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SEVERAL IDENTITIES FOR THE GENERALIZED APOSTOL-EULER AND APOSTOL-GENOCCHI POLYNOMIALS

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Abstract. The purpose of this paper is to give several symmetric identities on the generalized Apostol-Euler and Apostol-Genocchi polynomials by applying the generating functions. These extend some known identities.

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1. Introduction

The classical Bernoulli polynomials $B_n(x)$, the classical Euler polynomials $E_n(x)$ and the classical Genocchi polynomials $G_n(x)$, together with their familiar generalizations $B_n^{\alpha}(x)$, $E_n^{\alpha}(x)$ and $G_n^{\alpha}(x)$ of (real or complex) of order α are usually defined by means of the following generating functions; see [13]-[15] and the references cited therein.

$$\left(\frac{t}{e^t-1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} B_n^{\alpha}(x)\frac{t^n}{n!} \; (|\, t \,| < 2\pi; 1^{\alpha} = 1) \tag{1.1}$$

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$$\left(\frac{2}{e^t+1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} E_n^{\alpha}(x)\frac{t^n}{n!} \; (\mid t \mid < 2\pi; 1^{\alpha} = 1) \tag{1.2}$$

and

$$\left(\frac{2t}{e^t+1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} G_n^{\alpha}(x)\frac{t^n}{n!} \; (\mid t \mid < 2\pi; 1^{\alpha} = 1). \tag{1.3}$$

So that obviously

$$B_n(x) = B_n^1(x), E_n(x) = E_n^1(x) \textit{ and } G_n(x) = G_n^1(x), (n\varepsilon N), \tag{1.4}$$

where

$$N_0 = N \cup \{0\} \, (N = 1, 2, 3, \cdots).$$

For the classical Bernoulli numbers B_n, the classical Euler numbers E_n and the classical Genocchi numbers G_n we readily find from (1.6) that

$$B_n^1(0) = B_n(0) = B_n, E_n^1(0) = E_n(0) = E_n \textit{ and } G_n^1(0) = G_n(0) = G_n \; (n\varepsilon N), \tag{1.5}$$

respectively.

Some interesting analogous of the classical Bernoulli polynomials and numbers were first investigated by Apostol; see [1], [16] and the references therein. We begin here Apostol's definition as follows.

Definition 1.1. The Apostol-Bernoulli polynomials $B_n(x;\lambda)$ $(\lambda \varepsilon C)$ are defined by means of the following generating function

$$\frac{t}{\lambda e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_n(x;\lambda)\frac{t^n}{n!} \; (\mid t \mid < 2\pi; \textit{ when}\lambda = 1; t < \mid \log\lambda \mid \textit{ when } \lambda \neq 1) \tag{1.6}$$

with of course

$$B_n(x) = B_n(x;1) \textit{ and } B_n(\lambda) = B_n(0;\lambda), \tag{1.7}$$

where $B_n(\lambda)$ denotes the so called Apostol-Bernoulli numbers.

Recently Luo and Srivastava [6] further extended the Apostol-Bernoulli polynomials as the so called Apostol-Bernoulli polynomials of order α.

Definition 1.2. The Apostol-Bernoulli polynomials $B_n^{\alpha}(x;\lambda)$ $(\lambda \varepsilon C)$ of order $\alpha \varepsilon N_0$ are defined by means of the following generating function

$$\left(\frac{t}{\lambda e^t - 1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} B_n^{\alpha}(x;\lambda)\frac{t^n}{n!} \ (\mid t \mid < 2\pi;\ when\ \lambda = 1; t < \mid \log\lambda \mid\ when\ \lambda \neq 1) \tag{1.8}$$

with of course

$$B_n^{\alpha}(x) = B_n^{\alpha}(x;1) and B_n^{\alpha}(\lambda) = B_n^{\alpha}(0;\lambda), \tag{1.9}$$

where $B_n^{\alpha}(x;\lambda)$ denotes the so called Apostol-Bernoulli numbers of order α.

On the other hand Luo [7] gave an analogous extension of the generalized Euler polynomials as the so called Apostol-Euler polynomials of order α.

Definition 1.3. The Apostol-Euler polynomials $E_n^{\alpha}(x;\lambda)$ $(\lambda \varepsilon C)$ of order $\alpha \varepsilon N_0$ are defined by means of the following generating function

$$\left(\frac{2}{\lambda e^t + 1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} E_n^{\alpha}(x;\lambda)\frac{t^n}{n!} \ (\mid t \mid \log(-\lambda)) \tag{1.10}$$

with of course

$$E_n^{\alpha}(x) = E_n^{\alpha}(x;1)\ and\ E_n^{\alpha}(\lambda) = E_n^{\alpha}(0;\lambda), \tag{1.11}$$

where $E_n^{\alpha}(x;\lambda)$ denotes the so called Apostol-Euler numbers of order α.

On the subject of the Genocchi polynomials $G_n(x)$ and their various extensions a remarkably large number of investigations have appeared in the literature (see for example [2,3,4,5,8,9,10]). Moreover Luo (8-11) introduced and investigated the Apostol-Genocchi polynomials of (real or complex) oredr α which are defined as follows.

Definition 1.4. The Apostol-Genocchi polynomials $G_n^{\alpha}(x;\lambda)$ $(\lambda \varepsilon C)$ of order $\alpha \varepsilon N_0$ are defined by means of the following generating function

$$\left(\frac{2t}{\lambda e^t + 1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} G_n^{\alpha}(x;\lambda)\frac{t^n}{n!} \ (\mid t \mid \log(-\lambda)) \tag{1.12}$$

with of course

$$G_n^{\alpha}(x) = G_n^{\alpha}(x;1), G_n^{\alpha}(\lambda) = G_n^{\alpha}(0;\lambda) G_n(x;\lambda) = G_n^{(1)}(x;\lambda), G_n(\lambda) = G_n^{(1)}(\lambda), \tag{1.13}$$

where $G_n(\lambda), G_n^{\alpha}(\lambda)$ and $G_n(x;\lambda)$ denotes the so called Apostol-Genocchi numbers of order α and the Apostol-Genocchi polynomials respectively.

For each integer $k \geq 0$, $M_k(n) = \sum_{i=0}^{n}(-1)^i i^k$ is called sum of alternative integer powers. The exponential generating function for $M_k(n)$ is

$$\sum_{k=0}^{\infty} M_k(n)\frac{t^k}{k!} = 1 - e^t + e^{2t} + \ldots + (-1)^n e^{nt} = \frac{1-(-e^t)^{n+1}}{e^t+1}. \tag{1.14}$$

Definition 1.5. For an arbitrary real or complex parameter λ, the generalized sum of alternative integer powers $M_k(n;\lambda)$ is defined by the following generating functions

$$\sum_{k=0}^{\infty} M_k(n,\lambda)\frac{t^k}{k!} = \frac{1-\lambda(-e^t)^{n+1}}{\lambda e^t+1}. \tag{1.15}$$

The object of this paper is to present a systematic account of these families in a unified and generalized form. We develop some elementary properties and derive the some symmetric identities for the generalized Apostol-Euler and Apostol-Genocchi polynomials by using different analytical means on their respective generating functions. Recently, Zhang *et al* [20], Yang [18], Yang *et al* [19] and Pathan [12] also established symmetric identities of the generalized Bernoulli and Apostol-Bernoulli polynomials. Many of these properties of these generalized polynomials extend appropriately to generalized Euler and Genocchi polynomials.

2. Some symmetric identities for the Apostol-Euler polynomials

In this section, we give general symmetry identities for the generalized Apostol-Euler polynomials $E_n^{(\alpha)}(x;\lambda)$ by applying generating functions (1.10) and (1.15). Throughout this section α will be taken as an arbitrary real or complex parameter.

Theorem 2.1. *For all integers $a>0, b>0$ and $n \geq 0, \alpha \geq 1, \lambda \varepsilon C$, we have the following identity*

$$\sum_{k=0}^{n}\binom{n}{k} a^{n-k}b^{k+1}E_{n-k}^{(\alpha)}(bx;\lambda)\sum_{i=0}^{k}\binom{k}{i} M_i(a-1;\lambda)E_{k-i}^{(\alpha-1)}(ay;\lambda)$$

$$= \sum_{k=0}^{n}\binom{n}{k} b^{n-k}a^{k+1}E_{n-k}^{(\alpha)}(ax;\lambda)\sum_{i=0}^{k}\binom{k}{i} M_i(b-1;\lambda)E_{k-i}^{(\alpha-1)}(by;\lambda). \tag{2.1}$$

Proof. Let

$$g(t) = \frac{1}{a^\alpha b^{\alpha-1}}\left(\frac{2a}{\lambda e^{at}+1}\right)^{\alpha} e^{abxt}\left(\frac{1-\lambda(-e^{at})^b}{\lambda e^{bt}+1}\right)\left(\frac{2b}{\lambda e^{bt}+1}\right)^{\alpha} e^{abyt} \tag{2.2}$$

$$= \frac{1}{a^{\alpha} b^{\alpha-1}} \left(\sum_{n=0}^{\infty} E_n^{(\alpha)}(bx;\lambda) \frac{(at)^n}{n!} \right) \left(\sum_{n=0}^{\infty} M_n(a-1;\lambda) \frac{(bt)^n}{n!} \right) \left(\sum_{n=0}^{\infty} E_n^{(\alpha-1)}(ay;\lambda) \frac{(bt)^n}{n!} \right).$$

$$g(t) = \frac{1}{a^{\alpha} b^{\alpha}} \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^{k+1} E_{n-k}^{(\alpha)}(bx;\lambda) \sum_{i=0}^{k} \binom{k}{i} M_i(a-1;\lambda) E_{k-i}^{(\alpha-1)}(ay;\lambda) \right) \frac{t^n}{n!}. \tag{2.3}$$

Using the similar method, we have

$$g(t) = \frac{1}{a^{\alpha} b^{\alpha}} \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{n}{k} b^{n-k} a^{k+1} E_{n-k}^{(\alpha)}(ax;\lambda) \sum_{i=0}^{k} \binom{k}{i} M_i(b-1;\lambda) E_{k-i}^{(\alpha-1)}(by;\lambda) \right) \frac{t^n}{n!}. \tag{2.4}$$

Equating the coefficients of $\frac{t^n}{n!}$ in the last two equations (2.3) and (2.4), we get the desired result.

By setting λ=1 in Theorem 2.1, the result reduces to a similar known result of Yang [18., Eq.(9)].

Corollary 2.1. *For all integers $a > 0, b > 0$ and $n \geq 0$, $\alpha \geq 1$ $\lambda \varepsilon C$,*

$$\sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^{k+1} E_{n-k}^{(\alpha)}(bx) \sum_{i=0}^{k} \binom{k}{i} M_i(a-1;\lambda) E_{k-i}^{(\alpha-1)}(ay)$$

$$= \sum_{k=0}^{n} \binom{n}{k} b^{n-k} a^{k+1} E_{n-k}^{(\alpha)}(ax) \sum_{i=0}^{k} \binom{k}{i} M_i(b-1) E_{k-i}^{(\alpha-1)}(by). \tag{2.5}$$

Setting y=0 and $\alpha = 1$ in Theorem (2.1), we obtain the relation.

Corollary 2.2. *For all integers $a > 0, b > 0$ and $n \geq 0$, $\lambda \varepsilon C$,*

$$\sum_{i=0}^{n} \binom{n}{i} a^{i-1} b^{n-i} E_i(bx;\lambda) M_{n-i}(a-1;\lambda) = \sum_{i=0}^{n} \binom{n}{i} b^{i-1} a^{n-i} E_i(ax;\lambda) M_{n-i}(b-1;\lambda). \tag{2.6}$$

Setting $x = 0$ in (2.6), we have the relation

Corollary 2.3. *For all integers $a > 0, b > 0$ and $n \geq 0$, $\lambda \varepsilon C$,*

$$\sum_{i=0}^{n}\binom{n}{i}a^{i-1}b^{n-i}E_i(\lambda)M_{n-i}(a-1;\lambda)=\sum_{i=0}^{n}\binom{n}{i}b^{i-1}a^{n-i}E_i(\lambda)M_{n-i}(b-1;\lambda). \quad (2.7)$$

For $\lambda = 1$ in (2.7), the result reduces to similar result of Tuenter [17].

$$\sum_{i=0}^{n}\binom{n}{i}a^{i-1}b^{n-i}E_iM_{n-i}(a-1)=\sum_{i=0}^{n}\binom{n}{i}b^{i-1}a^{n-i}E_iM_{n-i}(b-1). \quad (2.8)$$

Setting b=1 in (2.6), we have

$$E_n(ax;\lambda)=\sum_{i=0}^{n}\binom{n}{i}a^{i-1}E_i(x;\lambda)M_{n-i}(a-1;\lambda). \quad (2.9)$$

On the other hand by setting $\lambda = 1$ in (2.9), the result reduces to similar result of Yang [18, Eq.(11)].

On the other hand by setting $x = 0$ in (2.9), we have a recurrence on the Apostol-Euler polynomials.

$$E_n(\lambda)=\sum_{i=0}^{n}\binom{n}{i}a^{i-1}E_i(\lambda)M_{n-i}(a-1;\lambda). \quad (2.10)$$

Theorem 2.2. *For each pair of positive integers a and b and all integers $n \geq 0, \alpha \geq 1, \lambda \varepsilon C$, we have the following identity*

$$\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{a-1}\sum_{j=0}^{b-1}(\lambda)^{i+j}a^kb^{n-k}E_k^{(\alpha)}(bx+\frac{b}{a}i;\lambda)E_{n-k}^{(\alpha)}(ay+\frac{a}{b}j;\lambda)$$

$$=\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{b-1}\sum_{j=0}^{a-1}(\lambda)^{i+j}b^ka^{n-k}E_k^{(\alpha)}(ax+\frac{a}{b}i;\lambda)E_{n-k}^{(\alpha)}(by+\frac{b}{a}j;\lambda). \quad (2.11)$$

Proof. Let $g(t)=\frac{(2a)^{\alpha}(2b)^{\alpha}e^{ab(x+y)t}(\lambda^a e^{abt}+1)(\lambda^b e^{abt}+1)}{(\lambda e^{at}+1)^{\alpha+1}(\lambda e^{bt}+1)^{\alpha+1}}$. Then the expression for $g(t)$ is symmetric in a and b and we can expand $g(t)$ into series in two ways to prove the theorem.

$$g(t)=\left(\frac{2a}{\lambda e^{at}+1}\right)^{\alpha}e^{abxt}\left(\frac{\lambda^a e^{abt}+1}{\lambda e^{bt}+1}\right)\left(\frac{2b}{\lambda e^{bt}+1}\right)^{\alpha}e^{abyt}\left(\frac{\lambda^b e^{abt}+1}{\lambda e^{at}+1}\right)$$

$$=\left(\frac{2a}{\lambda e^{at}+1}\right)^{\alpha}e^{abxt}\sum_{i=0}^{a-1}\lambda^i e^{bti}\left(\frac{2b}{\lambda e^{bt}+1}\right)^{\alpha}e^{abyt}\sum_{j=0}^{b-1}\lambda^j e^{atj}$$

$$=\sum_{i=0}^{a-1}\lambda^i\left(\frac{2a}{\lambda e^{at}+1}\right)^{\alpha}e^{(bx+\frac{b}{a}i)at}\sum_{j=0}^{b-1}\lambda^j\left(\frac{2b}{\lambda e^{bt}+1}\right)^{\alpha}e^{(ax+\frac{a}{b}j)bt}$$

$$=\left(\sum_{i=0}^{a-1}\lambda^i\sum_{k=0}^{\infty}E_k^{(\alpha)}(bx+\frac{b}{a}i;\lambda)\frac{(at)^k}{k!}\right)\left(\sum_{j=0}^{b-1}\lambda^j\sum_{n=0}^{\infty}E_n^{(\alpha)}(ay+\frac{a}{b}j;\lambda)\frac{(bt)^n}{n!}\right)$$

$$=\sum_{i=0}^{a-1}\sum_{j=0}^{b-1}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\lambda^{i+j}E_k^{(\alpha)}(bx+\frac{b}{a}i;\lambda)a^kb^nE_n^{(\alpha)}(ay+\frac{a}{b}j;\lambda)\frac{t^{n+k}}{n!k!}.$$

Replacing n by n-k in above equation, we get

$$g(t)=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{a-1}\sum_{j=0}^{b-1}a^kb^{n-k}E_k^{(\alpha)}(bx+\frac{b}{a}i;\lambda)E_{n-k}^{(\alpha)}(ay+\frac{a}{b}j;\lambda)\right)\frac{t^n}{n!}. \tag{2.12}$$

On the other hand

$$g(t)=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{b-1}\sum_{j=0}^{a-1}a^{n-k}b^kE_k^{(\alpha)}(ax+\frac{a}{b}i;\lambda)E_{n-k}^{(\alpha)}(by+\frac{b}{a}j;\lambda)\right)\frac{t^n}{n!}. \tag{2.13}$$

Equating the coefficients of $\frac{t^n}{n!}$ in the last two expressions for g(t) gives us the desired result.

By setting $\lambda = 1$ in Theorem (2.2), we immediately deduce the following result.

Corollary 2.4. *For all integers $a>0, b>0$ and $n\geq 0$, $\alpha\geq 1$. Then*

$$\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{a-1}\sum_{j=0}^{b-1}a^kb^{n-k}E_k^{(\alpha)}(bx+\frac{b}{a}i)E_{n-k}^{(\alpha)}(ay+\frac{a}{b}j)$$

$$=\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{b-1}\sum_{j=0}^{a-1}b^ka^{n-k}E_k^{(\alpha)}(ax+\frac{a}{b}i)E_{n-k}^{(\alpha)}(by+\frac{b}{a}j) \tag{2.14}$$

Setting $y=0, \alpha=1$ in Theorem 2.2, we have the following.

Corollary 2.5. *For all integers $a>0, b>0$ and $n\geq 0$, $\lambda\varepsilon C$. Then*

$$\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{a-1}\sum_{j=0}^{b-1}(\lambda)^{i+j}a^kb^{n-k}E_k(bx+\frac{b}{a}i;\lambda)E_{n-k}(\frac{a}{b}j;\lambda)$$

$$=\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{b-1}\sum_{j=0}^{a-1}(\lambda)^{i+j}b^ka^{n-k}E_k(ax+\frac{a}{b}i;\lambda)E_{n-k}(\frac{b}{a}j;\lambda) \tag{2.15}$$

When b=1 in (2.15), we have the relationship

$$\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{a-1}\lambda^{i}a^{k}E_{k}(x+\frac{i}{a};\lambda)E_{n-k}(\lambda)=\sum_{k=0}^{n}\binom{n}{k}\sum_{j=0}^{a-1}\lambda^{j}a^{n-k}E_{k}(ax;\lambda)E_{n-k}(\frac{j}{a};\lambda) \tag{2.16}$$

Substituting $\lambda = 1$ *in (2.16), we have the relationship*

$$\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{a-1}a^{k}E_{k}(x+\frac{i}{a})E_{n-k}=\sum_{k=0}^{n}\binom{n}{k}\sum_{j=0}^{a-1}a^{n-k}E_{k}(ax)E_{n-k}(\frac{j}{a}). \tag{2.17}$$

Theorem 2.3. *For each pair of positive integers a and b and all integers* $n \geq 0, \alpha \geq 1, \lambda \varepsilon C$, *we have the following identity*

$$\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{a-1}\sum_{j=0}^{b-1}(\lambda)^{i+j}a^{k}b^{n-k}E_{k}^{(\alpha)}(bx+\frac{b}{a}i+j;\lambda)E_{n-k}^{(\alpha)}(ay;\lambda)$$

$$=\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{b-1}\sum_{j=0}^{a-1}(\lambda)^{i+j}b^{k}a^{n-k}E_{k}^{(\alpha)}(ax+\frac{a}{b}i+j;\lambda)E_{n-k}^{(\alpha)}(by;\lambda). \tag{2.18}$$

Proof. The proof is analogous to Theorem 2.2, but we need to change the order of the summation of series. On the one hand

$$g(t)=\frac{(2a)^{\alpha}(2b)^{\alpha}e^{ab(x+y)t}(\lambda^{a}e^{abt}+1)(\lambda^{b}e^{abt}+1)}{(\lambda e^{at}+1)^{\alpha+1}(\lambda e^{bt}+1)^{\alpha+1}}$$

$$g(t)=\left(\frac{2a}{\lambda e^{at}+1}\right)^{\alpha}e^{abxt}\left(\frac{\lambda^{a}e^{abt}+1}{\lambda e^{bt}+1}\right)\left(\frac{2b}{\lambda e^{bt}+1}\right)^{\alpha}e^{abyt}\left(\frac{\lambda^{b}e^{abt}+1}{\lambda e^{at}+1}\right)$$

$$=\left(\frac{2a}{\lambda e^{at}+1}\right)^{\alpha}e^{abxt}\sum_{i=0}^{a-1}\lambda^{i}e^{bti}\left(\frac{2b}{\lambda e^{bt}+1}\right)^{\alpha}e^{abyt}\sum_{j=0}^{b-1}\lambda^{j}e^{atj}$$

$$=\sum_{i=0}^{a-1}\sum_{j=0}^{b-1}\lambda^{i+j}\left(\frac{2a}{\lambda e^{at}+1}\right)^{\alpha}e^{(bx+\frac{b}{a}i+j)at}\left(\frac{2b}{\lambda e^{bt}+1}\right)^{\alpha}e^{abyt}$$

$$=\left(\sum_{i=0}^{a-1}\sum_{j=0}^{b-1}\lambda^{i+j}\sum_{k=0}^{\infty}E_{k}^{(\alpha)}(bx+\frac{b}{a}i+j;\lambda)\frac{(at)^{k}}{k!}\right)\left(\sum_{n=0}^{\infty}E_{n}^{(\alpha)}(ay;\lambda)\frac{(bt)^{n}}{n!}\right)$$

$$=\sum_{i=0}^{a-1}\sum_{j=0}^{b-1}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\lambda^{i+j}E_{k}^{(\alpha)}(bx+\frac{b}{a}i+j;\lambda)a^{k}b^{n}E_{n}^{(\alpha)}(ay;\lambda)\frac{t^{n+k}}{n!k!}.$$

Replacing n by n-k in above equation, we get

$$g(t)=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{a-1}\sum_{j=0}^{b-1}a^{k}b^{n-k}E_{k}^{(\alpha)}(bx+\frac{b}{a}i+j;\lambda)E_{n-k}^{(\alpha)}(ay;\lambda)\right)\frac{t^{n}}{n!}. \tag{2.19}$$

On the other hand, we have

$$g(t)=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{b-1}\sum_{j=0}^{a-1}a^{n-k}b^{k}E_{k}^{(\alpha)}(ax+\frac{a}{b}i+j;\lambda)E_{n-k}^{(\alpha)}(by;\lambda)\right)\frac{t^{n}}{n!}. \tag{2.20}$$

Equating the coefficients of $\frac{t^n}{n!}$ in the last two expressions for g(t) gives us the desired result.

By setting $\lambda=1$ in Theorem 2.3, we immediately deduce the following result.

Corollary 2.6. *For all integers $a>0, b>0$ and $n\geq 0$, $\alpha\geq 1$. Then*

$$\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{a-1}\sum_{j=0}^{b-1}a^{k}b^{n-k}E_{k}^{(\alpha)}(bx+\frac{b}{a}i+j)E_{n-k}^{(\alpha)}(ay)$$

$$=\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{b-1}\sum_{j=0}^{a-1}b^{k}a^{n-k}E_{k}^{(\alpha)}(ax+\frac{a}{b}i+j)E_{n-k}^{(\alpha)}(by) \tag{2.21}$$

Setting $y=0, \alpha=1$ in Theorem 2.3, we have the following.

Corollary 2.7. *For all integers $a>0, b>0$ and $n\geq 0$, $\lambda \varepsilon C$. Then*

$$\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{a-1}\sum_{j=0}^{b-1}(\lambda)^{i+j}a^{k}b^{n-k}E_{k}(bx+\frac{b}{a}i+j;\lambda)E_{n-k}(\lambda)$$

$$=\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{b-1}\sum_{j=0}^{a-1}(\lambda)^{i+j}b^{k}a^{n-k}E_{k}(ax+\frac{a}{b}i+j;\lambda)E_{n-k}(\lambda). \tag{2.22}$$

When $b=1$ in (2.22), we have the relationship

$$\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{a-1}\lambda^{i}a^{k}E_{k}(x+\frac{i}{a};\lambda)E_{n-k}(\lambda)=\sum_{k=0}^{n}\binom{n}{k}\sum_{j=0}^{a-1}\lambda^{j}a^{n-k}E_{k}(ax+j;\lambda)E_{n-k}(\lambda). \tag{2.23}$$

Substituting $\lambda=1$ in (2.23), we have the relationship

$$\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{a-1}a^{k}E_{k}(x+\frac{i}{a})E_{n-k}=\sum_{k=0}^{n}\binom{n}{k}\sum_{j=0}^{a-1}a^{n-k}E_{k}(ax+j)E_{n-k}. \tag{2.24}$$

3. Some symmetric identities for the Apostol-Genocchi polynomials

In this section, we give general symmetry identities for the generalized Apostol-Genocchi polynomials $G_n^{(\alpha)}(x;\lambda)$ by applying generating functions (1.12) and (1.15). Throughout this section α will be taken as an arbitrary real or complex parameter.

Theorem 3.1 *For all integers $a>0, b>0$ and $n\geq 0, \alpha\geq 1, \lambda \varepsilon C$, we have the following identity*

$$\sum_{k=0}^{n}\binom{n}{k} a^{n-k}b^{k+1}G_{n-k}^{(\alpha)}(bx;\lambda)\sum_{i=0}^{k}\binom{k}{i}M_i(a-1;\lambda)G_{k-i}^{(\alpha-1)}(ay;\lambda)$$

$$=\sum_{k=0}^{n}\binom{n}{k} b^{n-k}a^{k+1}G_{n-k}^{(\alpha)}(ax;\lambda)\sum_{i=0}^{k}\binom{k}{i}M_i(b-1;\lambda)G_{k-i}^{(\alpha-1)}(by;\lambda) \tag{3.1}$$

Proof. Let

$$g(t)=\frac{1}{a^{\alpha}b^{\alpha-1}}\left(\frac{2at}{\lambda e^{at}+1}\right)^{\alpha}e^{abxt}\left(\frac{1-\lambda(-e^{at})^{b}}{\lambda e^{bt}+1}\right)\left(\frac{2bt}{\lambda e^{bt}+1}\right)^{\alpha}e^{abyt} \tag{3.2}$$

$$=\frac{1}{a^{\alpha}b^{\alpha-1}}\left(\sum_{n=0}^{\infty}G_n^{(\alpha)}(bx;\lambda)\frac{(at)^n}{n!}\right)\left(\sum_{n=0}^{\infty}M_n(a-1;\lambda)\frac{(bt)^n}{n!}\right)\left(\sum_{n=0}^{\infty}G_n^{(\alpha-1)}(ay;\lambda)\frac{(bt)^n}{n!}\right)$$

$$g(t)=\frac{1}{a^{\alpha}b^{\alpha}}\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^{k+1}G_{n-k}^{(\alpha)}(bx;\lambda)\sum_{i=0}^{k}\binom{k}{i}M_i(a-1;\lambda)G_{k-i}^{(\alpha-1)}(ay;\lambda)\right)\frac{t^n}{n!}. \tag{3.3}$$

Using similar plan, we have

$$g(t)=\frac{1}{a^{\alpha}b^{\alpha}}\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}b^{n-k}a^{k+1}G_{n-k}^{(\alpha)}(ax;\lambda)\sum_{i=0}^{k}\binom{k}{i}M_i(b-1;\lambda)G_{k-i}^{(\alpha-1)}(by;\lambda)\right)\frac{t^n}{n!}. \tag{3.4}$$

Equating the coefficients of $\frac{t^n}{n!}$ in the last two equations (3.3) and (3.4), we get the desired result.

By setting λ=1 in Theorem 3.1, the result reduces to a similar known result of Yang [18., Eq.(9)].

Corollary 3.1. *For all integers $a>0, b>0$ and $n\geq 0$, $\alpha\geq 1$ $\lambda\varepsilon C$,*

$$\sum_{k=0}^{n}\left(\begin{array}{c}n\\k\end{array}\right)a^{n-k}b^{k+1}G_{n-k}^{(\alpha)}(bx)\sum_{i=0}^{k}\left(\begin{array}{c}k\\i\end{array}\right)M_i(a-1;\lambda)G_{k-i}^{(\alpha-1)}(ay)$$

$$=\sum_{k=0}^{n}\left(\begin{array}{c}n\\k\end{array}\right)b^{n-k}a^{k+1}G_{n-k}^{(\alpha)}(ax)\sum_{i=0}^{k}\left(\begin{array}{c}k\\i\end{array}\right)M_i(b-1)G_{k-i}^{(\alpha-1)}(by) \tag{3.5}$$

Setting y=0 and $\alpha=1$ in Theorem (3.1), we obtain the relation.

Corollary 3.2. *For all integers $a>0, b>0$ and $n\geq 0$, $\lambda \varepsilon C$,*

$$\sum_{i=0}^{n}\left(\begin{array}{c}n\\i\end{array}\right)a^{i-1}b^{n-i}G_i(bx;\lambda)M_{n-i}(a-1;\lambda)=\sum_{i=0}^{n}\left(\begin{array}{c}n\\i\end{array}\right)b^{i-1}a^{n-i}G_i(ax;\lambda)M_{n-i}(b-1;\lambda) \tag{3.6}$$

Setting $x=0$ in (3.6), we have the relation.

Corollary 3.3. *For all integers $a>0, b>0$ and $n\geq 0$, $\lambda \varepsilon C$,*

$$\sum_{i=0}^{n}\left(\begin{array}{c}n\\i\end{array}\right)a^{i-1}b^{n-i}G_i(\lambda)M_{n-i}(a-1;\lambda)=\sum_{i=0}^{n}\left(\begin{array}{c}n\\i\end{array}\right)b^{i-1}a^{n-i}G_i(\lambda)M_{n-i}(b-1;\lambda) \tag{3.7}$$

For $\lambda=1$ in (3.7), the result reduces to similar result of Tuenter [17].

$$\sum_{i=0}^{n}\left(\begin{array}{c}n\\i\end{array}\right)a^{i-1}b^{n-i}G_iM_{n-i}(a-1)=\sum_{i=0}^{n}\left(\begin{array}{c}n\\i\end{array}\right)b^{i-1}a^{n-i}G_iM_{n-i}(b-1). \tag{3.8}$$

Setting $b=1$ in (3.6), we have

$$G_n(ax;\lambda)=\sum_{i=0}^{n}\left(\begin{array}{c}n\\i\end{array}\right)a^{i-1}G_i(x;\lambda)M_{n-i}(a-1;\lambda). \tag{3.9}$$

On the other hand by setting $\lambda=1$ in (3.9), the result reduces to similar result of Yang [18, Eq.(11)]. By setting $x=0$ in (3.9), we have a recurrence on the Apostol-Genocchi polynomials.

$$G_n(\lambda)=\sum_{i=0}^{n}\left(\begin{array}{c}n\\i\end{array}\right)a^{i-1}G_i(\lambda)M_{n-i}(a-1;\lambda). \tag{3.10}$$

Theorem 3.2. *For each pair of positive integers a and b and all integers $n \geq 0, \alpha \geq 1, \lambda \varepsilon C$, we have the following identity*

$$\sum_{k=0}^{n} \binom{n}{k} \sum_{i=0}^{a-1} \sum_{j=0}^{b-1} (\lambda)^{i+j} a^k b^{n-k} G_k^{(\alpha)}(bx + \frac{b}{a} i; \lambda) G_{n-k}^{(\alpha)}(ay + \frac{a}{b} j; \lambda)$$

$$= \sum_{k=0}^{n} \binom{n}{k} \sum_{i=0}^{b-1} \sum_{j=0}^{a-1} (\lambda)^{i+j} b^k a^{n-k} G_k^{(\alpha)}(ax + \frac{a}{b} i; \lambda) G_{n-k}^{(\alpha)}(by + \frac{b}{a} j; \lambda) \qquad (3.11)$$

Proof. Let $g(t) = \frac{(2at)^\alpha (2bt)^\alpha e^{ab(x+y)t} (\lambda^a e^{abt}+1)(\lambda^b e^{abt}+1)}{(\lambda e^{at}+1)^{\alpha+1} (\lambda e^{bt}+1)^{\alpha+1}}$. Then the expression for $g(t)$ is symmetric in a and b and we can expand $g(t)$ into series in two ways to prove the theorem.

$$g(t) = \left(\frac{2at}{\lambda e^{at}+1}\right)^\alpha e^{abxt} \left(\frac{\lambda^a e^{abt}+1}{\lambda e^{bt}+1}\right) \left(\frac{2bt}{\lambda e^{bt}+1}\right)^\alpha e^{abyt} \left(\frac{\lambda^b e^{abt}+1}{\lambda e^{at}+1}\right)$$

$$= \left(\frac{2at}{\lambda e^{at}+1}\right)^\alpha e^{abxt} \sum_{i=0}^{a-1} \lambda^i e^{bti} \left(\frac{2bt}{\lambda e^{bt}+1}\right)^\alpha e^{abyt} \sum_{j=0}^{b-1} \lambda^j e^{atj}$$

$$= \sum_{i=0}^{a-1} \lambda^i \left(\frac{2at}{\lambda e^{at}+1}\right)^\alpha e^{(bx+\frac{b}{a}i)at} \sum_{j=0}^{b-1} \lambda^j \left(\frac{2bt}{\lambda e^{bt}+1}\right)^\alpha e^{(ax+\frac{a}{b}j)bt}$$

$$= \left(\sum_{i=0}^{a-1} \lambda^i \sum_{k=0}^{\infty} G_k^{(\alpha)}(bx + \frac{b}{a} i; \lambda) \frac{(at)^k}{k!}\right) \left(\sum_{j=0}^{b-1} \lambda^j \sum_{n=0}^{\infty} G_n^{(\alpha)}(ay + \frac{a}{b} j; \lambda) \frac{(bt)^n}{n!}\right)$$

$$= \sum_{i=0}^{a-1} \sum_{j=0}^{b-1} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \lambda^{i+j} G_k^{(\alpha)}(bx + \frac{b}{a} i; \lambda) a^k b^n G_n^{(\alpha)}(ay + \frac{a}{b} j; \lambda) \frac{t^{n+k}}{n!k!}.$$

Replacing n by n-k in above equation, we get

$$g(t) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{n}{k} \sum_{i=0}^{a-1} \sum_{j=0}^{b-1} a^k b^{n-k} G_k^{(\alpha)}(bx + \frac{b}{a} i; \lambda) G_{n-k}^{(\alpha)}(ay + \frac{a}{b} j; \lambda)\right) \frac{t^n}{n!}. \qquad (3.12)$$

On the other hand

$$g(t) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{n}{k} \sum_{i=0}^{b-1} \sum_{j=0}^{a-1} a^{n-k} b^k G_k^{(\alpha)}(ax + \frac{a}{b} i; \lambda) G_{n-k}^{(\alpha)}(by + \frac{b}{a} j; \lambda)\right) \frac{t^n}{n!}. \qquad (3.13)$$

Equating the coefficients of $\frac{t^n}{n!}$ in the last two expressions for g(t) gives us the desired result.

By setting $\lambda = 1$ in Theorem 3.2, we immediately deduce the following result

Corollary 3.4. *For all integers $a>0, b>0$ and $n\geq 0$, $\alpha\geq 1$. Then*

$$\sum_{k=0}^{n}\begin{pmatrix} n \\ k \end{pmatrix}\sum_{i=0}^{a-1}\sum_{j=0}^{b-1}a^kb^{n-k}G_k^{(\alpha)}(bx+\frac{b}{a}i)G_{n-k}^{(\alpha)}(ay+\frac{a}{b}j)$$

$$=\sum_{k=0}^{n}\begin{pmatrix} n \\ k \end{pmatrix}\sum_{i=0}^{b-1}\sum_{j=0}^{a-1}b^ka^{n-k}G_k^{(\alpha)}(ax+\frac{a}{b}i)G_{n-k}^{(\alpha)}(by+\frac{b}{a}j) \tag{3.14}$$

Setting $y=0,\alpha=1$ in Theorem 3.2, we have the following.

Corollary 3.5. *For all integers $a>0, b>0$ and $n\geq 0$, $\lambda\varepsilon C$. Then*

$$\sum_{k=0}^{n}\begin{pmatrix} n \\ k \end{pmatrix}\sum_{i=0}^{a-1}\sum_{j=0}^{b-1}(\lambda)^{i+j}a^kb^{n-k}G_k(bx+\frac{b}{a}i;\lambda)G_{n-k}(\frac{a}{b}j;\lambda)$$

$$=\sum_{k=0}^{n}\begin{pmatrix} n \\ k \end{pmatrix}\sum_{i=0}^{b-1}\sum_{j=0}^{a-1}(\lambda)^{i+j}b^ka^{n-k}G_k(ax+\frac{a}{b}i;\lambda)G_{n-k}(\frac{b}{a}j;\lambda). \tag{3.15}$$

When $b=1$ in (3.15), we have the relationship

$$\sum_{k=0}^{n}\begin{pmatrix} n \\ k \end{pmatrix}\sum_{i=0}^{a-1}\lambda^ia^kG_k(x+\frac{i}{a};\lambda)G_{n-k}(\lambda)=\sum_{k=0}^{n}\begin{pmatrix} n \\ k \end{pmatrix}\sum_{j=0}^{a-1}\lambda^ja^{n-k}G_k(ax;\lambda)G_{n-k}(\frac{j}{a};\lambda). \tag{3.16}$$

Substituting $\lambda=1$ in (3.16), we have the relationship

$$\sum_{k=0}^{n}\begin{pmatrix} n \\ k \end{pmatrix}\sum_{i=0}^{a-1}a^kG_k(x+\frac{i}{a})G_{n-k}=\sum_{k=0}^{n}\begin{pmatrix} n \\ k \end{pmatrix}\sum_{j=0}^{a-1}a^{n-k}G_k(ax)G_{n-k}(\frac{j}{a}). \tag{3.17}$$

Theorem 3.3. *For each pair of positive integers a and b and all integers $n\geq 0,\alpha\geq 1,\lambda\varepsilon C$, we have the following identity*

$$\sum_{k=0}^{n}\begin{pmatrix} n \\ k \end{pmatrix}\sum_{i=0}^{a-1}\sum_{j=0}^{b-1}(\lambda)^{i+j}a^kb^{n-k}G_k^{(\alpha)}(bx+\frac{b}{a}i+j;\lambda)G_{n-k}^{(\alpha)}(ay;\lambda)$$

$$=\sum_{k=0}^{n}\begin{pmatrix} n \\ k \end{pmatrix}\sum_{i=0}^{b-1}\sum_{j=0}^{a-1}(\lambda)^{i+j}b^ka^{n-k}G_k^{(\alpha)}(ax+\frac{a}{b}i+j;\lambda)G_{n-k}^{(\alpha)}(by;\lambda). \tag{3.18}$$

Proof. The proof is analogous to Theorem 3.2, but we need to change the order of the summation of series. On the one hand

$$g(t)=\frac{(2at)^{\alpha}(2bt)^{\alpha}e^{ab(x+y)t}(\lambda^{a}e^{abt}+1)(\lambda^{b}e^{abt}+1)}{(\lambda e^{at}+1)^{\alpha+1}(\lambda e^{bt}+1)^{\alpha+1}}$$

$$\begin{aligned}
g(t)&=\left(\frac{2at}{\lambda e^{at}+1}\right)^{\alpha}e^{abxt}\left(\frac{\lambda^{a}e^{abt}+1}{\lambda e^{bt}+1}\right)\left(\frac{2bt}{\lambda e^{bt}+1}\right)^{\alpha}e^{abyt}\left(\frac{\lambda^{b}e^{abt}+1}{\lambda e^{at}+1}\right)\\
&=\left(\frac{2at}{\lambda e^{at}+1}\right)^{\alpha}e^{abxt}\sum_{i=0}^{a-1}\lambda^{i}e^{bti}\left(\frac{2bt}{\lambda e^{bt}+1}\right)^{\alpha}e^{abyt}\sum_{j=0}^{b-1}\lambda^{j}e^{atj}\\
&=\sum_{i=0}^{a-1}\sum_{j=0}^{b-1}\lambda^{i+j}\left(\frac{2at}{\lambda e^{at}+1}\right)^{\alpha}e^{(bx+\frac{b}{a}i+j)at}\left(\frac{2bt}{\lambda e^{bt}+1}\right)^{\alpha}e^{abyt}\\
&=\left(\sum_{i=0}^{a-1}\sum_{j=0}^{b-1}\lambda^{i+j}\sum_{k=0}^{\infty}G_{k}^{(\alpha)}(bx+\frac{b}{a}i+j;\lambda)\frac{(at)^{k}}{k!}\right)\left(\sum_{n=0}^{\infty}G_{n}^{(\alpha)}(ay;\lambda)\frac{(bt)^{n}}{n!}\right)\\
&=\sum_{i=0}^{a-1}\sum_{j=0}^{b-1}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\lambda^{i+j}G_{k}^{(\alpha)}(bx+\frac{b}{a}i+j;\lambda)a^{k}b^{n}G_{n}^{(\alpha)}(ay;\lambda)\frac{t^{n+k}}{n!k!}.
\end{aligned}$$

Replacing n by $n-k$ in above equation, we get

$$g(t)=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{a-1}\sum_{j=0}^{b-1}a^{k}b^{n-k}G_{k}^{(\alpha)}(bx+\frac{b}{a}i+j;\lambda)G_{n-k}^{(\alpha)}(ay;\lambda)\right)\frac{t^{n}}{n!}. \tag{3.19}$$

On the other hand

$$g(t)=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{b-1}\sum_{j=0}^{a-1}a^{n-k}b^{k}G_{k}^{(\alpha)}(ax+\frac{a}{b}i+j;\lambda)G_{n-k}^{(\alpha)}(by;\lambda)\right)\frac{t^{n}}{n!}. \tag{3.20}$$

Equating the coefficients of $\frac{t^n}{n!}$ in the last two expressions for g(t) gives us the desired result. By setting $\lambda=1$ in Theorem 3.3, we immediately deduce the following result.

Corollary 3.6. *For all integers $a>0, b>0$ and $n\geq 0$, $\alpha\geq 1$. Then*

$$\begin{aligned}
&\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{a-1}\sum_{j=0}^{b-1}a^{k}b^{n-k}G_{k}^{(\alpha)}(bx+\frac{b}{a}i+j)G_{n-k}^{(\alpha)}(ay)\\
&=\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{b-1}\sum_{j=0}^{a-1}b^{k}a^{n-k}G_{k}^{(\alpha)}(ax+\frac{a}{b}i+j)G_{n-k}^{(\alpha)}(by)
\end{aligned} \tag{3.21}$$

Setting $y=0, \alpha=1$ in Theorem 3.3, we have the following.

Corollary 3.7. *For all integers $a>0, b>0$ and $n\geq 0$, $\lambda \varepsilon C$. Then*

$$\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{a-1}\sum_{j=0}^{b-1}(\lambda)^{i+j}a^k b^{n-k}G_k(bx+\frac{b}{a}i+j;\lambda)G_{n-k}(\lambda)$$

$$=\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{b-1}\sum_{j=0}^{a-1}(\lambda)^{i+j}b^k a^{n-k}G_k(ax+\frac{a}{b}i+j;\lambda)G_{n-k}(\lambda). \tag{3.22}$$

When $b=1$ in (3.22), we have the relationship

$$\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{a-1}\lambda^i a^k G_k(x+\frac{i}{a};\lambda)G_{n-k}(\lambda)=\sum_{k=0}^{n}\binom{n}{k}\sum_{j=0}^{a-1}\lambda^j a^{n-k}G_k(ax+j;\lambda)G_{n-k}(\lambda). \tag{3.23}$$

Substituting $\lambda=1$ in (3.23), we have the relationship

$$\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{a-1}a^k G_k(x+\frac{i}{a})G_{n-k}=\sum_{k=0}^{n}\binom{n}{k}\sum_{j=0}^{a-1}a^{n-k}G_k(ax+j)G_{n-k}. \tag{3.24}$$

Conflict of Interests

The authors declare that there is no conflict of interests.

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