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EXISTENCE OF SOLUTIONS FOR ELLIPTIC EQUATIONS HAVING NATURAL GROWTH TERMS IN MUSIELAK-ORLICZ SPACES

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Abstract: This paper is devoted to the study of the existence of solutions in Musielak-Orlicz spaces for a strongly non-linear elliptic equation with natural growth condition on the non-linearity.

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1. Introduction

Let Ω be a bounded subset of $\mathbb{R}^N$ $(N \geq 2)$. Consider the following non-linear Dirichlet problem

$$A(u) + g(x,u,\nabla u) = f, \tag{1.1}$$

where $A(u) = -\text{div}a(x,u,\nabla u)$ is a Leray-Lions Operator defined on $D(A) \subset W_0^1 L_\varphi(\Omega) \to W^{-1}L_\psi(\Omega)$ with φ and ψ are two complementary Musielak-Orlicz functions, and where g is a non-linearity which satisfies, for all $s \in \mathbb{R}, \xi \in \mathbb{R}^N$ and almost all $x \in \Omega$, the classical sign condition, i.e. $g(x,s,\xi)s \geq 0$, and the following natural growth condition:

$$|g(x,s,\xi)| \leq b(|s|)(c(x) + \varphi(x,|\xi|)), \tag{1.2}$$

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where $b : \mathbb{R} \to \mathbb{R}$ is a continuous and non-decreasing function and $c(.)$ is a given non-negative function in $L^1(\Omega)$. We study the problem (1.1) in the variational case i.e.

$$f \in W^{-1}E_{\psi}(\Omega).$$

In Orlicz spaces, Gossez [16] solved (1.1) in the case where g depends only on x and u. If g depends also on ∇u, the problem (1.1) has been solved by Benkirane and Elmahi in [5] and [6] by making some restrictions. In [5], g is supposed to satisfy a "non-natural" growth condition, and in [6], g is supposed to satisfy a natural growth condition but the result is restricted to N-function satisfying a Δ_2-condition. Elmahi and Meskine [15] proved the existence of solutions for (1.1) without assuming a Δ_2-condition on the N-function.

In the framework of variable exponent Sobolev spaces, E. Azroul, A. Barbara and H. Hjiaj have shown, in [2], the existence of solutions for the elliptic problem (1.1) where the second member f is firstly taken in $W^{-1,p'(x)}(\Omega)$ and then in $L^1(\Omega)$.

In Musielak-Orlicz spaces, the existence results for (1.1), where the non-linearity g depends only on x and u, have recently been proved by Benkirane and Sidi El Vally in [12]. If g depends also on ∇u, Benkirane, Blali and Sidi El Vally [3] have solved (1.1) in the case where the Musielak-orlicz function complementary to φ satisfies the Δ_2-condition.

It is our purpose in this paper to study the problem (1.1) in context of Musielak-Orlicz spaces, in the variational case i.e. $f \in W^{-1}E_{\psi}(\Omega)$, without assuming a Δ_2-condition on φ and its complementary. Our result generalizes that of Elmahi and Meskine in [15] and that of Benkirane, Blali and Sidi El Vally [3].

The study of nonlinear partial differential equations in this type of spaces is strongly motivated by numerous phenomena of physics, namely the problems related to non-Newtonian fluids of strongly inhomogeneous behavior with a high ability of increasing their viscosity under a different stimulus, like shear rate, magnetic or electric field [20].

As an example of equations to which the present result can be applied, we give

$$-\mathrm{div}\left(\frac{m(x,|\nabla u|)}{|\nabla u|}.\nabla u\right) + u\varphi(x,|\nabla u|) = f,$$

where m is the derivative of φ with respect to t.

The paper is Organized as follows: after introduction in section 1, we give in section 2 some preliminaries and lemmas that we will use in the proof of the theorem of existence for solution which is the main result in the section 3.

2. PRELIMINARIES

Musielak-orlicz function. Let Ω be an open subset of $\mathbb{R}^N$ and let φ be a real-valued function defined in $\Omega\times\mathbb{R}_+$ and satisfying the following conditions:

(a)**:** $\varphi(x,.)$ is an N-function for all $x\in\Omega$ (i.e. convex, nondecreasing, continuous, $\varphi(x,0)=0$, $\varphi(x,t)>0$ for all $t>0$, $\lim_{t\to 0}\frac{\varphi(x,t)}{t}=0$ and $\lim_{t\to\infty}\frac{\varphi(x,t)}{t}=\infty$);

(b)**:** $\varphi(.,t)$ is a measurable function for all $t\geq 0$.

A function φ which satisfies the conditions (a) and (b) is called a Musielak-orlicz function.

For a Musielak-orlicz function φ we put $\varphi_x(t)=\varphi(x,t)$ and we associate its nonnegative reciprocal function φ_x^{-1}, with respect to t, that is

$$\varphi_x^{-1}(\varphi(x,t))=\varphi(x,\varphi_x^{-1}(t))=t.$$

The Musielak-orlicz function φ is said to satisfy the Δ_2-condition if for some $k>0$, and a non negative function h, integrable in Ω, we have

$$\varphi(x,2t)\leq\ k\varphi(x,t)+h(x)\ \text{ for all }\ x\in\Omega \text{ and all } t\geq 0. \tag{2.1}$$

When (2.1) holds only for $t\geq t_0>0$, then φ is said to satisfy the Δ_2-condition near infinity.

Let φ and γ be two Musielak-orlicz functions, we say that φ dominate γ, and we write $\gamma\prec\varphi$, near infinity (resp. globally) if there exist two positive constants c and t_0 such that for almost all $x\in\Omega$:

$$\gamma(x,t)\leq\varphi(x,ct) \text{ for all } t\geq t_0 \quad (\text{resp. for all } t\geq 0 \text{ i.e. } t_0=0).$$

We say that γ grows essentially less rapidly than φ at 0 (resp. near infinity), and we write $\gamma\prec\prec\varphi$, If for every positive constant c we have

$$\lim_{t\to 0}\left(\sup_{x\in\Omega}\frac{\gamma(x,ct)}{\varphi(x,t)}\right)=0 \quad (\text{resp.} \lim_{t\to\infty}\left(\sup_{x\in\Omega}\frac{\gamma(x,ct)}{\varphi(x,t)}\right)=0).$$

Remark 1. [12] *If $\gamma \prec\prec \varphi$ near infinity, then $\forall \varepsilon > 0$ there exist $k(\varepsilon) > 0$ such that for almost all $x \in \Omega$ we have*

$$\gamma(x,t) \leq k(\varepsilon)\,\varphi(x,\varepsilon t) \text{ for all } t \geq 0. \tag{2.2}$$

Musielak-Orlicz space. For a Musielak-Orlicz function φ and a measurable function $u : \Omega \to \mathbb{R}$ we define the functional

$$\rho_{\varphi,\Omega}(u) = \int_{\Omega} \varphi(x, |u(x)|)\,dx.$$

The set $K_\varphi(\Omega) = \{u : \Omega \to \mathbb{R} \text{ measurable } / \, \rho_{\varphi,\Omega}(u) < \infty\}$ is called the Musielak-Orlicz class (or generalized orlicz class). The Musielak-Orlicz space (or generalized orlicz space) $L_\varphi(\Omega)$ is the vector space generated by $K_\varphi(\Omega)$, that is, $L_\varphi(\Omega)$ is the smallest linear space containing the set $K_\varphi(\Omega)$. Equivalently:

$$L_\varphi(\Omega) = \left\{ u : \Omega \to \mathbb{R} \text{ measurable} \Big/ \rho_{\varphi,\Omega}\left(\frac{u}{\lambda}\right) < \infty \text{ for some } \lambda > 0 \right\}.$$

For a Musielak-Orlicz function φ we put

$$\psi(x,s) = \sup_{t \geq 0}(st - \varphi(x,t)).$$

ψ is called the Musielak-orlicz function complementary (or conjugate) to φ in the sense of Young with respect to s.

We say that a sequence of functions $u_n \in L_\varphi(\Omega)$ is modular convergent to $u \in L_\varphi(\Omega)$ if there exists a constant $\lambda > 0$ such that

$$\lim_{n\to\infty} \rho_{\varphi,\Omega}\left(\frac{u_n - u}{\lambda}\right) = 0.$$

This implies convergence for $\sigma(\Pi L_\varphi, \Pi L_\psi)$ (Lemma 4.7 of [12]).

In the space $L_\varphi(\Omega)$ we define the following two norms:

$$\|u\|_{\varphi,\Omega} = \inf\left\{ \lambda > 0 \Big/ \int_{\Omega} \varphi\left(x, \frac{|u(x)|}{\lambda}\right) dx \leq 1 \right\},$$

which is called the Luxemburg norm, and the so-called Orlicz norm by

$$|||u|||_{\varphi,\Omega} = \sup_{\|v\|_\psi \le 1} \int_\Omega |u(x)\, v(x)|\, dx,$$

where ψ is the Musielak-Orlicz function complementary to φ. These two norms are equivalents [23]. $K_\varphi(\Omega)$ is a convex subset of $L_\varphi(\Omega)$.

The closure in $L_\varphi(\Omega)$ of the set of bounded measurable functions with compact support in $\overline{\Omega}$ is denoted by $E_\varphi(\Omega)$, it is a separable space and $(E_\psi(\Omega))^* = L_\varphi(\Omega)$ [23].
We have $E_\varphi(\Omega) = K_\varphi(\Omega)$ if and only if $K_\varphi(\Omega) = L_\varphi(\Omega)$ if and only if φ satisfy the Δ_2-condition (2.1) for large values of t or for all values of t, according to whether Ω has finite measure or not.

We define

$$W^1L_\varphi(\Omega) = \{u \in L_\varphi(\Omega) : D^\alpha u \in L_\varphi(\Omega), \quad \forall |\alpha| \le 1\}$$

and

$$W^1E_\varphi(\Omega) = \{u \in E_\varphi(\Omega) : D^\alpha u \in E_\varphi(\Omega), \quad \forall |\alpha| \le 1\},$$

where $\alpha = (\alpha_1, \ldots, \alpha_N)$, $|\alpha| = |\alpha_1| + \cdots + |\alpha_N|$ and $D^\alpha u$ denote the distributional derivatives. The space $W^1L_\varphi(\Omega)$ is called the Musielak-Orlicz-Sobolev space. Let

$$\overline{\rho}_{\varphi,\Omega}(u) = \sum_{|\alpha| \le 1} \rho_{\varphi,\Omega}(D^\alpha u) \text{ and } \|u\|^1_{\varphi,\Omega} = \inf\left\{\lambda > 0 : \overline{\rho}_{\varphi,\Omega}\left(\frac{u}{\lambda}\right) \le 1\right\} \text{ for } u \in W^1L_\varphi(\Omega).$$

These functionals are convex modular and a norm on $W^1L_\varphi(\Omega)$ respectively. The pair $\langle W^1L_\varphi(\Omega), \|u\|^1_{\varphi,\Omega}\rangle$ is a Banach space if φ satisfies the following condition [23]:

$$\text{there exists a constant } c > 0 \text{ such that } \inf_{x \in \Omega} \varphi(x,1) \ge c. \tag{2.3}$$

The space $W^1L_\varphi(\Omega)$ is identified to a subspace of the product $\Pi_{|\alpha|\le 1} L_\varphi(\Omega) = \Pi L_\varphi$; this subspace is $\sigma(\Pi L_\varphi, \Pi E_\psi)$ closed.
We denote by $\mathfrak{D}(\Omega)$ the Schwartz space of infinitely smooth functions with compact support in Ω and by $\mathfrak{D}(\overline{\Omega})$ the restriction of $\mathfrak{D}(\mathbb{R}^N)$ on Ω. The space $W_0^1L_\varphi(\Omega)$ is defined as the $\sigma(\Pi L_\varphi, \Pi E_\psi)$ closure of $\mathfrak{D}(\Omega)$ in $W^1L_\varphi(\Omega)$ and the space $W_0^1E_\varphi(\Omega)$ as the (norm) closure of the Schwartz space $\mathfrak{D}(\Omega)$ in $W^1L_\varphi(\Omega)$.

For two complementary Musielak-Orlicz functions φ and ψ, we have [23]:

(2.4) $i)$ The Young inequality: $t.s \leq \varphi(x,t) + \psi(x,s)$ for all $t,s \geq 0,\ x \in \Omega$.

$ii)$ The Hölder inequality:

$$\left| \int_\Omega u(x)\, v(x)\, dx \right| \leq \|u\|_{\varphi,\Omega}\, |||v|||_{\psi,\Omega}, \text{ for all } u \in L_\varphi(\Omega), v \in L_\psi(\Omega). \tag{2.5}$$

We say that a sequence of functions u_n converges to u for the modular convergence in $W^1L_\varphi(\Omega)$ (respectively in $W_0^1L_\varphi(\Omega)$) if, for some $\lambda > 0$,

$$\lim_{n\to\infty} \overline{\rho}_{\varphi,\Omega}\left(\frac{u_n - u}{\lambda}\right) = 0.$$

The following spaces of distributions will also be used:

$$W^{-1}L_\psi(\Omega) = \left\{ f \in \mathfrak{D}'(\Omega) : f = \sum_{|\alpha|\leq 1} (-1)^{|\alpha|} D^\alpha f_\alpha \text{ where } f_\alpha \in L_\psi(\Omega) \right\}$$

and

$$W^{-1}E_\psi(\Omega) = \left\{ f \in \mathfrak{D}'(\Omega) : f = \sum_{|\alpha|\leq 1} (-1)^{|\alpha|} D^\alpha f_\alpha \text{ where } f_\alpha \in E_\psi(\Omega) \right\}.$$

Lemma 2. [11] *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^N$ and let φ and ψ be two complementary Musielak-Orlicz functions which satisfy the following conditions:*

(i) *There exist a constant $c > 0$ such that $\inf_{x\in\Omega} \varphi(x,1) \geq c$; [(2.3)]*

(ii) *There exist a constant $A > 0$ such that for all $x,y \in \Omega$ with $|x-y| \leq \frac{1}{2}$ we have*

$$\frac{\varphi(x,t)}{\varphi(y,t)} \leq t^{\left(\frac{A}{\log\left(\frac{1}{|x-y|}\right)}\right)} \text{ for all } \quad t \geq 1; \tag{2.6}$$

(2.7) (iii) *If $D \subset \Omega$ is a bounded measurable set, then $\int_D \varphi(x,1)\, dx < \infty$;*

(2.8) (iv) *There exist a constant $C > 0$ such that $\psi(x,1) \leq C$ a.e in Ω.*

Under this assumptions, $\mathfrak{D}(\Omega)$ is dense in $L_\varphi(\Omega)$ with respect to the modular topology, $\mathfrak{D}(\Omega)$ is dense in $W_0^1L_\varphi(\Omega)$ for the modular convergence and $\mathfrak{D}(\overline{\Omega})$ is dense in $W^1L_\varphi(\Omega)$ for the modular convergence.

Consequently, the action of a distribution S in $W^{-1}L_{\psi}(\Omega)$ on an element u of $W_0^1L_{\varphi}(\Omega)$ is well defined. It will be denoted by $\langle S,u\rangle$.

Truncation Operator. For $k>0$ we define the truncation at height k: $T_k:\mathbb{R}\to\mathbb{R}$ by:

$$T_k(s)=\begin{cases} s & \text{if} \quad |s|\le k, \\ k\dfrac{s}{|s|} & \text{if} \quad |s|>k. \end{cases} \tag{2.9}$$

Lemma 3. [12] *Let $F:\mathbb{R}\to\mathbb{R}$ be uniformly Lipschitzian, with $F(0)=0$. Let φ be a Musielak-Orlicz function and let $u\in W_0^1L_{\varphi}(\Omega)$. Then $F(u)\in W_0^1L_{\varphi}(\Omega)$. Moreover, if the set D of discontinuity points of F' is finite, we have*

$$\frac{\partial}{\partial x_i}F(u)=\begin{cases} F'(u)\frac{\partial u}{\partial x_i} & a.e\ in \quad \{x\in\Omega : u(x)\notin D\} \\ 0 & a.e\ in \quad \{x\in\Omega : u(x)\in D\}. \end{cases}$$

Lemma 4. *Let $(f_n), f\subset L^1(\Omega)$ such that:*

i) $f_n\ge 0$ *a.e in Ω;*

ii) $f_n\to f$ *a.e in Ω;*

iii) $\int_{\Omega} f_n(x)\,dx\to\int_{\Omega} f(x)\,dx.$

then $f_n\to f$ strongly in $L^1(\Omega)$.

Proof. We have $|f-f_n|=2(f-f_n)^+-(f-f_n)$, where $g^+=\sup(g,0)$ for all measurable function g. If $f(x)>f_n(x)$ then $(f-f_n)^+(x)=f(x)-f_n(x)\le f(x)$, consequently $0\le(f-f_n)^+\le f$. Since $(f-f_n)^+\to 0$ a.e. in Ω then by using Lebesgue's theorem we have $(f-f_n)^+\to 0$ strongly in $L^1(\Omega)$. In view of (iii) we obtain
$\int_{\Omega}|f-f_n|\,dx\to 0$, which shows that $f_n\to f$ strongly in $L^1(\Omega)$ as required.

Lemma 5. *Suppose the Musielak-Orlicz function φ does not satisfy the Δ_2-condition. Then*

$$\{u\in L_{\varphi}/d(u,E_{\varphi})<1\}\subset K_{\varphi}\subset\overline{\{u\in L_{\varphi}/d(u,E_{\varphi})<1\}}$$

where $d(u,E_{\varphi})=\inf\limits_{v\in E_{\varphi}}\|u-v\|_{\varphi}$.

Proof. It is easily adapted from that given in Theorem 10.1 of [21].

Lemma 6. ***(The Nemytskii operator)*** *Let Ω be an open susbset of $\mathbb{R}^N$ with finite measure and let φ and ψ be two Musielak-Orlicz functions. Let $f:\Omega\times\mathbb{R}^p\to\mathbb{R}^q$ be a Caratheodory function such that for a.e. $x\in\Omega$ and all $s\in\mathbb{R}^p$:*

$$|f(x,s)| \leq c(x)+k_1\ \psi_x^{-1}\ \varphi(x,k_2|s|), \tag{2.10}$$

where k_1, k_2 are real positive constants and $c\in E_\psi(\Omega)$.
Then The Nemytskii operator N_f defined by $N_f(u)(x)=f(x,u(x))$, is continuous from $(\mathscr{P}(E_\varphi(\Omega),\frac{1}{k_2}))^p=\Pi\{u\in L_\varphi(\Omega):d(u,E_\varphi(\Omega))<\frac{1}{k_2}\}$ into $(L_\psi(\Omega))^q$ for the modular convergence. Furthermore if $c\in E_\gamma(\Omega)$ and $\gamma\prec\prec\psi$ then N_f is strongly continuous from $(\mathscr{P}(E_\varphi(\Omega),\frac{1}{k_2}))^p$ into $(E_\gamma(\Omega))^q$.

Proof. Let $\lambda\geq 2k_1$ such that $\frac{2c}{\lambda}\in K_\psi(\Omega)$ and let $u=(u_1,\dots,u_p)\in(\mathscr{P}(E_\varphi(\Omega),\frac{1}{k_2}))^p$ i.e. $d(u_i,E_\varphi(\Omega))<\frac{1}{k_2}$, then $\int_\Omega \varphi(x,k_2|u(x)|)\,dx\leq 1$ (by using Lemma 5). We have

$$\begin{aligned}\psi(x,\frac{|f(x,u(x))|}{\lambda}) &\leq \psi(x,\frac{c(x)}{\lambda}+\frac{1}{2}\psi_x^{-1}\varphi(x,k_2|u(x)|))\\ &\leq \frac{1}{2}\psi(x,\frac{2c(x)}{\lambda})+\frac{1}{2}\varphi(x,k_2|u(x)|).\end{aligned}$$

Integrating over Ω, we deduce that $|f(x,u)|\in L_\psi(\Omega)$ and thus $f(x,u)\in(L_\varphi(\Omega))^q$.

On the other hand, assume that $u_n\to u$ strongly in $(L_\varphi(\Omega))^p$ with $u\in(\mathscr{P}(E_\varphi(\Omega),\frac{1}{k_2}))^p$. Let $\alpha>0$ such that $d(k_2|u|,E_\varphi(\Omega))<\alpha<1$, by using Lemma 5, we have

$$\frac{k_2}{\alpha}|u|\in K_\varphi(\Omega).$$

For $\lambda \geq 4k_1$ such that $\frac{4c}{\lambda} \in K_\psi(\Omega)$ we have

$$\begin{aligned}
&\psi\left(x, \frac{|f(x,u_n(x)) - f(x,u(x))|}{\lambda}\right) \\
&\leq \psi\left(x, \frac{2c(x) + k_1\psi_x^{-1}\varphi(x,k_2|u_n(x)|) + k_1\psi_x^{-1}\varphi(x,k_2|u(x)|)}{\lambda}\right) \\
&\leq \psi\left(x, \frac{1}{2}\frac{4c(x)}{\lambda} + \frac{1}{4}\psi_x^{-1}\varphi(x,k_2|u_n(x)|) + \frac{1}{4}\psi_x^{-1}\varphi(x,k_2|u(x)|)\right) \\
&\leq \frac{1}{2}\psi\left(x, \frac{4c(x)}{\lambda}\right) + \frac{1}{4}\varphi\left(x,k_2|u_n(x)|\right) + \frac{1}{4}\varphi\left(x,k_2|u(x)|\right) \\
&\leq \frac{1}{2}\psi\left(x, \frac{4c(x)}{\lambda}\right) + \frac{1}{4}(1-\alpha)\varphi\left(x, \frac{k_2}{1-\alpha}|u_n(x) - u(x)|\right) + \frac{1}{4}\alpha\varphi\left(x, \frac{k_2}{\alpha}|u(x)|\right) + \frac{1}{4}\varphi\left(x,k_2|u(x)|\right) \\
&\leq \frac{1}{2}\psi\left(x, \frac{4c(x)}{\lambda}\right) + \varphi\left(x, \frac{k_2}{1-\alpha}|u_n(x) - u(x)|\right) + \varphi\left(x, \frac{k_2}{\alpha}|u(x)|\right),
\end{aligned}$$

we used the fact that $\varphi(x,k_2|u(x)|) \leq \varphi(x, \frac{k_2}{\alpha}|u(x)|)$.

Note that $\psi(x, \frac{4c}{\lambda}),\ \varphi(x, \frac{k_2}{\alpha}|u|) \in L^1(\Omega)$ and $\int_\Omega \varphi(x, \frac{k_2}{1-\alpha}|u_n(x) - u(x)|)\,dx \to 0$ as $n \to \infty$.

Consequently, for $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|E| < \delta \Rightarrow \int_E \psi(x, \frac{4c(x)}{\lambda})\,dx < \varepsilon, \int_E \varphi(x, \frac{k_2}{\alpha}|u(x)|)\,dx < \frac{\varepsilon}{4}$$
$$\text{and} \quad \int_E \varphi(x, \frac{k_2}{1-\alpha}|u_n(x) - u(x)|)\,dx < \frac{\varepsilon}{4}, \quad \forall n \geq n_0.$$

Thus

$$|E| < \delta \Rightarrow \int_E \psi(x, \frac{|f(x,u_n(x)) - f(x,u(x))|}{\lambda})\,dx < \varepsilon, \quad \forall n \geq n_0.$$

For a subsequence, we can assume that $u_n \to u$ almost everywhere in Ω. So $f(x,u_n) \to f(x,u)$ and $\psi\left(x, \frac{|f(x,u_n) - f(x,u)|}{\lambda}\right) \to 0$ almost everywhere in Ω. By using Vitali's theorem, we deduce that

$$\int_\Omega \psi(x, \frac{|f(x,u_n(x)) - f(x,u(x))|}{\lambda})\,dx \to 0 \text{ as } n \to 0$$

which implies that $f(x,u_n) \to f(x,u)$ in $(L_\psi(\Omega))^q$ for the modular convergence. Since the limit $f(x,u)$ is independent of the subsequence, this convergence is, also, true for the sequence.

Now we shall prove that N_f is bounded in the ball $(B_{L_\varphi(\Omega)}(0,\frac{1}{k_2}))^p$.
Let $u \in (L_\varphi(\Omega))^p$ with $\|u\|_{\varphi,\Omega} \le \frac{1}{k_2}$ and let $\lambda \ge 2k_1$ such that

$$\int_\Omega \psi\left(x, \frac{2c(x)}{\lambda}\right) dx \le 1.$$

Then

$$\int_\Omega \psi\left(x, \frac{|f(x,u(x))|}{\lambda}\right) dx \le \frac{1}{2}\int_\Omega \psi\left(x, \frac{2c(x)}{\lambda}\right) dx + \frac{1}{2}\int_\Omega \varphi(x,k_2|u(x)|)\, dx \le 1.$$

Consequently $\|f(x,u)\|_{\psi,\Omega} \le \lambda,\ \forall u \in (B_{L_\varphi(\Omega)}(0,\frac{1}{k_2}))^p$.

Finally, we assume that $c \in E_\gamma(\Omega)$ with $\gamma \prec\prec \psi$. Let $u \in (\mathscr{P}(E_\varphi(\Omega),\frac{1}{k_2}))^p$ and we shall prove that $f(x,u) \in (E_\gamma(\Omega))^q$. Remark that $\psi_x^{-1}\varphi(x,k_2|u|) \in L_\psi(\Omega) \subset E_\gamma(\Omega)$. By using (2.10) and the fact that $c \in E_\gamma(\Omega)$ we obtain $f(x,u) \in (E_\gamma(\Omega))^q$.
Now, we assume that $u_n \to u$ strongly in $(L_\varphi(\Omega))^p$ with $u \in (\mathscr{P}(E_\varphi(\Omega),\frac{1}{k_2}))^p$ and $u_n \to u$ (for a subsequence) almost everywhere in Ω.
Let α such that $d(k_2|u|, E_\varphi(\Omega)) < \alpha < 1$. For a fixed $\varepsilon > 0$ we have

$$\frac{|f(x,u_n(x)) - f(x,u(x))|}{\varepsilon} \le \frac{1}{2}\frac{4c(x)}{\varepsilon} + \frac{1}{4}\frac{4k_1}{\varepsilon}\psi_x^{-1}\varphi(x,k_2|u_n(x)|) + \frac{1}{4}\frac{4k_1}{\varepsilon}\psi_x^{-1}\varphi(x,k_2|u(x)|).$$

Then

$$\gamma\left(x, \frac{|f(x,u_n(x)) - f(x,u(x))|}{\varepsilon}\right) \le \frac{1}{2}\gamma\left(x, \frac{4c(x)}{\varepsilon}\right) + \frac{1}{4}\gamma\left(x, \frac{4k_1}{\varepsilon}\psi_x^{-1}\varphi(x,k_2|u_n(x)|)\right) + \frac{1}{4}\gamma\left(x, \frac{4k_1}{\varepsilon}\psi_x^{-1}\varphi(x,k_2|u(x)|)\right).$$

Since $\gamma \prec\prec \psi$ and remark 1 then there exists $k(\varepsilon) \ge 0$ such that

$$\gamma\left(x, \frac{4k_1}{\varepsilon}t\right) \le k(\varepsilon)\,\psi(x,t), \quad \forall t \ge 0.$$

Then:

$$\gamma\left(x, \frac{|f(x,u_n(x)) - f(x,u(x))|}{\varepsilon}\right) \le \frac{1}{2}\gamma\left(x, \frac{4c(x)}{\varepsilon}\right) + \frac{1}{4}k(\varepsilon)\,\varphi\left(x,k_2|u_n(x)|\right) + \frac{1}{4}k(\varepsilon)\,\varphi\left(x,k_2|u(x)|\right).$$

and thus

$$\gamma\left(x, \frac{|f(x,u_n(x)) - f(x,u(x))|}{\varepsilon}\right) \le \frac{1}{2}\gamma\left(x, \frac{4c(x)}{\varepsilon}\right) + k(\varepsilon)\,\varphi\left(x, \frac{k_2}{1-\alpha}|u_n(x)|\right) + k(\varepsilon)\,\varphi\left(x, \frac{k_2}{\alpha}|u(x)|\right).$$

By using the same technique as above and Vitali's theorem we conclude that

$$\int_{\Omega} \gamma\left(x, \frac{|f(x,u_n(x)) - f(x,u(x))|}{\varepsilon}\right) dx \to 0 \quad \text{as} \quad n \to \infty.$$

Then, there exists n_0 such that for $n \geq n_0$, we have

$$\int_{\Omega} \gamma\left(x, \frac{|f(x,u_n(x)) - f(x,u(x))|}{\varepsilon}\right) dx \leq 1.$$

And so

$$||f(x,u_n) - f(x,u)||_{\gamma,\Omega} \leq \varepsilon \text{ for all } n \geq n_0.$$

Finally $f(x,u_n) \to f(x,u)$ strongly in $(E_\gamma(\Omega))^q$.

3. The main result

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^N$ $(N \geq 2)$, and let φ and γ be two Musielak-Orlicz functions such that φ and its complementary ψ satisfies conditions of Lemma 2 and $\gamma \prec\prec \varphi$. Let $A : D(A) \subset W_0^1 L_\varphi(\Omega) \to W^{-1} L_\psi(\Omega)$ be a mapping (not everywhere defined) given by

$$A(u) = -\mathrm{div} a(x,u,\nabla u),$$

where $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a caratheodory function satisfying, for a.e $x \in \Omega$ and for all $s \in \mathbb{R}$ and all ξ ,$\xi_* \in \mathbb{R}^N$, $\xi \neq \xi_*$:

$$|a(x,s,\xi)| \leq k_1 \left(c(x) + \psi_x^{-1}(\gamma(x,k_2|s|)) + \psi_x^{-1}\left(\varphi(x,k_3|\xi|\right)\right) \tag{3.1}$$

$$\left(a(x,s,\xi) - a(x,s,\xi_*)\right)(\xi - \xi_*) > 0 \tag{3.2}$$

$$a(x,s,\xi)\,\xi \geq \alpha\,\varphi(x,|\xi|) \tag{3.3}$$

where $c(.)$ belongs to $E_\psi(\Omega)$, $c \geq 0$ and $k_i > 0$, $i = 1,2,3$, $\alpha \in \mathbb{R}_+^*$.
Furthermore, let $g : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ be a caratheodory function such that, for a.e $x \in \Omega$ and for all $s \in \mathbb{R}$, $\xi \in \mathbb{R}^N$

$$g(x,s,\xi)\,s \geq 0 \tag{3.4}$$

$$|g(x,s,\xi)| \leq b(|s|)\,(c'(x) + \varphi(x,|\xi|)) \tag{3.5}$$

where $b:\mathbb{R}\to\mathbb{R}$ is a continuous and non-decreasing function and $c'(.)$ is a given non-negative function in $L^1(\Omega)$.

Finally we assume that

$$f\in W^{-1}E_{\psi}(\Omega) \tag{3.6}$$

Consider the following elliptic problem, with Dirichlet boundary condition,

$$\begin{cases} u\in W_0^1L_{\varphi}(\Omega), g(x,u,\nabla u)\in L^1(\Omega), g(x,u,\nabla u)u\in L^1(\Omega)\\ \langle A(u),v\rangle+\int_{\Omega} g(x,u,\nabla u)\, v\,dx=\langle f,v\rangle\\ \text{for all}\quad v\in W_0^1L_{\varphi}(\Omega)\cap L^{\infty}(\Omega)\quad\text{and for } v=u. \end{cases} \tag{3.7}$$

We shall prove the following existence theorem:

Theorem 7. *Assume that (3.1)-(3.6) hold true, then there exists at least one solution of the elliptic problem (3.7).*

Proof.

Step 1 : **A priori estimates**.

Consider the following approximate problems:

$$\begin{cases} u_n\in W_0^1L_{\varphi}(\Omega)\\ \langle A(u_n),v\rangle+\int_{\Omega} g_n(x,u_n,\nabla u_n)\, v\,dx=\langle f,v\rangle,\quad \forall v\in W_0^1L_{\varphi}(\Omega), \end{cases} \tag{3.8}$$

where $g_n(x,s,\xi)=T_n(g(x,s,\xi))$.

Note that $g_n(x,s,\xi)\,s\geq 0$, $|g_n(x,s,\xi)|\leq|g(x,s,\xi)|$ and $|g_n(x,s,\xi)|\leq n$. Since g_n is bounded for any fixed $n>0$, there exists at least one solution u_n of (3.8). (see Proposition 1 of [19] and Theorem 4.4 of [12])

Using in (3.8) the test function $v=u_n$, we get

$$\int_{\Omega} a(x,u_n,\nabla u_n)\,\nabla u_n\,dx\leq\langle f,u_n\rangle.$$

Consequently, by Theorem 4.4 of [12], one has that (u_n) is bounded in $W_0^1L_\varphi(\Omega)$, $(a(x,u_n,\nabla u_n))_n$ is bounded in $(L_\psi(\Omega))^N$, and

$$\int_\Omega g_n(x,u_n,\nabla u_n)\, u_n\, dx \leq C, \tag{3.9}$$

Where C is a real constant which does not depend on n. Passing to a subsequence, if necessary, we can assume that

$$u_n \rightharpoonup u \text{ weakly in } W_0^1L_\varphi(\Omega) \text{ for } \sigma(\Pi L_\varphi,\Pi E_\psi),\ \text{strongly in } E_\varphi(\Omega) \text{ and a.e. in } \Omega, \tag{3.10}$$

$$a(x,u_n,\nabla u_n) \rightharpoonup h \text{ and } a(x,T_k(u_n),\nabla T_k(u_n)) \rightharpoonup h_k \text{ weakly in } (L_\psi(\Omega))^N$$

$$\text{for } \sigma(\Pi L_\psi,\Pi E_\varphi) \text{ for some } h \text{ and } h_k \in (L_\psi(\Omega))^N.$$

Step 2 : **Almost everywhere convergence of the gradients**.

Let $\mu(t) = te^{\sigma t^2}, \sigma > 0$. It is well known that when $\sigma \geq (\frac{b(k)}{2\alpha})^2$ one has

$$\mu'(t) - \frac{b(k)}{\alpha}|\mu(t)| \geq \frac{1}{2} \qquad \text{for all } \ t \in \mathbb{R}, \tag{3.11}$$

where $k > 0$ is a fixed real number which will be used as a level of the truncation.

Let $(v_j) \subset \mathfrak{D}(\Omega)$ be a sequence which converges to u for the modular convergence in $W_0^1L_\varphi(\Omega)$ and set $\theta_n^j = T_k(u_n) - T_k(v_j), \theta^j = T_k(u) - T_k(v_j)$ and $z_n^j = \mu(\theta_n^j)$.

Using in (3.8) the test function z_n^j, we get

$$\langle A(u_n), z_n^j\rangle + \int_\Omega g_n(x,u_n,\nabla u_n) z_n^j\, dx = \langle f, z_n^j\rangle.$$

Denote by $\varepsilon_i(n,j)$ $(i = 0,1,2,\dots)$ various sequences of real numbers which tend to 0 when n and $j \to \infty$, i.e.

$$\lim_{j\to\infty}\lim_{n\to\infty} \varepsilon_i(n,j) = 0.$$

In view of (3.10), we have $z_n^j \to \mu(\theta^j)$ weakly in $W_0^1L_\varphi(\Omega)$ for $\sigma(\Pi L_\varphi,\Pi E_\psi)$ as $n \to \infty$ and then $\langle f, z_n^j\rangle \to \langle f, \mu(\theta^j)\rangle$ as $n \to \infty$. Using, now, the modular convergence of v_j, we get $\langle f, \mu(\theta^j)\rangle \to 0$ as $j \to \infty$ so that

$$\langle f, z_n^j\rangle = \varepsilon_0(n,j).$$

Since $g_n(x,u_n,\nabla u_n)z_n^j \geq 0$ on the subset $\{x\in\Omega : |u_n|>k\}$ we have

$$\langle A(u_n), z_n^j\rangle + \int_{\{|u_n|\leq k\}} g_n(x,u_n,\nabla u_n)\, z_n^j\, dx \leq \varepsilon_0(n,j). \tag{3.12}$$

The first term of the left-hand side of (3.12) reads as

$$\begin{aligned}
\langle A(u_n), z_n^j\rangle &= \int_{\{|u_n|\leq k\}} a(x,u_n,\nabla u_n)\,[\nabla T_k(u_n)-\nabla T_k(v_j)]\,\mu'(\theta_n^j)\,dx \\
&\quad - \int_{\{|u_n|>k\}} a(x,u_n,\nabla u_n)\,\nabla T_k(v_j)\,\mu'(\theta_n^j)\,dx \\
&= \int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\,[\nabla T_k(u_n)-\nabla T_k(v_j)]\,\mu'(\theta_n^j)\,dx \\
&\quad - \int_{\{|u_n|>k\}} a(x,u_n,\nabla u_n)\,\nabla T_k(v_j)\,\mu'(\theta_n^j)\,dx,
\end{aligned}$$

and then

$$\begin{aligned}
\langle A(u_n), z_n^j\rangle &= \int_\Omega [a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\ T_k(v_j)\chi_j^s)] \\
&\qquad \times[\nabla T_k(u_n)-\nabla T_k(v_j)\chi_j^s]\,\mu'(\theta_n^j)\,dx \\
&\quad + \int_\Omega a(x,T_k(u_n),\nabla T_k(v_j)\chi_j^s)\,[\nabla T_k(u_n)-\nabla T_k(v_j)\chi_j^s]\,\mu'(\theta_n^j)\,dx \\
&\quad - \int_{\Omega\setminus\Omega_j^s} a(x,T_k(u_n),\nabla T_k(u_n))\,\nabla T_k(v_j)\,\mu'(\theta_n^j)\,dx \\
&\quad - \int_{\{|u_n|>k\}} a(x,u_n,\nabla u_n)\,\nabla T_k(v_j)\,\mu'(\theta_n^j)\,dx,
\end{aligned} \tag{3.13}$$

where χ_j^s denotes the characteristic function of the subset $\Omega_j^s = \{x\in\Omega : |\nabla T_k(v_j)|\leq s\}$.

We shall pass to the limit in n and in j for s and m fixed in the last three terms of the right-hand side of (3.13). We start with the fourth term,

Observe that, since

$$|\nabla T_k(v_j)\chi_{\{|u_n|>k\}}\mu'(\theta_n^j)| \leq \mu'(2k)|\nabla T_k(v_j)| \leq \mu'(2k)\|\nabla v_j\|_\infty = a_j \in \mathbb{R}$$

we have

$$\nabla T_k(v_j)\chi_{\{|u_n|>k\}}\mu'(\theta_n^j) \to \nabla T_k(v_j)\chi_{\{|u|>k\}}\mu'(\theta^j) \text{ strongly in } (E_\varphi(\Omega))^N \text{ as } n\to\infty$$

and hence

$$\int_{\{|u_n|>k\}} a(x,u_n,\nabla u_n)\nabla T_k(v_j)\mu'(\theta_n^j)\,dx \to \int_{\{|u|>k\}} h\nabla T_k(v_j)\mu'(\theta^j)\,dx \text{ as } n\to\infty,$$

Observe that

$$|\nabla T_k(v_j)\chi_{\{|u|>k\}}\mu'(\theta^j)| \le \mu'(2k)|\nabla T_k(v_j)| \le \mu'(2k)|\nabla v_j|$$

then, by using the modular convergence of $|\nabla v_j|$ in $L_\varphi(\Omega)$ and the Vitali's theorem, we get

$$\nabla T_k(v_j)\chi_{\{|u|>k\}}\mu'(\theta^j) \to 0$$

for the modular convergence in $(L_\varphi(\Omega))^N$ and thus

$$\int_{\{|u|>k\}} h\nabla T_k(v_j)\mu'(\theta^j)\,dx \to 0 \text{ as } j\to\infty.$$

We have then proved that

$$\int_{\{|u_n|>k\}} a(x,u_n,\nabla u_n)\ \nabla T_k(v_j)\ \mu'(\theta_n^j)\,dx = \varepsilon_1(n,j). \tag{3.14}$$

The second term on the right hand side of (3.13) tends to by letting $n\to\infty$

$$\int_\Omega a(x,T_k(u),\nabla T_k(v_j)\chi_j^s)\ [\nabla T_k(u)-\nabla T_k(v_j)\chi_j^s]\ \mu'(\theta^j)\,dx$$

Since $a(x,T_k(u_n),\nabla T_k(v_j)\chi_j^s)\ \mu'(\theta_n^j) \to a(x,T_k(u),\nabla T_k(v_j)\chi_j^s)\ \mu'(\theta^j)$ strongly in $(E_\psi(\Omega))^N$ as $n\to\infty$, by Lemma 6, while $\nabla T_k(u_n) \rightharpoonup \nabla T_k(u)$ weakly in $(L_\varphi(\Omega))^N$, by (3.10). Since $\nabla T_k(v_j)\chi_j^s \to \nabla T_k(u)\chi^s$ strongly in $(E_\varphi(\Omega))^N$ as $j\to\infty$, where χ^s denotes the characteristic function of $\Omega^s=\{x\in\Omega : |\nabla T_k(u)|\le s\}$, it is easy to see that

$$\int_\Omega a(x,T_k(u),\nabla T_k(v_j)\chi_j^s)\ [\nabla T_k(u)-\nabla T_k(v_j)\chi_j^s]\ \mu'(\theta^j)\,dx \to 0 \ \text{ as } j\to\infty,$$

and thus

$$\int_\Omega a(x,T_k(u_n),\nabla T_k(v_j)\chi_j^s)\ [\nabla T_k(u_n)-\nabla T_k(v_j)\chi_j^s]\ \mu'(\theta_n^j)\,dx = \varepsilon_2(n,j). \tag{3.15}$$

Concerning the third term on the right-hand side of (3.13), we have

$$-\int_{\Omega\setminus\Omega_j^s} a(x,T_k(u_n),\nabla T_k(u_n))\ \nabla T_k(v_j)\ \mu'(\theta_n^j)\,dx \to -\int_{\Omega\setminus\Omega_j^s} h_k\ \nabla T_k(v_j)\ \mu'(\theta^j)\,dx.$$

as $n\to\infty$ by the fact that $\nabla T_k(v_j)$ belongs to $(E_\varphi(\Omega))^N$. Using now, the modular convergence of (∇v_j) in $(L_\varphi(\Omega))^N$ we get

$$-\int_{\Omega\setminus\Omega_j^s} h_k\ \nabla T_k(v_j)\ \mu'(\theta^j)\,dx \to -\int_{\Omega\setminus\Omega_s} h_k\ \nabla T_k(u)\,dx \ \text{ as } j\to\infty,$$

and thus

$$-\int_{\Omega\setminus\Omega_j^s} a(x,T_k(u_n),\nabla T_k(u_n))\ \nabla T_k(v_j)\ \mu'(\theta_n^j)\,dx = \varepsilon_3(n,j) - \int_{\Omega\setminus\Omega_s} h_k\ \nabla T_k(u)\,dx. \tag{3.16}$$

Now combining equations (3.14), (3.15), and (3.16), we obtain

$$\begin{aligned}\langle A(u_n),z_n^j\rangle &= \int_\Omega [a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(v_j)\chi_j^s)] \\ &\quad\times[\nabla T_k(u_n)-\nabla T_k(v_j)\chi_j^s]\ \mu'(\theta_n^j)\,dx - \int_{\Omega\setminus\Omega_s} h_k\ \nabla T_k(u)\,dx + \varepsilon_4(n,j).\end{aligned} \tag{3.17}$$

We now turn to the second term of the left-hand side of (3.12). We have

$$\begin{aligned}\left|\int_{\{|u_n|\le k\}} g_n(x,u_n,\nabla u_n) z_n^j\,dx\right| &= \left|\int_{\{|u_n|\le k\}} g_n(x,T_k(u_n),\nabla T_k(u_n))\ z_n^j\,dx\right| \\ &\le \int_\Omega b(k)\ c'(x)\ |\mu(\theta_n^j)|\,dx + b(k)\int_\Omega \varphi(x,|\nabla T_k(u_n)|)\ |\mu(\theta_n^j)|\,dx \\ &\le \frac{b(k)}{\alpha}\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\ \nabla T_k(u_n)\ |\mu(\theta_n^j)|\,dx + \varepsilon_5(n,j).\end{aligned}$$

The first term of the right-hand side of this inequality reads as

$$
\begin{aligned}
(3.18)\quad & \frac{b(k)}{\alpha}\int_{\Omega}[a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(v_j)\chi_j^s)] \\
&\quad\times[\nabla T_k(u_n)-\nabla T_k(v_j)\chi_j^s]\,|\mu(\theta_n^j)|\,dx \\
&+\frac{b(k)}{\alpha}\int_{\Omega}a(x,T_k(u_n),\nabla T_k(v_j)\chi_j^s)[\nabla T_k(u_n)-\nabla T_k(v_j)\chi_j^s]\,|\mu(\theta_n^j)|\,dx \\
&-\frac{b(k)}{\alpha}\int_{\Omega}a(x,T_k(u_n),\nabla T_k(u_n))\,\nabla T_k(v_j)\chi_j^s\,|\mu(\theta_n^j)|\,dx
\end{aligned}
$$

and, as above, it is easy to see that

$$\frac{b(k)}{\alpha}\int_{\Omega}a(x,T_k(u_n),\nabla T_k(v_j)\chi_j^s)[\nabla T_k(u_n)-\nabla T_k(v_j)\chi_j^s]\,|\mu(\theta_n^j)|\,dx=\varepsilon_6(n,j)$$

and that

$$-\frac{b(k)}{\alpha}\int_{\Omega}a(x,T_k(u_n),\nabla T_k(u_n))\,\nabla T_k(v_j)\chi_j^s\,|\mu(\theta_n^j)|\,dx=\varepsilon_7(n,j).$$

So that

$$
\begin{aligned}
&\left|\int_{\{|u_n|\le k\}} g_n(x,u_n,\nabla u_n)z_n^j\,dx\right| \\
&\qquad\le\frac{b(k)}{\alpha}\int_{\Omega}[a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(v_j)\chi_j^s)] \\
&\qquad\qquad\times[\nabla T_k(u_n)-\nabla T_k(v_j)\chi_j^s]\,|\mu(\theta_n^j)|\,dx\;+\;\varepsilon_8(n,j).
\end{aligned}
$$

Combining this inequality with (3.12) and (3.17), we obtain

$$
\begin{aligned}
&\int_{\Omega}[a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(v_j)\,\chi_j^s)]\times[\nabla T_k(u_n)-\nabla T_k(v_j)\,\chi_j^s] \\
&\qquad\times\left[\mu'(\theta_n^j)-\frac{b(k)}{\alpha}|\mu(\theta_n^j)|\right]dx\le\varepsilon_9(n,j)+\int_{\Omega\setminus\Omega_s}h_k\,\nabla T_k(u)\,dx.
\end{aligned}
$$

Consequently, by using (3.11), we conclude that

$$\int_\Omega [a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(v_j)\chi_j^s)] \times [\nabla T_k(u_n) - \nabla T_k(v_j)\chi_j^s]\,dx \le 2\,\varepsilon_9(n,j) + 2\int_{\Omega\setminus\Omega_s} h_k\,\nabla T_k(u)\,dx. \tag{3.19}$$

On the other hand

$$\begin{aligned}
&\int_\Omega [a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u)\chi^s)] \times [\nabla T_k(u_n) - \nabla T_k(u)\chi^s]\,dx \\
&\quad = \int_\Omega [a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(v_j)\,\chi_j^s)] \times [\nabla T_k(u_n) - \nabla T_k(v_j)\,\chi_j^s]\,dx \\
&\quad\quad + \int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\,[\nabla T_k(v_j)\,\chi_j^s - \nabla T_k(u)\,\chi^s]\,dx \\
&\quad\quad - \int_\Omega a(x,T_k(u_n),\nabla T_k(u)\,\chi^s)\,[\nabla T_k(u_n) - \nabla T_k(u)\,\chi^s]\,dx \\
&\quad\quad + \int_\Omega a(x,T_k(u_n),\nabla T_k(v_j)\,\chi_j^s)\,[\nabla T_k(u_n) - \nabla T_k(v_j)\,\chi_j^s]\,dx.
\end{aligned}$$

We will pass to the limit in n and in j in the last three terms of the right-hand side of the above equality. Similar tools as in (3.13) and (3.18) give

$$\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))[\nabla T_k(v_j)\chi_j^s - \nabla T_k(u)\chi^s]\,dx = \varepsilon_{10}(n,j)$$

and

$$\int_\Omega a(x,T_k(u_n),\nabla T_k(u)\chi^s)[\nabla T_k(u_n) - \nabla T_k(u)\chi^s]\,dx = \varepsilon_{11}(n,j)$$

and

$$\int_\Omega a(x,T_k(u_n),\nabla T_k(v_j)\chi_j^s)[\nabla T_k(u_n) - \nabla T_k(v_j)\chi_j^s]\,dx = \varepsilon_{12}(n,j), \tag{3.20}$$

which imply that

$$\int_{\Omega} [a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u)\chi^s)] \times [\nabla T_k(u_n) - \nabla T_k(u)\chi^s]\,dx$$
$$= \int_{\Omega} [a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(v_j)\chi_j^s)] \times [\nabla T_k(u_n) - \nabla T_k(v_j)\chi_j^s]\,dx$$
$$+ \varepsilon_{13}(n,j).$$

For $r \leq s$ one has

$$\begin{aligned}
0 &\leq \int_{\Omega_r} [a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u))] \times [\nabla T_k(u_n) - \nabla T_k(u)]\,dx \\
&\leq \int_{\Omega_s} [a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u))] \times [\nabla T_k(u_n) - \nabla T_k(u)]\,dx \\
&= \int_{\Omega_s} [a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u)\chi^s)] \times [\nabla T_k(u_n) - \nabla T_k(u)\chi^s]\,dx \\
&\leq \int_{\Omega} [a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u)\chi^s)] \times [\nabla T_k(u_n) - \nabla T_k(u)\chi^s]\,dx \\
&= \int_{\Omega} [a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(v_j)\chi_j^s)] \times [\nabla T_k(u_n) - \nabla T_k(v_j)\chi_j^s]\,dx \\
&\quad + \varepsilon_{13}(n,j) \\
&\leq \varepsilon_{14}(n,j) + 2 \int_{\Omega\setminus\Omega_s} h_k\, \nabla T_k(u)\,dx.
\end{aligned}$$

This implies that, by passing at first to the limit sup over n and then over j,

$$\begin{aligned}
0 &\leq \limsup_{n\to\infty} \int_{\Omega_r} [a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u))] \\
&\qquad \times [\nabla T_k(u_n) - \nabla T_k(u)]\,dx \\
&\leq 2 \int_{\Omega\setminus\Omega_s} h_k\, \nabla T_k(u)\,dx.
\end{aligned}$$

Using the fact that $h_k \nabla T_k(u) \in L^1(\Omega)$ and letting $s \to \infty$, we get, since $|\Omega\setminus\Omega_s| \to 0$,

$$\int_{\Omega_r} [a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u))] \times [\nabla T_k(u_n) - \nabla T_k(u)]\,dx \to 0 \text{ as } n \to \infty.$$

As in [5], we deduce that there exists a subsequence still denoted by u_n such that

$$\nabla u_n \longrightarrow \nabla u \text{ a.e in } \Omega, \tag{3.21}$$

which implies that

$$a(x,u_n,\nabla u_n) \rightharpoonup a(x,u,\nabla u) \text{ weakly in } (L_\psi(\Omega))^N \text{ for } \sigma(\Pi L_\psi, \Pi E_\varphi).$$

Step 3 : **Modular convergence of the truncations**.

We turn now to equation(3.19), we can write

$$\begin{aligned}\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\,\nabla T_k(u_n)\,dx \leq & \int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\,\nabla T_k(v_j)\,\chi_j^s\,dx \\ & + \int_\Omega a(x,T_k(u_n),\nabla T_k(v_j)\,\chi_j^s) \\ & \qquad \times [\nabla T_k(u_n) - \nabla T_k(v_j)\,\chi_j^s]\,dx + 2\,\varepsilon_9(n,j) \\ & + 2\int_{\Omega\setminus\Omega_s} h_k\,\nabla T_k(u)\,dx,\end{aligned}$$

which implies, by using (3.20), that

$$\begin{aligned}\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\;\nabla T_k(u_n)\,dx \leq & \int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\;\nabla T_k(v_j)\;\chi_j^s\,dx + \varepsilon_{15}(n,j) \\ & + 2\int_{\Omega\setminus\Omega_s} h_k\;\nabla T_k(u)\,dx.\end{aligned}$$

Passing to the limit sup over n in both sides of this inequality yields

$$\begin{aligned}\limsup_{n\to\infty}\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\;\nabla T_k(u_n)\,dx \leq & \int_\Omega a(x,T_k(u),\nabla T_k(u))\;\nabla T_k(v_j)\;\chi_j^s\,dx \\ & + \lim_{n\to\infty}\varepsilon_{15}(n,j) + 2\int_{\Omega\setminus\Omega_s} h_k\;\nabla T_k(u)\,dx,\end{aligned}$$

in which, we can pass to the limit in j, to obtain

$$\begin{aligned}\limsup_{n\to\infty}\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\;\nabla T_k(u_n)\,dx \leq & \int_\Omega a(x,T_k(u),\nabla T_k(u))\;\nabla T_k(u)\;\chi^s\,dx \\ & + 2\int_{\Omega\setminus\Omega_s} h_k\;\nabla T_k(u)\,dx.\end{aligned}$$

Letting $s \to \infty$ gives

$$\limsup_{n\to\infty} \int_\Omega a(x, T_k(u_n), \nabla T_k(u_n))\, \nabla T_k(u_n)\, dx \le \int_\Omega a(x, T_k(u), \nabla T_k(u))\, \nabla T_k(u)\, dx$$

On the other hand we have, by using Fatou's lemma,

$$\int_\Omega a(x, T_k(u), \nabla T_k(u))\, \nabla T_k(u)\, dx \le \liminf_{n\to\infty} \int_\Omega a(x, T_k(u_n), \nabla T_k(u_n))\, \nabla T_k(u_n)\, dx,$$

Which implies that

$$\int_\Omega a(x, T_k(u_n), \nabla T_k(u_n))\, \nabla T_k(u_n)\, dx \to \int_\Omega a(x, T_k(u), \nabla T_k(u))\, \nabla T_k(u)\, dx$$

as $n \to \infty$ and, by using Lemma 4, we conclude that

$$a(x, T_k(u_n), \nabla T_k(u_n))\, \nabla T_k(u_n) \to a(x, T_k(u), \nabla T_k(u))\, \nabla T_k(u) \ \text{ in } L^1(\Omega). \tag{3.22}$$

This implies, by using (3.3), that

$$T_k(u_n) \to T_k(u) \ \text{ in } W_0^1 L_\varphi(\Omega) \ \text{ for the modular convergence.}$$

Step 4 : **Equi-integrability of the non-linearities and passage to the limit**.

We shall prove that $g_n(x, u_n, \nabla u_n) \to g(x, u, \nabla u)$ strongly in $L^1(\Omega)$ by using Vitali's theorem. Since $g_n(x, u_n, \nabla u_n) \to g(x, u, \nabla u)$ a.e in Ω, thanks to (3.21), it suffices to prove that $g_n(x, u_n, \nabla u_n)$ are uniformly equi-integrable in Ω.

Let $E \subset \Omega$ be a measurable subset of Ω. We have for any $m > 0$,

$$\begin{aligned}\int_E |g_n(x, u_n, \nabla u_n)|\, dx &= \int_{E\cap\{|u_n|\le m\}} |g_n(x, u_n, \nabla u_n)|\, dx + \int_{E\cap\{|u_n|>m\}} |g_n(x, u_n, \nabla u_n)|\, dx \\ &\le \frac{b(m)}{\alpha} \int_E a(x, T_m(u_n), \nabla T_m(u_n))\, \nabla T_m(u_n)\, dx \\ &\quad + b(m) \int_E c'(x)\, dx + \frac{1}{m} \int_\Omega g_n(x, u_n, \nabla u_n)\, u_n\, dx.\end{aligned}$$

By virtue of the strong convergence (3.22) and the fact that $c' \in L^1(\Omega)$, there exists $\eta > 0$ such that

$$|E| < \eta \ \text{ implies } \int_E |g_n(x, u_n, \nabla u_n)|\, dx \le \varepsilon, \quad \forall n,$$

Which shows that $g_n(x,u_n,\nabla u_n)$ are uniformly equi-integrable in Ω as required.

In order to pass to the limit, we have, by going back to approximate equations (3.8),

$$\int_\Omega a(x,u_n,\nabla u_n)\ \nabla w\,dx + \int_\Omega g_n(x,u_n,\nabla u_n)\ w\,dx = \langle f,w\rangle$$

for all $w \in \mathfrak{D}(\Omega)$, in which, we can easily pass to the limit as $n \to \infty$ to get

$$\int_\Omega a(x,u,\nabla u)\ \nabla w\,dx + \int_\Omega g(x,u,\nabla u)\ w\,dx = \langle f,w\rangle. \tag{3.23}$$

Let $v \in W_0^1 L_\varphi(\Omega) \cap L^\infty(\Omega)$. By Theorem 2.5 of [11], there exists a sequence $(w_j) \subset \mathfrak{D}(\Omega)$ such that $w_j \to v$ in $W_0^1 L_\varphi(\Omega)$ for the modular convergence and $\|w_j\|_{\infty,\Omega} \le (N+1)\|v\|_{\infty,\Omega}$ for all $j \in \mathbb{N}$. Taking $w = w_j$ in (3.23) and letting $j \to \infty$ yields

$$\int_\Omega a(x,u,\nabla u)\ \nabla v\,dx + \int_\Omega g(x,u,\nabla u)\ v\,dx = \langle f,v\rangle.$$

By choosing $v = T_k(u)$ in the last equality, we get

$$\int_\Omega a(x,u,\nabla u)\ \nabla T_k(u)\,dx + \int_\Omega g(x,u,\nabla u)\ T_k(u)\,dx = \langle f,T_k(u)\rangle. \tag{3.24}$$

From (3.9), we deduce by Fatou's lemma that $g(x,u,\nabla u)\ u \in L^1(\Omega)$ and since $|g(x,u,\nabla u)\ T_k(u)| \le g(x,u,\nabla u)\ u$ and $T_k(u) \to u$ in $W_0^1 L_\varphi(\Omega)$ for the modular convergence and a.e. in Ω as $k \to \infty$, it is easy to pass to the limit in both sides of (3.24) (by using Lebesgue theorem) to obtain

$$\int_\Omega a(x,u,\nabla u)\ \nabla u\,dx + \int_\Omega g(x,u,\nabla u)\ u\,dx = \langle f,u\rangle.$$

This completes the proof of Theorem 7.

Conflict of Interests

The authors declare that there is no conflict of interests.

REFERENCES

[1] R.A. Adams, Sobolev spaces, Academic Press, New York, (1975).

[2] E. Azroul, A. Barbara and H. Hjiaj, Strongly nonlinear $p(x)$-elliptic problems with L^1-data, African Diaspora J. Math. 16 (2014), 1-22.

[3] A. Benkirane, F. Blali, M. Sidi El Vally (Ould Mohamedhen Val), An existence theorem for a strongly nonlinear elliptique problem in Musielak-Orlicz spaces, (Submited).

[4] A. Benkirane, J. Douieb and M. Ould Mohamedhen Val, An approximation theorem in Musielak-Orlicz-Sobolev spaces, Comm. Math. 51 (2011), 109-120.

[5] A. Benkirane, A. Elmahi, Almost everywhere convergence of gradients of solutions to elliptic equations in Orlicz spaces and application, Nonlinear Anal. 28 (1997), 1769-1784.

[6] A. Benkirane, A. Elmahi, An existence theorem for a strongly non-linear elliptic problem in Orlicz spaces, Nonlinear Anal. 36 (1999), 11-24.

[7] A. Benkirane, A. Elmahi, A strongly non-linear elliptic equation having natural growth terms and L^1 data, Nonlinear Anal. 39 (2000), 403-411.

[8] A. Benkirane, A. Elmahi and D. Meskine, An existence theorem for a class of elliptic problems in L^1, Appl. Math. 29 (2002), 439-457.

[9] A. Benkirane, J.-P. Gossez, An approximation theorem for higher order Orlicz-Sobolev spaces, Stud. Math. 92 (1989), 231-255.

[10] A. Benkirane, M. Sidi El Vally (Ould Mohamedhen Val), An existence result for nonlinear elliptic equations in Musielak-Orlicz-Sobolev spaces, Bull. Belg. Math. Soc. Simon Stevin, 20 (2013), 57-75.

[11] A. Benkirane, M. Sidi El Vally (Ould Mohamedhen Val), Some approximation properties in Musielak-Orlicz-Sobolev spaces, Thai.J. Math. 10 (2012), 371-381.

[12] A. Benkirane, M. Sidi El Vally (Ould Mohamedhen val), Variational inequalities in Musielak-Orlicz-Sobolev spaces, to appear in Bull. Belg. Math. Soc. Simon Stevin.

[13] A. Bensoussan, L. Boccardo and F. Murat, On a nonlinear partial differential equation having natural growth terms and unbounded solution, Ann. Inst. Henri Poincare, 5 (1988), pp.347-364.

[14] H. Brzis, Analyse fonctionnelle. Thorie et applications, Dunod, Paris, (1999).

[15] A. Elmahi, D. Meskine, Existence of solutions for elliptic equations having natural growth terms in Orlicz spaces, Abstr. Appl. Anal. 12 (2004), 1031-1045.

[16] J.-P. Gossez, A strongly non-linear elliptic problem in Orlicz-Sobolev spaces, Proc. Am. Math. Soc. Symp. Pure Math. 45 (1986), 455-462.

[17] J.-P. Gossez, Nonlinear elliptic boundary value problems for equations with rapidly (or slowly) increasing coefficients, Trans. Am. Math. Soc. 190 (1974), 163-205.

[18] J.-P. Gossez, Some approximation properties in Orlicz-Sobolev spaces, Stud. Math. 74 (1982), 17-24.

[19] J.-P. Gossez, V. Mustonen, Variational inequalities in Orlicz-Sobolev spaces, Nonlinear Anal. 11 (1987), 379-392.

[20] P. Gwiazda, A. Świerczewska-Gwiazda, On non-newtonian fluids with the property of rapid thickening under different stimulus, Math. Models Methods Appl. Sci. 18 (2008), 1073-1092.

[21] M.A. Krasnosel'skiĭ, Ya.B. Rutickiĭ, convex functions and Orlicz spaces, Noordhoff, Groningen, (1961).

[22] J. Leray, J.-L. Lions, Quelques rsulatats de Višik sur les problmes elliptiques nonlinaires par les mthodes de Minty-Browder, Bull. Soc. Math. France 93 (1965), 97-107.

[23] J. Musielak, Modular spaces and Orlicz spaces, Lecture Notes in Math. (1983).

[24] M. Sidi El Vally, Strongly nonlinear elliptic problems in Musielak-Orlicz-Sobolev spaces, Adv. Dyn. Sys. Appl. 8 (2013), 115-124.