Available online at http://scik.org
J. Math. Comput. Sci. 2 (2012), No. 3, 637-655
ISSN: 1927-5307

SOLUTION OF TWO POINT BOUNDARY VALUE PROBLEM BY NINE DEGREE SPLINE AND SUPERPOSITION METHODS

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Abstract. In this paper, we will try to solve a two point boundary value problem for linear ordinary differential equation by using ninth degree lacunary spline function of the type (0,1,3,5,7) and supperposition methods. Numerical illustrations with their tables are given to show the applicability and efficiency of our construction.

Keywords: Two point boundary value problem; Spline function; superposition method; Taylor's series, Maximum error; Convergence analysis.

2000 AMS Subject Classification: 34B05; 65L11

1. Introduction:

In the present paper, we will try to solve a two point boundary value problem for linear ordinary differential equation of the type:

$$y''(x) + f_1(x)y'(x) + f_2(x)y(x) = r(x),\ y(x_0) = y_0,\ y(x_n) = y_n \tag{1}$$

Where f_1(x) , f_2(x) and $r(x)$ are continuous functions of x by using ninth degree lacunary spline function of the type $(0, 1, 3, 5, 7)$, and we have concentrated our trial on discussing

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Received February 9, 2012

theory of existence and uniqueness of spline function of degree nine, and also we have studied convergence and error bounds. Our method for solving boundary value problem (1) is depended on converting the given linear ordinary differential equation (1) from boundary value to two initial value problems, and this method known as the method of superposition. Finally we have given an example to compare the convergence of defined lacunary spline function to the solution of the given two point boundary value problem. Many authors studied the solution of boundary value problem but for different methods and different types of lacunary splines, see [1-4].

2. Construction of the Spline Function:

We consider an interval $[0,1]$ and subdivided by a mesh of $(n+1)$ points $\{x_i\}$ into n equal parts defined by: $x_i = x_0 + ih$, where $h = \dfrac{1}{n}$ for i=0,1,2,...,n ,and $0 = x_0 < x_1 < x_2 < ... < x_n = 1$, and h is the length of each subintervals. Let $y = f(x)$ be a smooth function defined on $[0,1]$ and $y = f(x) \in C^{n-1}[0,1]$, $n \geq 2$. We define a nine degree spline interpolation $S_i(x)$ for one dimensional on the interval $[x_i, x_{i+1}]$ for i=0,1,2,...,n-1 as:

(2)

$$\begin{aligned} S_i(x) &= y(x_i) + (x-x_i)y'(x_i) + (x-x_i)^2 a_{i,2} + \frac{(x-x_i)^3}{3!}y'''(x_i) + (x-x_i)^4 a_{i,4} \\ &+ \frac{(x-x_i)^5}{5!}y^{(5)}x_i + (x-x_i)^6 a_{i,6} + \frac{(x-x_i)^7}{7!}y^{(7)}x_i + (x-x_i)^8 a_{i,8} + (x-x_i)^9 a_{i,9} \end{aligned}$$

where $a_{i,j}$, $j =$ 2,4,6,8,9 are knowns to be determinded

3. Existence and Uniqueness of the Spline Function:

In this section we present the existence and uniqueness of our defined lacunary spline function of degree nine (2) which is used to solve the given boundary value problem (1) and also we give the conditions that guarantee the existence and uniqueness of the given lacunary spline function.

Theorem 1: Given the real numbers $y(x_i)$,$y'(x_i)$,$y^{(3)}(x_i)$,$y^{(5)}(x_i)$ and $y^{(7)}(x_i)$ for i=0,1,2,...,n,there exists a unique spline of degree nine as given in the equations (2) such that:

$S(x_i) = y(x_i)$

$$S^{(r)}(x_i) = y^{(r)}(x_i), for\ r = 1, 3, 5, 7\ and\ i = 0, 1, ..., n. \tag{3}$$

The spline function S(x) is defined as: $S(x) = S_i(x)$, where $x \in [x_i, x_{i+1}]$ for i=0, 1,2, ..., n-1. Where the coefficients of this polynomial are to be determined by the following conditions:

(4)

$$S_i(x_{i+1}) = S_{i+1}(x_{i+1}) = y(x_{i+1}), S_i^{(r)}(x_{i+1}) = S_{i+1}^{(r)}(x_{i+1}) = y^{(r)}(x_{i+1}), for\ r = 1, 3, 5, 7.$$

Proof: We try to find uniquely the coefficients $a_{i,j}$, j=2, 4, 6, 8, 9 of $S_i(x)$ for i=1,2...,n-1 which is defined in equation (2). From equation (2) we have:

$$\begin{aligned} S_i(x) &= y(x_i) + (x - x_i)y'(x_i) + (x - x_i)^2 a_{i,2} + \frac{(x - x_i)^3}{3!} y'''(x_i) + (x - x_i)^4 a_{i,4} \\ &+ \frac{(x - x_i)^5}{5!} y^{(5)}(x_i) + (x - x_i)^6 a_{i,6} + \frac{(x - x_i)^7}{7!} y^{(7)}(x_i) + (x - x_i)^8 a_{i,8} + (x - x_i)^9 a_{i,9} \end{aligned}$$

And the first, third, fifth, and seventh derivatives of $S_i(x)$ can be taken easily. From the conditions of equation (4) we get the following equations:

$$h^2 a_{i,2} + h^4 a_{i,4} + h^6 a_{i,6} + h^8 a_{i,8} + h^9 a_{i,9} = y_{i+1} - y_i - hy_i' - \frac{h^3}{6} y_i^{(3)} - \frac{h^5}{120} y_i^{(5)} - \frac{h^7}{5040} y_i^{(7)}, \tag{5}$$

$$2ha_{i,2} + 4h^3 a_{i,4} + 6h^5 a_{i,6} + 8h^7 a_{i,8} + 9h^8 a_{i,9} = y_{i+1}' - y_i' - \frac{h^2}{2} y_i^{(3)} - \frac{h^4}{24} y_i^{(5)} - \frac{h^6}{720} y_i^{(7)}, \tag{6}$$

$$24ha_{i,4} + 120h^3 a_{i,6} + 336h^5 a_{i,8} + 504h^6 a_{i,9} = y_{i+1}^{(3)} - y_i^{(3)} - \frac{h^2}{2} y_i^{(5)} - \frac{h^4}{24} y_i^{(7)}, \tag{7}$$

$$720ha_{i,6} + 6720h^3 a_{i,8} + 15120h^4 a_{i,9} = y_{i+1}^{(5)} - y_i^{(5)} - \frac{h^2}{2} y_i^{(7)}, \tag{8}$$

and

$$40320ha_{i,8} + 181440h^2 a_{i,9} = y_{i+1}^{(7)} - y_i^{(7)}. \tag{9}$$

The equations (5)-(9) forms a system and the coefficient matrix of this system in the unknowns ai,j , j=2, 4, 6, 8, 9,i=0, 1, 2, ..., n-1 is non singular matrix and hence the coefficients $a_{i,j}$, j=2, 4, 6, 8, 9,i=0, 1, 2, ..., n-1 are uniquely determined. By solving the equations (5)-(9) we can specify uniquely the equation of the coefficients $a_{i,j}$, for j=2, 4, 6, 8, 9, i=0, 1, 2, ..., n-1 as follows:

$$\begin{aligned} a_{i,2} &= \frac{153}{62h^2}[y_{i+1}-y_i] - \frac{1}{124h}[91y'_{i+1}+215y'_i] + \frac{h}{1488}[29y^{(3)}_{i+1} - 95y^{(3)}_i] \\ &- \frac{5h^3}{8928}[y^{(5)}_{i+1} - \frac{37}{25}y^{(3)}_i] + \frac{19h^{(5)}}{1071360}[\frac{115}{133}y^{(7)}_{i+1} - y^{(7)}_i], \end{aligned} \tag{10}$$

$$\begin{aligned} a_{i,4} &= -\frac{63}{31h^4}[y_{i+1}-y_i] + \frac{63}{62h^3}[y'_{i+1}+y'_i] - \frac{4}{93h}[y^3_{i+1} \\ &+ \frac{47}{16}y^{(3)}_i] + \frac{17h}{11160}[y^{(5)}_{i+1} - \frac{121}{34}y^{(5)}_i] - \frac{5h^{(3)}}{107136}[y^{(7)}_{i+1} - \frac{37}{25}y^{(7)}_i], \end{aligned} \tag{11}$$

$$\begin{aligned} a_{i,6} &= \frac{21}{31h^6}[y_{i+1}-y_i] - \frac{21}{62h^5}[y'_{i+1}+y'_i] + \frac{7}{248h^3}[y^{(3)}_{i+1} + y^{(3)}_i] \\ &- \frac{2}{1395h}[y^{(5)}_{i+1} + \frac{47}{16}y^{(5)}_i] + \frac{19h}{107136}[\frac{29}{95}y^{(7)}_{i+1} - y^{(7)}_i], \end{aligned} \tag{12}$$

$$\begin{aligned} a_{i,8} &= -\frac{9}{62h^8}[y_{i+1}-y_i] + \frac{9}{124h^7}[y'_{i+1}+y'_i] - \frac{3}{496h^5}[y^{(3)}_{i+1} + y^{(3)}_i] \\ &+ \frac{3}{4960h^3}[y^{(5)}_{i+1} + y^{(5)}_i] - \frac{13}{357120h}[y^{(7)}_{i+1} + \frac{215}{91}y^{(7)}_i], \end{aligned} \tag{13}$$

and

$$\begin{aligned} a_{i,9} &= \frac{1}{31h^9}[y_{i+1}-y_i] - \frac{1}{62h^8}[y'_{i+1}+y'_i] + \frac{1}{744h^6}[y^{(3)}_{i+1} + y^{(3)}_i] \\ &- \frac{1}{7440h^4}[y^{(5)}_{i+1} + y^{(5)}_i] + \frac{17}{1249920h^2}[y^{(7)}_{i+1} + y^{(7)}_i], \end{aligned} \tag{14}$$

Hence the theorm is completed.

4. Reduction of Linear Boundary Value Problems to Initial Value Problems:

Method of Superposition[5]: The techniques based on transforming linear ordinary differential equations from boundary value to initial value problems. For linear ordinary differential equations, it is in general possible to reduce the boundary value problem to two or more initial value problems. Using one of the Taylor's method, shooting method, or Runge-Kutta method, the initial value problems can be solved. Combining these solutions then gives the solution of the original boundary value problem. Here we use Taylor's approximations method to the initial value problems. Now, below we explain the technique of the method. Consider a second order linear ordinary differential equation:

$$y''(x) + f_1(x)y'(x) + f_2(x)y(x) = r(x) \tag{15}$$

Subject to the boundary conditions:

$$y(x_0) = y_0,\ y(x_n) = y_n \tag{16}$$

Where $f_1(x),\ f_2(x)$ and r(x) are continuous functions of x, the continuity of $f_1(x),\ f_2(x)$ and r(x) assures the existence and uniqueness of the solution of equation (15).
To transform equation (15) and (16) into an initial value problem, we assume:

$$y(x) = y_1(x) + \mu\, y_2(x) \tag{17}$$

Where μ is a constant to be determined. By substituting $y(x)$ from equation (17) in equation (15), we get:

$$[y_1(x) + \mu\, y_2(x)]'' + f_1(x)[y_1(x) + \mu\, y_2(x)]' + f_2(x)[y_1(x) + \mu\, y_2(x)] = r(x)$$

or

$$[y_1''(x) + f_1(x)y_1'(x) + f_2(x)y_1(x) - r(x)] + \mu\, [y_2''(x) + f_1(x)y_2' + f_2(x)y_2(x)] = 0 \tag{18}$$

From equation (18) we obtain:

$$y_1''(x) + f_1(x)y_1'(x) + f_2(x)y_1(x) = r(x) \tag{19}$$

and

$$y_2''(x) + f_1(x)y_2'(x) + f_2(x)y_2(x) = 0 \tag{20}$$

The first boundary condition in equation (16) is next transformed to:

$$y(x_0) = y_1(x_0) + \mu\, y_2(x_0) \; or \; y_0 = y_1(x_0) + \mu\, y_2(x_0)$$

From which:

$$y_1(x_0) = y_0 \tag{21}$$

and

$$y_2(x_0) = 0 \tag{22}$$

Differentiating equation (17) and setting x equal to x_0, we get:

$$y'(x_0) = y_1'(x_0) + \mu\, y_2'(x_0) \tag{23}$$

If the two unknown boundary conditions are set equal to:

$$y_1'(x_0) = 0 \; and \; y_2'(x_0) = 1 \tag{24}$$

Then equation (23) gives:

$$y'(x_0) = \mu \tag{25}$$

Thus the unknown constant μ is identified as the missing initial slope. As a final step, the boundary condition at the second point is transformed to:

$$y(x_n) = y_1(x_n) + \mu\, y_2(x_n) \; or \; y_n = y_1(x_n) + \mu y_2(x_n)$$

From which:

$$\mu = \frac{y_n - y_1(x_n)}{y_2(x_n)} \tag{26}$$

Thus the solution of equation (15) consists of the following steps:

(1) We solve equation (19) with the initial conditions given by equations (21) and (24) from $x = x_0$ to $x = x_n$, the value of $y_1(x_n)$ is obtained.

(2)We solve equation (20) with the initial conditions given by equations (22) and (24)

from $x = x_0$ to $x = x_n$, the value of $y_2(x_n)$ is obtained.
(3)From equation (26) we calculate μ , which according to equation (25) the missing initial slope is.
(4)The solution of the original differential equation (15) can be calculated from equation (17).

5. Convergence and Error Bounds:

In this section we show a criterion for convergence of the given initial value problems (18) - (19) and the solution of the given boundary value problems (1), and the convergence for non interpolating points of the second and fourth derivatives of the given lacunary spline functions.

Theorem 2: Let $y_1(x)$ be a solution of the initial value problem (19) which is found by Taylor's series expansion formula where $y_1(x) \in C^9[0,1]$ and $y_1^{(r)}(x_i)$ be given for r=0, 1, 3, 5, 7 and i=0, 1, 2, ..., n. Let $S_i(x)$ be a unique spline function of degree nine which is defined by equation (2). Then for $x \in [x_i, x_{i+1}]$, i=0, 1, 2, ..., n-1

$$||S_i(x) - y_1(x)||_\infty \leq \frac{23h^9}{98247} W_9(h)$$

Where $W_9(h)$ is the module of continuity which is defined by:

$$W_9(h) = \max\{|y^{(9)}(s) - y^{(9}(t)| : |s-t| \leq h \forall s,t \in [0,1]\}, \; and \; ||f(x)||_\infty = \max\{|f(x)| : \forall x \in [0,1]\}.$$

Proof: Let x be any point in the interval $[x_i, x_{i+1}]$ where i=0, 1, 2, , n-1.
We try to find $||S_i(x) - y_1(x)||_\infty$.
Using Taylor's series expansion about $x = x_i$ for $y_{i+1}^{(r)}$, r=0, 1, 3, 5, 7 of the equations

(10)-(14) and after some simple algebraic operations consequently we get:

$$a_{i,2} = \frac{1}{2}y_i'' + \frac{17h^7}{2499840}y^{(9)}(\theta_0) - \frac{(13h^7)}{714240}y^{(9)}(\theta_1) + \frac{(29h^7)}{1071360}y^{(9)}(\theta_2)$$
$$- \frac{(5h^7)}{214272}y^{(9)}(\theta_3) + \frac{(23h^7)}{2999808}y^{(9)}(\theta_4).$$

where $x_i < \theta_0, \theta_1, \theta_2, \theta_3, \theta_4 < x_{i+1}$,

$$a_{i,4} = \frac{1}{24}y_i^{(4)} - \frac{h^5}{178560}y^{(9)}(\alpha_1) + \frac{(h^5)}{39680}y^{(9)}(\alpha_2) - \frac{(h^5)}{16740}y^{(9)}(\alpha_3)$$
$$+ \frac{(17h^5)}{267840}y^{(9)}(\alpha_4) - \frac{(5h^5)}{214272}y^{(9)}(\alpha_5).$$

where $x_i < \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5 < x_{i+1}$,

$$a_{i,6} = \frac{1}{720}y_i^{(6)} + \frac{h^3}{535680}y^{(9)}(\gamma_1) - \frac{(h^3)}{119040}y^{(9)}(\gamma_2)$$
$$+ \frac{(7h^3)}{178560}y^{(9)}(\gamma_3) - \frac{h^3}{16740}y^{(9)}(\gamma_4) + \frac{29h^3}{1071360}y^{(9)}(\gamma_5).$$

where $x_i < \gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5 < x_{i+1}$,

$$a_{i,8} = \frac{1}{40320}y_i^{(8)} - \frac{h}{2499840}y^{(9)}(\delta_1) + \frac{h}{555520}y^{(9)}(\delta_2)$$
$$- \frac{(h)}{119040}y^{(9)}(\delta_3) + \frac{h}{39680}y^{(9)}(\delta_4) - \frac{13h}{714240}y^{(9)}(\delta_5).$$

where $x_i < \delta_1, \delta_2, \delta_3, \delta_4, \delta_5 < x_{i+1}$,

$$a_{i,9} = \frac{1}{11249280}y^{(9)}(\beta_0) - \frac{1}{2499840}y^{(9)}(\beta_1) + \frac{1}{535680}y^{(9)}(\beta_2)$$
$$- \frac{(1)}{178560}y^{(9)}(\beta_3) + \frac{17}{2499840}y^{(9)}(\beta_4).$$

where $x_i < \beta_0, \beta_1, \beta_2, \beta_3, \beta_4 < x_{i+1}$, for i = 0,1,... , n-1

And also Taylor's expansion formula about $x = x_i$ for $y = y_1$, where $y = y_1 \in C^9[0,1]$ is of the form:

$$\begin{aligned} y_1(x) &= y_1(x_i) + (x-x_i)y_1'(x_i) + \frac{(x-x_i)^2}{2}y_1''(x_i) + \frac{(x-x_i)^3}{6}y_1'''(x_i) \\ &+ \frac{(x-x_i)^4}{24}y_1^{(4)}(x_i) + \frac{(x-x_i)^5}{120}y_1^{(5)}(x_i) + \frac{(x-x_i)^6}{720}y_1^{(6)}(x_i) \\ &+ \frac{(x-x_i)^7}{5040}y_1^{(7)}(x_i) + \frac{(x-x_i)^8}{40320}y_1^{(8)}(x_i) + \frac{(x-x_i)^9}{362880}y_1^{(9)}(\rho). \end{aligned}$$

where $x_i < \rho < x_{i+1}$. Substituting equations $a_{i,2}$, $a_{i,4}$,$a_{i,6}$,$a_{i,8}$ and $a_{i,9}$ in equation (2) and taking the absolute value of $S_i(x) - y_1(x)$, we deduce:

$$\begin{aligned} |S_i(x) - y_1(x)| &\leqslant \frac{(89h^9)}{2142720}|y^{(9)}(\theta_i) - y^{(9)}(\theta_j)| \\ &+ \frac{(19h^9)}{214272}|y^{(9)}(\alpha_m) - y^{(9)}(\alpha_n)| + \frac{(73h^9)}{1071360}|y^{(9)}(\gamma_k) - y^{(9)}(\gamma_l)| \\ &+ \frac{(3h^9)}{111104}|y^{(9)}(\delta_k) - y^{(9)}(\delta_q)| + \frac{(197h^9)}{22498560}|y^{(9)}(\beta_s) - y^{(9)}(\beta_t)|, \end{aligned}$$

$\forall x \in [x_i, x_{i+1}]$,where $x_i < \theta_i, \theta_j, \alpha_m, \alpha_n, \gamma_k, \gamma_l, \delta_k, \delta_q, \beta_s, \beta_t < x_{i+1}$

or

$$||S_i(x) - y_i(x)|| \leqslant \frac{23h^9}{98247}W_9(h).$$

Thus the proof is completed.

Theorem 3: Let $y_2(x)$ be a solution of the initial value problem (20) which is found by Taylor's series expansion formula where $y_2(x) \in C^9[0,1]$ and $y_2^{(r)}(x_i)$ be given for r=0,1,3,5,7 and i=0,1,2,...,n. Let $S_i(x)$ be a unique spline function of degree nine which is defined by equation (2). Then for $x \in [x_i, x_{i+1}]$,i=0,1,2,...,n-1

$$||S_i(x) - y_2(x)||_\infty \leqslant \frac{23h^9}{98247}W_9(h)$$

Proof: The result is obtained by the same way as in the proof of the theorem 2.

Theorem 4: Let $y(x)$ be a solution of the boundary value problem (15) defined by

equation (17) and $S_i(x)$ be a unique spline function of degree nine which is defined by equation (2) and is a solution of the boundary value problem (15), then: $||S_i(x)-y(x)||_\infty \leqslant \frac{1}{1-|\mu|}\frac{23h^9}{98247}W_9(h),\ |\mu| \neq 1,$
where μ is a constant and it's determined by equation (26).

Proof: From equation (17) we have: $y(x) = y_1(x) + \mu\ y_2(x)$, where x is any point in the interval [0, 1].

$$\begin{aligned}
||S_i(x) - y(x)||_\infty &= ||S_i(x) - (y_1(x) + \mu y_2(x))||_\infty \\
&= ||S_i(x) - y_1(x) + (-\mu)y_2(x))||_\infty \\
&\leq ||S_i(x) - y_1(x)||_\infty + ||(-\mu)y_2(x))||_\infty \\
&\leq \frac{23h^9}{98247}W_9(h) + |-\mu|||y_2(x)||_\infty \\
\therefore ||S_i(x) - y(x)||_\infty &\leq \frac{23h^9}{98247}W_9(h) + |\mu|||y_2(x)||_\infty
\end{aligned}$$

Since $S_i(x)$ and $y(x)$ are solutions of the equations (15), so

$$\begin{aligned}
S_i''(x) + f_1(x)S_i'(x) + f_2(x)S_i(x) &= r(x),\ \textit{and} \\
y''(x) + f_1(x)y'(x) + f_2(x)y(x) &= r(x)
\end{aligned}$$

From which

$$S_i''(x) + f_1(x)S_i'(x) + f_2(x)S_i(x) = y''(x) + f_1(x)y'(x) + f_2(x)y(x)$$

or

$$S_i''(x) - y''(x) + f_1(x)(S_i'(x) - y'(x)) + f_2(x)(S_i(x) - y(x)) = 0$$

or

$$(S_i(x) - y(x))'' + f_1(x)(S_i(x) - y(x))' + f_2(x)((S_i(x) - y(x)) = 0 \tag{27}$$

This means that $S_i(x)-y(x)$ is a solution of equation (20), but $y_2(x)$ is also a solution of equation (20), so

$$y_2''(x)+f_1(x)y_2'(x)+f_2(x)y_2(x)=0 \tag{28}$$

From equations (27) and (28) we get: $S_i(x)-y(x)=y_2(x)$, hence

$$\begin{aligned}
||S_i(x)-y(x)||_\infty &\leq \frac{23h^9}{98247}W_9(h)+|\mu|||y_2(x)||_\infty \\
&\leq \frac{23h^9}{98247}W_9(h)++|\mu|||S_i(x)-y(x)||_\infty \\
(1-|\mu|)||S_i(x)-y(x)||_\infty &\leq \frac{23h^9}{98247}W_9(h)
\end{aligned}$$

or

$$||S_i(x)-y(x)||_\infty \leq \frac{1}{(1-|\mu|)}\frac{23h^9}{98247}W_9(h)$$

Thus the proof is finished

Theorem 5: Let $y_1(x)$ be a solution of the initial value problem (19) which is found by Taylor's series expansion formula where $y_1(x)\in C^9[0,1]$ and $y_1^{(r)}(x_i)$ be given for r=0,1,3,5,7 and i=0,1,2,...,n. Let $S_i(x)$ be a unique spline function of degree nine which is defined by equation (2), then for $x\in[x_i,x_{i+1}]$,i=0,1,2, ... ,n-1

$||S_i''(x)-y_1''(x)||_\infty \leqslant \dfrac{60h^7}{11249}W_9(h)$,

and

$||S_i^{(4)}(x)-y_1^{(4)}(x)||_\infty \leqslant \dfrac{4397h^5}{44640}W_9(h)$,

where $W_9(h)$ is the module of continuity which is defined by:

$W_9(h)=\max\{|y^{(9)}(s)-y^{(9)}(t)|:|s-t|\leq h\ \forall s,t\in[0,1]\}$, and

$||f(x)||_\infty = \max\{|f(x)| : \forall x \in [0,1]\}$.

Prrof: Let x be any point in the interval $[x_i, x_{i+1}]$ where i=0,1,2,...,n-1. First we try to find $\|S_i''(x) - y_1''(x)\|_\infty$.

$$\begin{aligned} S_i''(x) &= 2a_{i,2} + (x-x_i)y_1^{(3)}(x_i) + 12(x-x_i)^2 a_{i,4} + \frac{(x-x_i)^3}{6}y_1^{(5)}(x_i) \\ &+ 30(x-x_i)^4 a_{i,6} + \frac{(x-x_i)^5}{120}y_1^{(7)}(x_i) + 56(x-x_i)^6 a_{i,8} + 72(x-x_i)^7 a_{i,9}. \end{aligned}$$

Using Taylor's series expansion about $x = x_i$ for $y_{i+1}^{(r)}$,r=0,1,3,5,7 of the equations (10) - (14) and substituting these equations in equation $S_i''(x)$ after simplifying we obtain:

$$\begin{aligned} S_i''(x) &= y_1''(x_i) + \frac{17h^7}{1249920}y_1^{(9)}(\theta_0) - \frac{13h^7}{357120}y_1^{(9)}(\theta_1) + \frac{29h^7}{535680}y_1^{(9)}(\theta_2) \\ &- \frac{5h^7}{107136}y_1^{(9)}(\theta_3) + \frac{23h^7}{1499904}y_1^{(9)}(\theta_4) + (x-x_i)y_1^{(3)}(x_i) + (x-x_i)^2\{\frac{1}{2}y_1^{(4)}(x_i) - \frac{h^5}{14880}y_1^{(9)}(\phi_1) \\ &+ \frac{3h^5}{9920}y_1^{(9)}(\phi_2) - \frac{h^5}{1395}y_1^{(9)}(\phi_3) + \frac{17h^5}{22320}y_1^{(9)}(\phi_4) - \frac{5h^5}{17856}y_1^{(9)}(\phi_5)\} + \frac{(x-x_i)^3}{6}y_1^{(5)}(x_i) \\ &+ (x-x_i)^4\{\frac{1}{24}y_1^{(6)}(x_i) + \frac{h^3}{17856}y_1^{(9)}(\gamma_1) - \frac{h^3}{3968}y_1^{(9)}(\gamma_2) + \frac{7h^3}{5952}y_1^{(9)}(\gamma_3) - \frac{h^3}{558}y_1^{(9)}(\gamma_4) \\ &+ \frac{29h^3}{35712}y_1^{(9)}(\gamma_5)\} + \frac{(x-x_i)^5}{120}y_1^{(7)}(x_i) + (x-x_i)^6\{\frac{1}{720}y_1^{(8)}(x_i) - \frac{h}{44640}y_1^{(9)}(\delta_1) + \frac{h}{9920}y_1^{(9)}(\delta_2) \\ &- \frac{7h}{14880}y_1^{(9)}(\delta_3) + \frac{7h}{4060}y_1^{(9)}(\delta_4) - \frac{91h}{89280}y_1^{(9)}(\delta_5)\} + (x-x_i)^7\{\frac{1}{156240}y_1^{(9)}(\sigma_0) - \frac{1}{34720}y_1^{(9)}(\sigma_1) \\ &+ \frac{1}{7440}y_1^{(9)}(\sigma_2) - \frac{1}{2480}y_1^{(9)}(\sigma_3) + \frac{1}{34720}y_1^{(9)}(\sigma_4)\}. \end{aligned}$$

where $x_i < \theta_k, \phi_k, \gamma_k, \delta_k, \sigma_k < x_{i+1}$, for k=0,1,2,3,4,5.

Again using Taylor's expansion formula on $y_1''(x_i)$ about $x = x_i$, and taking the absolute value of $(S_i^{(2)}(x) - y_1''(x_i))$ after simplifying and removing the similar terms, we get:

$$\begin{aligned} |S_i''(x) - y_1(x_i)| &\leq \frac{(89h^7)}{1071360}W_9(h) + \frac{(19h^7)}{17856}W_9(h) + \frac{73}{35712}W_9(h) \\ &+ \frac{(3h^7)}{1984}W_9(h) + \frac{(197h^7)}{312480}W_9(h), \forall x \in [x_i, x_{i+1}], \end{aligned}$$

or

$$||S_i''(x) - y_1''(x)||_\infty \leq \frac{60h^7}{11249}W_9(h).$$

It remains to find $||S_i^{(4)}(x) - y_1^{(4)}(x)||_\infty$.

$$\begin{aligned} S_i^{(4)}(x) &= 24a_{i,4} + (x-x_i)y_1^{(5)}(x_i) + 362(x-x_i)^2 a_{i,6} + \frac{(x-x_i)^3}{6}y_1^{(7)}(x_i) \\ &+ 1680(x-x_i)^4 a_{i,8} + 3024(x-x_i)^5 a_{i,9} \end{aligned}$$

Again in similar method, using Taylor's series expansion about $x = x_i$ for $y_{i+1}^{(r)}$, r=0,1,3,4,5,7 of the equations (11) - (14) and substituting these equations in equation $S_i^{(4)}(x)$ and taking the absolute value of $S_i^{(4)}(x) - y_1^{(4)}(x)$, after simplifying we get:

$$|S_i^{(4)}(x) - y_1^{(4)}(x)| \leq \frac{19h^5}{8928}W_9(h) + \frac{73h^5}{2976}W_9(h) + \frac{45h^5}{992}W_9(h) + \frac{197h^5}{7440}W_9(h)\ \forall x \in [x_i, x_{i+1}],$$

or

$$||S_i^{(4)}(x) - y_1^{(4)}(x)||_\infty \leq \frac{4397h^5}{44640}W_9(h)$$

Hence the proof is completed.

Theorem 6: Let $y_2(x)$ be a solution of the initial value problem (20) which is found by Taylor's series expansion formula where $y_2(x) \in C^9[0,1]$ and $y_2^{(r)}(x_i)$ be given for r=0,1,3,5,7 and i=0,1,2,...,n. Let $S_i(x)$ be a unique spline function of degree nine which is defined by equation (2), then for $x \in [x_i, x_{i+1}]$,i=0,1,2, ... ,n-1

$$||S_i^{(2)}(x) - y_2^{(2)}(x)||_\infty \leqslant \frac{60h^7}{11249}W_9(h),$$

and

$$||S_i^{(4)}(x) - y_2^{(4)}(x)||_\infty \leqslant \frac{4397h^5}{44640}W_9(h).$$

Proof: The proof is obtained by the same way as in the proof of the theorem 5.

Theorem 7: Let $y(x) \in C^9[0,1]$ be a solution of the boundary value problem (15) defined by equation (17) and $S_i(x)$ be a unique spline function of degree nine which is defined by equation (2) and is a solution of the boundary value problem (15), then for $x \in [x_i, x_{i+1}]$ i = 0,1,2,...,n-1:
$||S_i''(x) - y''(x)||_\infty \leqslant \frac{1}{1-|\mu|}\frac{60h^7}{11249}W_9(h).$
and
$||S_i^{(4)}(x) - y^{(4)}(x)||_\infty \leqslant \frac{1}{1-|\mu|}\frac{4397h^5}{44640}W_9(h).$
where μ is a constant and it's determined by equation (26), $|\mu| \neq 1$,

Proof: Let x be any point in the interval $[x_i, x_{i+1}]$ where i=0,1,2...,n-1. First we begin by $||S_i''(x) - y''(x)||_\infty$. Differentiating equation (17) twice with respect to x yields:
$y'(x) = y_1''(x) + \mu\, y_2''(x)$

$$\begin{aligned} ||S_i''(x) - y''(x)||_\infty &= ||S_i''(x) - (y_1''(x) + \mu y_2''(x))||_\infty \\ &= ||S_i''(x) - y_1''(x) + (-\mu) y_2''(x))||_\infty \\ &\leq ||S_i''(x) - y_1''(x)||_\infty + ||(-\mu) y_2''(x))||_\infty \\ &\leq \frac{60h^7}{11249}W_9(h) + |-\mu| ||y_2''(x)||_\infty \; \{using theorm 4\} \end{aligned}$$

$$\therefore ||S_i''(x) - y''(x)||_\infty \leq \frac{60h^7}{11249}W_9(h) + |\mu| ||y_2(x)||_\infty .$$

Since $S_i(x)$ and $y_(x)$ are solutions of the equations (15), so

$$S_i''(x) + f_1(x)S_i'(x) + f_2(x)S_i(x) = r(x),$$

And

$$y''(x) + f_1(x)y'(x) + f_2(x)y(x) = r(x),$$

From which

$$(S_i(x) - y(x))'' + f_1(x)(S_i(x) - y(x))' + f_2(x)(S_i(x) - y(x)) = 0 \tag{29}$$

This means that $S_i(x) - y(x)$ is a solution of equation (20), but $y_2(x)$ is also a solution of equation (20), so

$$y_2''(x) + f_1(x)y_2'(x) + f_2(x)y_2(x) = r(x), \tag{30}$$

From equations (29) and (30) we deduce: $S_i''(x) - y''(x) = y_2''(x)$,

then

$\|S_i''(x) - y''(x)\|_\infty \leq \dfrac{(60h^7)}{11249} W_9(h) + |\mu| \|S_i''(x) - y''(x)\|_\infty$

or

$\|S_i''(x) - y''(x)\|_\infty \leq \dfrac{1}{1-|\mu|} \dfrac{(60h^7)}{11249} W_9(h)$

We can prove the second part of theorem by the same way.

Example (1): We consider a boundary value problem:

$y''(x) + y(x) + 1 = 0$ where $x \in [0, 1]$ and $y(0) = 0,\ y(1) = 0$.

Solution: $y''(x)+y(x)+1 = 0 \Rightarrow y''(x)+y(x) = -1$, comparing this equation with equation (15) we get: $f_1(x) = 0$,$f_2(x) = 1$ and $r(x) = -1$. Now, we transform the given boundary problems to two initial value problems respectively as follows: $y_1''(x) + y_1(x) = -1$, $y_1(0) = 0$, $y_1'(0) = 0$, and $y_2''(x) + y_2(x) = -1$, $y_2(0) = 0$, $y_2'(0) = 1$.

If we take $h = 0.1$, then $n = 10$. We solve the two initial value problems by using Taylor's series approximation, and then the general solution y(x)of the given boundary value problems computed by the equation (17) and μ is calculated by equation (26). The values of $y(x)$, $S_i(x)$ and the maximum absolute error for $S_i^{(r)}(x) - y^{(r)}(x)$,r = 0,2,4,at the various points of x in the interval [0, 1] are tabulated below in tables (1) and (2) respectively.

TABLE 1. The values of $y(x), S_i(x)$.

x	$y(x)$	$S_i(x)$
0	0	0
0.1	0.04954340936180	0.04954340936180
0.2	0.08860012791019	0.08860012791018
0.3	0.11677991382384	0.11677991382384
0.4	0.13380120399694	0.13380120399694
0.5	0.13949392732454	0.13949392732455
0.6	0.13380120399694	0.13380120399693
0.7	0.11677991382384	0.11677991382384
0.8	0.08860012791018	0.08860012791019
0.9	0.04954340936180	0.04954340936180
1	0	0

TABLE 2. maximum absolute error for $S_i^{(r)}(x) - y^{(r)}(x)$, r = 0,2,4.

x	$\|S_i(x) - y(x)\|_\infty$	$\|S_i''(x) - y''(x)\|_\infty$
0	0	0
0.1	$1.554312234475219 * 10^{-17}$	$4.99600361081320 * 10^{-14}$
0.2	$2.77555756156289 * 10^{-17}$	$4.44089209850063 * 10^{-14}$
0.3	0	$3.77475828372553 * 10^{-15}$
0.4	$5.5511151232578 * 10^{-17}$	$1.59872115546023 * 10^{-14}$
0.5	$2.7555756156289 * 10^{-17}$	$1.57651669496772 * 10^{-14}$
0.6	0	$1.33226762955019 * 10^{-15}$
0.7	$1.38777878078145 * 10^{-17}$	$5.15143483426073 * 10^{-14}$
0.8	$1.38777878078145 * 10^{-17}$	$2.28705943072782 * 10^{-14}$
0.9	$6.93889390390723 * 10^{-18}$	$3.46389583683049 * 10^{-14}$
1	$3.39320042888803 * 10^{-17}$	$4.57411886145564 * 10^{-14}$

x	$\|S_i^{(4)}(x) - y^{(4)}(x)\|_\infty$
0	$3.040012686028604 * 10^{-12}$
0.1	$6.68689548177781 * 10^{-11}$
0.2	$6.77604639065521 * 10^{-11}$
0.3	$1.26987309556625 * 10^{-11}$
0.4	$1.26632038188745 * 10^{-11}$
0.5	$2.68594035901515 * 10^{-11}$
0.6	$1.03070885160150 * 10^{-11}$
0.7	$2.86177748165528 * 10^{-11}$
0.8	$3.42061934333060 * 10^{-11}$
0.9	$3.48086004464676 * 10^{-11}$
1	$2.40609754342813 * 10^{-11}$

Example (2): Consider the boundary value problem: $y''(x) - y(x) = 2x$ where $x \in [0,1]$ and $y(0) = 0$,$y(1) = 1$.

Sloution:

$y''(x) - y(x) = 2x$, Comparing this equation with equation (15) we get: $f_1(x) = 0$, $f_2(x) = -1$,and $r(x) = 2x$.

Now, we transform the given boundary problems to two initial value problems respectively as follows: $y_1''(x) - y_1(x) = 2x$, $y_1(0) = 0$,$y_1'(0) = 0$, and $y_2''(x) - y_2(x) = 0$, $y_2(0) = 0$, $y_2'(0) = 1$,and the general solution y(x) of the given boundary value problems computed by the equation (17). If we take $h = 0.1$, then $n = 10$. The values of $y(x)$, $S_i(x)$ and the maximum absolute error for $S_i^{(r)}(x) - y^{(r)}(x)$, $r = 0, 2, 4$, at the various points of x in the interval [0, 1] are tabulated below in tables (3) and (4) respectively:

TABLE 3. The values of $y(x), S_i(x)$.

x	$y(x)$	$S_i(x)$
0	0	0
0.1	0.08110524286717	0.08110524286717
0.2	0.16402304787652	0.16402304787652
0.3	0.24958328429024	0.24958328429024
0.4	0.33864226771213	0.33864226771213
0.5	0.43209133038205	0.43209133038205
0.6	0.53086574192849	0.53086574192849
0.7	0.63595406986146	0.63595406986146
0.8	0.74840807348831	0.74840807348831
0.9	0.86935323027442	0.86935323027442
1	1	1

TABLE 4. The maximum absolute error for $S_i^{(r)}(x) - y^{(r)}(x)$, r = 0,2,4.

x	$\|S_i(x) - y(x)\|_\infty$	$\|S_i''(x) - y''(x)\|_\infty$
0	0	0
0.1	$2.77555756156289 * 10^{-17}$	$5.46784839627890 * 10^{-14}$
0.2	$1.11022302462516 * 10^{-16}$	$9.27036225562006 * 10^{-14}$
0.3	0	$6.68354260824344 * 10^{-15}$
0.4	0	$7.84927678409986 * 10^{-14}$
0.5	$5.55111512312578 * 10^{-17}$	$6.88338275267597 * 10^{-14}$
0.6	0	$8.14903700074865 * 10^{-15}$
0.7	0	$1.11022302462516 * 10^{-14}$
0.8	$1.11022302462516 * 10^{-16}$	$9.62563362350011 * 10^{-14}$
0.9	0	$6.17284001691587 * 10^{-14}$
1	$1.11022302462516 * 10^{-16}$	$7.34967642301854 * 10^{-14}$

x	$\|S_i^{(4)}(x) - y^{(4)}(x)\|_\infty$
0	$2.640000000000000 * 10^{-12}$
0.1	$9.00821639504557 * 10^{-11}$
0.2	$9.61398183285667 * 10^{-11}$
0.3	$6.54065690497418 * 10^{-11}$
0.4	$2.66204835952522 * 10^{-11}$
0.5	$6.09634565051920 * 10^{-11}$
0.6	$7.61818386152413 * 10^{-11}$
0.7	$9.72935065846059 * 10^{-11}$
0.8	$9.61102308849604 * 10^{-11}$
0.9	$1.60334856502686 * 10^{-10}$
1	$1.49811940630684 * 10^{-11}$

References

[1] A.Saxena, E.Venturino, Solving two point boundary value problems by means of deficient quartic splines, Babes - Bolyai Mathematica, Vol.32, 1987, 60 - 70 .

[2] J.Rashidinia, R.Mohammadi and R.Jalilian, Qunitic spline solution of boundary Value problems in the plate deflection theory, Computer science and Engineering and Electrical engineering ,Vol.16(1),2009, 53-59.

[3] N.A.A Rahman , Lacunary spline interpolation and two point boundary value Problem, Annales Univ.Sci.Budapest, Vol.36,1993, 235-246.

[4] R.A Usmani, Finite difference methods for a certain two point boundary value Problem, Indian J.pure appl. Math., Vol.14 (3),1993, 398-411.

[5] T.Y Na, Computational methods in engineering boundary value problems , 1979, Academic press, INC, New York.