Figure 10) or $i-3$ (refer figure 11) and hence $v \in S_1$. Then $|d(x,v)-d(y,v)| = 2$ which implies the vertex v, 2-resolves the vertices x and y.

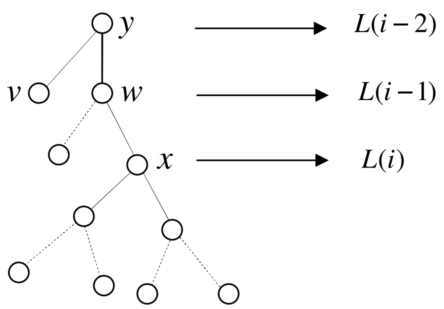

FIGURE 10. Illustration of proof of Lemma 6.1 (Case 1, Subcase 1.2, item 2)

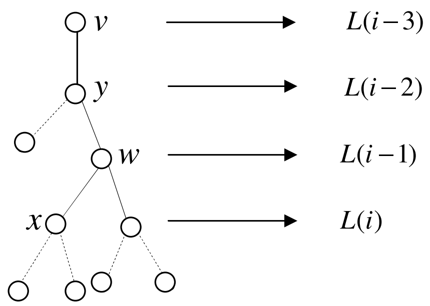

FIGURE 11. Illustration of proof of Lemma 6.1 (Case 1, Subcase 1.2, item 2)

Case 2: $d(x,y) = l \geq 4$.

Now,there exists a unique shortest path between x and y of length $l \geq 4$ (refer figure 12). Let w be a vertex on this path adjacent to x . Then $|d(y,w)-d(x,w)| = (l-1)-1 = l-2 \geq 2$. Hence w 2-resolves x and y. Similarly, we can prove that S_2 is a 2-resolving set.

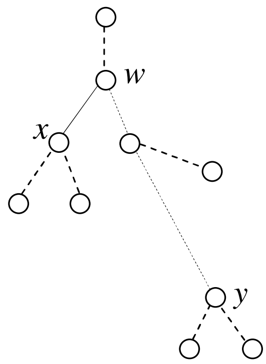

FIGURE 12. Illustration of proof of Lemma 6.1 (Case 2)

Remark 6.2. For a graph G with $n \geq 2$ vertices and a 2-resolving set S, S and $V(G)-S$ are always partitions of G. Since S and $V(G)-S$ are the partitions in tree (T,c), the vertices in any level of (T,c) should be in either S or $V(T,c)-S$. Since by Lemma 3.5, no two vertices in $V(T,c)-S$ are adjacent, the level of the vertices in $V(T,c)-S$ of (T,c) are also not adjacent. Hence the level of vertices in $V(T,c)-S$ is at least distance two apart. Now the vertices in S have no such restrictions. But for S to be a set with minimum cardinality, the vertices in two adjacent levels are not in S except when they are terminal vertices. From Lemma 4.2, there are only two ways for terminal vertices to be in $V(T,c)-S$. Accordingly, we have two possibilities as follows.

(1) By Lemma 4.2, case (1), let a terminal vertex say v, incident on a exterior major vertex say u is in $V(T,c)-S$ and u with all the remaining terminal vertices incident on u is in S. Let v be in even(or odd) level of (T,c). Then by above argument, all the vertices that are in even(or odd) level of (T,c) must be in $V(T,c)-S$ and all the remaining vertices of (T,c) are in S. The same argument can be extended to all the terminal vertices in (T,c). This leads to the definition of $S_1(or\ S_2)$ stated in Lemma 6.1.

(2) By Lemma 4.2, case(2) let all the terminal vertices incident on a major vertex u be in $V(T,c)-S$. Let these terminal vertices be in even (or odd) level of (T,c). Then by above argument all the vertices that are in odd(or even) level of (T,c) must be in $V(T,c)-S$ and all the remaining vertices of (T,c) are in S. The same argument can be extended

to all the terminal vertices in (T,c). This leads to the definition of $S_1(or\ S_2)$ stated in Lemma 6.1.

Also there exists no subset $S \subset V(G)$, with $|S| < Minimun(|S_1|,|S_2|)$ will not be a 2-resolving set of G.Hence S_1 and S_2 are the only minimum 2-resolving set of G.

Theorem 6.3. If (T,c) is a tree, then $\beta_2(T,c) = min\{|S_1|,|S_2|\}$ where S_1 and S_2 are the 2-resolving set as stated in Lemma 6.2.

Proof: The proof directly follows from the Remark 6.2.

Corolary 6.4. If P_n is a path on $n \geq 2$ vertices, then $\beta_2(P_n) = \lfloor \frac{n}{2} \rfloor$.

Proof: Since P_n is a tree, it can be viewed as a rooted tree with center as a root. Then by Lemma 6.1, $S_1 = L(odd)$ and $S_2 = L(even)$ are the only two 2-resolving sets of P_n. We have the following two cases:

Case 1: If n is odd then there are exactly two vertices in each level from 1 to $e(c)$ and hence $|S_1| = \lfloor \frac{n}{2} \rfloor$ and $|S_2| = \lceil \frac{n}{2} \rceil$. Thus by Theorem 6.3, $\beta_2(P_n) = \lfloor \frac{n}{2} \rfloor$.

Case 2: If n is even then there are exactly two vertices in each level from 1 to $e(c) - 1$ and one vertex in the level $e(c)$. Hence $|S_1| = \frac{n}{2}$ and $|S_2| = \frac{n}{2}$. Thus by Theorem 6.3, $\beta_2(P_n) = \frac{n}{2}$. But in case of even integer, $\lfloor \frac{n}{2} \rfloor = \lceil \frac{n}{2} \rceil = \frac{n}{2}$. Hence $\beta_2(P_n) = \lfloor \frac{n}{2} \rfloor$

Corollary 6.5. For a cycle $C_n, n \geq 3$, $\beta_2(C_n) = \lceil \frac{n}{2} \rceil$.

Proof: Consider a cycle $C_n = \{v_1, v_2, \dots v_n, v_1\}$. Let G be a graph with $V(G) = V(C_n)$ and $E(G) = E(C_n) - \{v_1 v_n\}$. Then G is a path on n vertices and by corollary 6.4, $\beta_2(G) = \lfloor \frac{n}{2} \rfloor$. Now, add the edge $\{v_1 v_n\}$ to G to obtain C_n. We have two cases:

Case 1: If n is even, then the 2-metric basis of G (same as in corollary 6.4) will be the metric basis of C_n also and hence $\beta_2(C_n) = \lfloor \frac{n}{2} \rfloor = \lceil \frac{n}{2} \rceil$(since n is even) .

Case 2: If n is odd, then the 2-metric basis S of G (same as in corollary 6.4) will not be the metric basis for C_n since both $v_1, v_n \in V(C_n) - S$, a contradiction to lemma 3.5. Hence either $v_1 \in S$ or $v_n \in S$ and thus $\beta_2(C_n) = |S| + 1 = \lfloor \frac{n}{2} \rfloor + 1 = \lceil \frac{n}{2} \rceil$.

Hence the proof.

7. 2-metric dimension of unicyclic graphs

In this section, we establish certain bound for the unicyclic graphs using the result of 2-metric dimension of trees.

Theorem 7.1. If (T,c) be a tree of order at least 3 and e is an edge of $(\bar{T},c)$, then $\beta_2(T,c) \leq \beta_2(T+e,c) \leq \beta_2(T,c)+1$.

Proof: Let (T,c) be a tree and let S be a 2-metric basis of (T,c) as stated in Theorem 6.3. Let $x,y \in V$ be any two arbitrary vertices. Then there exists a unique path $P: x = x_1, x_2, \ldots x_{l-1}, x_l = y$ between x and y of length $l-1$ in (T,c). Now, add an edge e between the vertices x and y in (T,c) and let $C: x = x_1, x_2, \ldots x_{l-1}, x_l = y, x$ be the unique cycle of $(T,c)+e$. Then we consider the following three cases:

Case 1: If $x,y \in S$ in P then $l \geq 2$ and S satisfies definition 3.1 in C and it follows that the vertices of S constitute a 2-metric basis of $(T+e,c)$. Hence $\beta_2(T+e,c) = \beta_2(T,c)$.

Case 2: If $x \in S$ and $y \in V(G)-S$ in P then, in C, $x, x_3, \ldots, x_{l-1} \in S$(by Lemma 6.1) and $|d(x_{l-1},x_2) - d(x_{l-1},y)| \geq 2$. Hence $\beta_2(T,c) = \beta_2(T+e,c)$.

Case 3: If $x,y \in V(G)-S$ in P then, in C, $x_2, x_4, \ldots x_{l-1} \in S$. This contradicts Lemma 6.1 since x and y are adjacent in $(T+e,c)$. Thus $x \in S$ or $y \in S$ and $\beta_2(T+e,c) = \beta_2(T,c)+1$. This completes the proof.

Conflict of Interests

The authors declare that there is no conflict of interests.

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