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ZWEIER DIFFERENCE SEQUENCE SPACES DEFINED BY A SEQUENCE OF MODULI

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Abstract. In this article, we introduce the sequence spaces $\mathscr{Z}_0(F,\triangle)$ and $\mathscr{Z}_\infty(F,\triangle)$ for the sequence of moduli $F=(f_k)$ and give some inclusion relations.

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1. Introduction

Let N, R and C be the sets of all natural, real and complex numbers respectively. We write

$$\omega = \{x=(x_k) : x_k \in R \; or \; C\},$$

the space of all real or complex sequences. Let l_∞, c and c_0 be the linear spaces of bounded, convergent and null sequences respectively, normed by

$$||x||_\infty = \sup_k |x_k|, \text{ where } k \in N.$$

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Each linear subspace of ω, for example, $\lambda, \mu \subset \omega$ is called a sequence space. A sequence space λ with linear topology is called a K-space provided each of maps $p_i \longrightarrow C$ defined by $p_i(x) = x_i$ is continuous for all $i \in N$. A K-space λ is called an FK-space provided λ is a complete linear metric space.

An FK-space whose topology is normable is called a BK-space. Let λ and μ be two sequence spaces and $A = (a_{nk})$ is an infinite matrix of real or complex numbers (a_{nk}), where $n,k \in N$. Then we say that A defines a matrix mapping from λ to μ, and we denote it by writting $A : \lambda \longrightarrow \mu$. If for every sequence $x = (x_k) \in \lambda$ the sequence $Ax = \{(Ax)_n\}$, the A transform of x is in μ, where

$$(Ax)_n = \sum_k a_{nk}x_k, \ (n \in N). \tag{1.1}$$

By $(\lambda : \mu)$, we denote the class of matrices A such that $A : \lambda \longrightarrow \mu$. Thus, $A \in (\lambda : \mu)$ if and only if series on the right side of (1.1) converges for each $n \in N$ and every $x \in \lambda$. The approach of constructing new sequence spaces by means of the matrix domain of a particular limitation method have been recently employed by Altay, Başar and Mursaleen [1], Başar and Altay [2], Malkowsky [8], Ng and Lee [10], and Wang [15]. Şengönül [14] defined the sequence $y = (y_i)$ which is frequently used as the Z^p transform of the sequence $x = (x_i)$, *i.e,*

$$y_i = px_i + (1-p)x_{i-1},$$

where $x_{-1} = 0, p \neq 1, 1 < p < \infty$ and Z^p denotes the matrix $Z^p = (z_{ik})$ defined by

$$z_{ik} = \begin{cases} p, (i = k), \\ 1-p, (i-1 = k); (i,k \in N), \\ 0, \text{otherwise}. \end{cases}$$

Following Başar and Altay [2], Şengönül [14] introduced the Zweier sequence spaces $\mathscr{Z}$ and $\mathscr{Z}_0$ as follows

$$\mathscr{Z} = \{x = (x_k) \in \omega : Z^p x \in c\},$$

$$\mathscr{Z}_0 = \{x = (x_k) \in \omega : Z^p x \in c_0\}.$$

Here we list below some of the results of [14] which we will use as a reference in order to establish some of the results of this article.

Theorem 1.1. [14, Theorem 2.1.] *The sets $\mathscr{Z}$ and $\mathscr{Z}_0$ are the linear spaces with the co-ordinate wise addition and scalar multiplication which are the BK-spaces with the norm*

$$||x||_{\mathscr{Z}} = ||x||_{\mathscr{Z}_0} = ||Z^p x||_c.$$

Theorem 1.2. [14, Theorem 2.2.] *The sequence spaces $\mathscr{Z}$ and $\mathscr{Z}_0$ are linearly isomorphic to the spaces c and c_0 respectively, i.e $\mathscr{Z} \cong c$ and $\mathscr{Z}_0 \cong c_0$.*

Theorem 1.3. [14, Theorem 2.3.] *The inclusions $\mathscr{Z}_0 \subset \mathscr{Z}$ strictly hold for $p \neq 1$.*

The idea of difference sequence spaces was introduced by Kizmaz [5]. In 1981, Kizmaz [5] defined the sequence spaces

$$l_\infty(\triangle) = \{x = (x_k) \in \omega : (\triangle x_k) \in l_\infty\},$$

$$c(\triangle) = \{x = (x_k) \in \omega : (\triangle x_k) \in c\},$$

and

$$c_0(\triangle) = \{x = (x_k) \in \omega : (\triangle x_k) \in c_0\},$$

where $\triangle x = (x_k - x_{k+1})$ and $\triangle^0 x = (x_k)$. These are Banach spaces with the norm

$$||x||_\triangle = |x_1| + ||\triangle x||_\infty.$$

The idea of modulus was structured in 1953 by Nakano; see [9] and the references therein. A function $f : [0,\infty) \longrightarrow [0,\infty)$ is called a modulus if

(1) f(t) = 0 if and only if t = 0,

(2) f(t+u)$\leq$ f(t)+ f(u) for all t, u$\geq$0,

(3) f is increasing, and

(4) f is continuous from the right at zero.

Let X be a sequence space. Ruckle [11-13] defined the sequence space X(f) as

$$X(f) = \{x = (x_k) : (f(|x_k|)) \in X\}$$

for a modulus f. Kolk [6-7] gave an extension of X(f) by considering a sequence of moduli $F = (f_k)$, that is

$$X(F) = \{x = (x_k) : (f_k(|x_k|)) \in X\}.$$

After then Gaur and Mursaleen[4] defined the following sequence spaces

$$l_\infty(F, \triangle) = \{x = (x_k) : \triangle x \in l_\infty(F)\},$$

$$c_0(F, \triangle) = \{x = (x_k) : \triangle x \in c_0(F)\}$$

for a sequence of moduli $F = (f_k)$ and gave the necessary and sufficient conditions for the inclusion relations between $X(\triangle)$ and $Y(F, \triangle)$, where $X, Y = l_\infty$ or c_0.

Lemma 1.4. [3, Lemma 1.2.] *The condition* $\sup_k f_k(t) < \infty,\ t > 0$ *holds if and only if there is a point* $t_0 > 0$ *such that* $\sup_k f_k(t_0) < \infty$.

Lemma 1.5. [3, Lemma 1.3.] *The condition* $\inf_k f_k(t) > 0$ *holds if and only if there exists a point* $t_0 > 0$ *such that* $\inf_k f_k(t_0) > 0$.

2. Main results

In this section, we introduce the following classes of sequence spaces.

$$\mathscr{Z}_\infty(F, \triangle) = \{x = (x_k) \in \omega : \triangle x \in \mathscr{Z}_\infty(F)\},$$

$$\mathscr{Z}_0(F, \triangle) = \{x = (x_k) \in \omega : \triangle x \in \mathscr{Z}_0(F)\}.$$

Theorem 2.1. *For a sequence* $F = (f_k)$ *of moduli, the following statements are equivalent:*

(a) $\mathscr{Z}_\infty(\triangle) \subseteq \mathscr{Z}_\infty(F, \triangle)$,

(b) $\mathscr{Z}_0(\triangle) \subset \mathscr{Z}_\infty(F, \triangle)$,

(c) $\sup_k f_k(t) < \infty,\ (t > 0)$.

Proof. (a) implies (b) is obvious.

(b) implies (c). Let $\mathscr{Z}_0(\triangle) \subset \mathscr{Z}_\infty(F, \triangle)$.

Suppose that (c) is not true. Then by Lemma 1.4 $\sup_k f_k(t) = \infty$ for all $t > 0$, and, therefore there is a sequence (k_i) of positive integers such that

$$f_{k_i}(\frac{1}{i}) > i,\ for\ i = 1,2,3...... \tag{2.1}$$

Define $x = (x_k)$ as follows

$$x_k = \begin{cases} \frac{1}{i}, \text{ if } k = k_i, i = 1,2,3.......; \\ 0,\ otherwise. \end{cases}$$

Then $x \in \mathscr{Z}_0(\triangle)$ but by (2.1), $x \notin \mathscr{Z}_\infty(F,\triangle)$ which contradicts (b). Hence (c) must hold.

(c) implies (a). Let (c) be satisfied and $x \in \mathscr{Z}_\infty(\triangle)$. If we suppose that $x \notin \mathscr{Z}_\infty(F,\triangle)$, then $\sup_k f_k(|\triangle x_k|) = \infty$ for $\triangle x \in \mathscr{Z}_\infty$. If we take t = $|\triangle x|$, then $\sup_k f_k(t) = \infty$ which contradicts (c). Hence $\mathscr{Z}_\infty(\triangle) \subseteq \mathscr{Z}_\infty(F,\triangle)$.

Theorem 2.2. *If $F = (f_k)$ is a sequence of moduli, then the following statements are equivalent:*

(a) $\mathscr{Z}_0(F,\triangle) \subseteq \mathscr{Z}_0(\triangle)$,

(b) $\mathscr{Z}_0(F,\triangle) \subset \mathscr{Z}_\infty(\triangle)$,

(c) $\inf_k f_k(t) > 0,\ (t > 0)$.

Proof. (a) implies (b) is obvious.

(b) implies (c). Let $\mathscr{Z}_0(F,\triangle) \subset \mathscr{Z}_\infty(\triangle)$.

Suppose that (c) does not hold. Then, by lemma 1.5 ,

$$\inf_k f_k(t) = 0, (t > 0), \tag{2.2}$$

and therefore there is a sequence (k_i) of positive integers such that

$$f_{k_i}(i^2) < \frac{1}{i} \text{ for } i = 1,2,.........$$

Define the sequence $x = (x_k)$ by

$$x_k = \begin{cases} i^2, \text{if } k = k_i,\ i = 1,2,3.......; \\ 0,\ otherwise. \end{cases}$$

By (2.2) $x \in \mathscr{Z}_0(F,\triangle)$ but $x \notin \mathscr{Z}_\infty(\triangle)$ which contradicts (b). Hence (c) must hold.

(c) implies (a). Let (c) holds and $x \in \mathscr{Z}_0(F,\triangle)$, that is, $\lim_k f_k(|\triangle x_k|) = 0$. Suppose that $x \notin \mathscr{Z}_0(\triangle)$. Then for some number $\varepsilon_0 > 0$ and positive integer k_0 we have $|\triangle x_k| \geq \varepsilon_0$ for $k \geq k_o$. Therefore $f_k(\varepsilon_0) \leq f_k(|\triangle x_k|)$ for $k \geq k_0$ and hence $\lim_k f_k(\varepsilon_0) = 0$ which contradicts (c). Thus $\mathscr{Z}_0(F,\triangle) \subseteq \mathscr{Z}_0(\triangle)$.

Theorem 2.3. *The inclusion $\mathscr{Z}_\infty(F,\triangle) \subseteq \mathscr{Z}_0(\triangle)$ holds if and only if*

$$\lim_k f_k(t) = \infty \, for \; t > 0. \tag{2.3}$$

Proof. Let $\mathscr{Z}_\infty(F,\triangle) \subseteq \mathscr{Z}_0(\triangle)$ such that $\lim_k f_k(t) = \infty$ for t >0 does not hold. Then there is a number $t_0 > 0$ and a sequence (k_i) of positive integers such that

$$f_{k_i}(t_0) \leq M < \infty \tag{2.4}$$

Define the sequence $x = (x_k)$ by

$$x_k = \begin{cases} t_0, \text{ if } k = k_i,\ i = 1,2,3.......; \\ 0,\ otherwise. \end{cases}$$

Thus $x \in \mathscr{Z}_\infty(F,\triangle)$, by (2.4). But $x \notin \mathscr{Z}_0(\triangle)$, so that (2.3) must hold If $\mathscr{Z}_\infty(F,\triangle) \subseteq \mathscr{Z}_0(\triangle)$. Conversely, let (2.3) hold. If $x \in \mathscr{Z}_\infty(F,\triangle)$, then $f_k(|\triangle x_k|) \leq M < \infty$ for k = 1,2,3.......Suppose that $x \notin \mathscr{Z}_0(\triangle)$. Then for some number $\varepsilon_0 > 0$ and positive integer k_0 we have $|\triangle x_k| \geq \varepsilon_0$ for $k \geq k_0$. Therefore $f_k(\varepsilon_0) \leq f_k(|\triangle x_k|) \leq M$ for k$\geq k_0$ which contradicts (2.3). Hence $x \in \mathscr{Z}_0(\triangle)$.

Theorem 2.4. *The inclusion $\mathscr{Z}_\infty(\triangle) \subseteq \mathscr{Z}_0(F,\triangle)$ holds, if and only if*

$$\lim_k f_k(t) = 0 \, for \; t > 0. \tag{2.5}$$

Proof. Suppose that $\mathscr{Z}_\infty(\triangle) \subseteq \mathscr{Z}_0(F,\triangle)$ but (2.5) does not hold.

Then

$$\lim_k f_k(t_0) = l \neq 0 \tag{2.6}$$

for some $t_0 > 0$. Define the sequence $x = (x_k)$ by

$$x_k = t_0 \sum_{v=0}^{k-1} (-1) \begin{bmatrix} k-v \\ k-v \end{bmatrix}$$

for $k = 1,2,3,\cdots$. Then $x \notin \mathscr{Z}_0(F,\triangle)$, by (2.6). Hence (2.5) holds. Conversly , let $x \in \mathscr{Z}_\infty(\triangle)$ and suppose that (2.5) holds. Then $|\triangle x_k| \leq M < \infty$ for k = 1,2,3.....

Therefore $f_k(|\triangle x_k|) \leq f_k(M)$ for $k = 1,2,3.....$ and $\lim_k f_k(|\triangle x_k|) \leq \lim_k f_k(M) = 0$, by (2.5). Hence $x \in \mathscr{Z}_0(F,\triangle)$.

Conflict of Interests

The author declares that there is no conflict of interests.

References

[1] B. Altay, F. Başar and Mursaleen, On the Euler sequence space which include the spaces l_p and l_∞, Inform. Sci. 176 (2006), 1450-1462,.

[2] F. Başar and B. Altay, On the spaces of sequences of p-bounded variation and related matrix mappings, Ukrainion Math. J. (2003).

[3] C.A. Bektas, R. Colak, Generalized difference sequence spaces defined by a sequence of moduli, Soochow J. Math. 29 (2003), 215-220.

[4] A.K. Gaur and Mursaleen, Difference sequence spaces defined by a sequence of moduli, Demonstratio Math. 31 (1998), 275-278.

[5] H. Kizmaz, On certain sequence spaces, Canadian Math. Bull. 24 (1981), 169-176.

[6] E. Kolk, On strong boundedness and summability with respect to a sequence of modulii, Acta Comment. Univ. Tartu, 960 (1993), 41-50.

[7] E. Kolk, Inclusion theorems for some sequence spaces defined by a sequence of modulii, Acta Comment. Univ. Tartu, 970 (1994), 65-72.

[8] E. Malkowsky, Recent results in the theory of matrix transformation in sequence spaces, Math. Vesnik 49 (1997), 187-196.

[9] H. Nakano, Concave modulars, J. Math Soc. Japan. 5 (1953), 29-49.

[10] P.N. Ng and P.Y. Lee, Cesaro sequence spaces of non-absolute type, Comment. Math. Pracc. Math. 20 (1978), 429-433.

[11] W.H. Ruckle, On perfect Symmetric BK-spaces, Math. Ann. 175 (1968), 121-126.

[12] W.H. Ruckle, Symmetric coordinate spaces and symmetric bases, Canad. J. Math. 19 (1967), 828-838.

[13] W.H. Ruckle, FK-spaces in which the sequence of coordinate vectors is bounded, Canad. J. Math. 25 (1973), 973-975.

[14] M. Şengönül, On The Zweier Sequence Space, Demonstratio Math. 1 (2007), 181-196.

[15] C.S. Wang, On Nörlund sequence spaces, Tamkang J. Math.9 (1978), 269-274.