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RELATIONSHIPS BETWEEN ALEXANDROV (FUZZY) TOPOLOGIES AND UPPER APPROXIMATION OPERATORS

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Abstract. In this paper, we investigate the properties of Alexandrov (fuzzy) topologies, fuzzy preorders and upper approximation operators in complete residuated lattices. Moreover, we investigate the relations among Alexandrov (fuzzy) topologies, fuzzy preorders and upper approximation operators. We give their examples.

Keywords: complete residuated lattices; fuzzy preorder; upper approximation operators; Alexander (fuzzy) topologies

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1. Introduction

Hájek [2] introduced a complete residuated lattice which is an algebraic structure for many valued logic. Pawlak [11,12] introduced the rough set theory as a formal tool to deal with imprecision and uncertainty in the data analysis. By using the concepts of lower and upper approximation operators, information systems and decision rules are investigated in complete

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residuated lattices [1,2,9,10,13,14]. Kim [6,7] investigated the properties of Alexandrov (fuzzy) topologies, fuzzy preorders and join-preserving maps in complete residuated lattices.

In this paper, we investigate the properties of Alexandrov (fuzzy) topologies, fuzzy preorders and upper approximation operators in complete residuated lattices. Moreover, we investigate the relations among Alexandrov (fuzzy) topologies, fuzzy preorders and upper approximation operators. We give their examples.

2. Preliminaries

Definition 2.1. [1-3] A structure $(L,\vee,\wedge,\odot,\to,\perp,\top)$ is called a *complete residuated lattice* iff it satisfies the following properties:

(L1) $(L,\vee,\wedge,\perp,\top)$ is a complete lattice where $\perp$ is the bottom element and $\top$ is the top element;

(L2) $(L,\odot,\top)$ is a monoid;

(L3) It has an adjointness, *i.e.*,

$$x \le y \to z \text{ iff } x \odot y \le z.$$

An operator $^*: L \to L$ defined by $a^* = a \to \perp$ is called *strong negations* if $a^{**} = a$.

$$\top_x(y) = \begin{cases} \top, & \text{if } y = x, \\ \perp, & \text{otherwise.} \end{cases} \quad \top^*_x(y) = \begin{cases} \perp, & \text{if } y = x, \\ \top, & \text{otherwise.} \end{cases}$$

In this paper, we assume that $(L,\vee,\wedge,\odot,\to,^*,\perp,\top)$ be a complete residuated lattice with a strong negation *.

Definition 2.2. [6,7] Let X be a set. A function $R_X : X \times X \to L$ is called a *fuzzy preorder* if it satisfies the following conditions

(E1) reflexive if $R_X(x,x) = 1$ for all $x \in X$,

(E2) transitive if $R_X(x,y) \odot R_X(y,z) \le R_X(x,z)$, for all $x,y,z \in X$'

Lemma 2.3. [1,2] *Let* $(L,\vee,\wedge,\odot,\to,^*,\perp,\top)$ *be a complete residuated lattice with a strong negation* *. *For each* $x,y,z,x_i,y_i \in L$, *the following properties hold.*

(1) *If* $y \le z$, *then* $x \odot y \le x \odot z$.

(2) *If* $y \le z$, *then* $x \to y \le x \to z$ *and* $z \to x \le y \to x$.

(3) $x \to y = \top$ *iff* $x \le y$.

(4) $x \to \top = \top$ *and* $\top \to x = x$.

(5) $x \odot y \le x \wedge y$.

(6) $x \odot (\bigvee_{i\in\Gamma} y_i) = \bigvee_{i\in\Gamma}(x \odot y_i)$ *and* $(\bigvee_{i\in\Gamma} x_i) \odot y = \bigvee_{i\in\Gamma}(x_i \odot y)$.

(7) $x \to (\bigwedge_{i\in\Gamma} y_i) = \bigwedge_{i\in\Gamma}(x \to y_i)$ *and* $(\bigvee_{i\in\Gamma} x_i) \to y = \bigwedge_{i\in\Gamma}(x_i \to y)$.

(8) $\bigvee_{i\in\Gamma} x_i \to \bigvee_{i\in\Gamma} y_i \ge \bigwedge_{i\in\Gamma}(x_i \to y_i)$ *and* $\bigwedge_{i\in\Gamma} x_i \to \bigwedge_{i\in\Gamma} y_i \ge \bigwedge_{i\in\Gamma}(x_i \to y_i)$.

(9) $(x \to y) \odot x \le y$ *and* $(y \to z) \odot (x \to y) \le (x \to z)$.

(10) $x \to y \le (y \to z) \to (x \to z)$ *and* $x \to y \le (z \to x) \to (z \to y)$.

(11) $\bigwedge_{i\in\Gamma} x_i^* = (\bigvee_{i\in\Gamma} x_i)^*$ *and* $\bigvee_{i\in\Gamma} x_i^* = (\bigwedge_{i\in\Gamma} x_i)^*$.

(12) $(x \odot y) \to z = x \to (y \to z) = y \to (x \to z)$ *and* $(x \odot y)^* = x \to y^*$.

(13) $x^* \to y^* = y \to x$ *and* $(x \to y)^* = x \odot y^*$.

(14) $y \to z \le x \odot y \to x \odot z$.

Definition 2.4. [5] A map $\mathscr{H} : L^X \to L^Y$ is called an *upper approximation operator* if it satisfies the following conditions, for all $A, A_i \in L^X$, and $\alpha \in L$,

(H1) $\mathscr{H}(\alpha \odot A) = \alpha \odot \mathscr{H}(A)$,

(H2) $\mathscr{H}(\bigvee_{i\in I} A_i) = \bigvee_{i\in I} \mathscr{H}(A_i)$,

(H3) $A \le \mathscr{H}(A)$,

(H4) $\mathscr{H}(\mathscr{H}(A)) \le \mathscr{H}(A)$.

Example 2.5. Let $R \in L^{X\times X}$ be a fuzzy preorder. Define $\mathscr{H}_R : L^X \to L^X$ as follows

$$\mathscr{H}_R(A)(y) = \bigvee_{x\in X} (A(x) \odot R(x,y)).$$

Since $\mathscr{H}_R(\alpha \odot A) = \alpha \odot \mathscr{H}_R(A)$ and $\mathscr{H}_R(\bigvee_{i\in\Gamma} A_i) = \bigvee_{i\in\Gamma} \mathscr{H}_R(A_i)$,

$$\begin{aligned}
&\mathscr{H}_R(A)(y) \ge A(y) \odot R(y,y) = A(y),\\
&\mathscr{H}_R(\mathscr{H}_R(A))(z) = \textstyle\bigvee_{y\in X}(\mathscr{H}_R(A)(y) \odot R(y,z))\\
&= \textstyle\bigvee_{y\in X}((\bigvee_{x\in X}(A(x) \odot R(x,y))) \odot R(y,z))\\
&\le \textstyle\bigvee_{x\in X}(A(x) \odot R(x,z)) = \mathscr{H}_R(A)(z).
\end{aligned}$$

then $\mathscr{H}_R$ is an upper approximation operator.

Definition 2.6. [5] A subset $\tau \subset L^X$ is called an *Alexandrov topology* if it satisfies satisfies the following conditions.

(O1) $\alpha_X \in \tau$ where $\alpha_X(x) = \alpha$ for each $x \in X$ and $\alpha \in L$.

(O2) If $A_i \in \tau$ for $i \in \Gamma$, $\bigvee_{i\in\Gamma} A_i, \bigwedge_{i\in\Gamma} A_i \in \tau$.

(O3) $\alpha \odot A \in \tau$ for all $\alpha \in L$ and $A \in \tau$.

(O4) $\alpha \to A \in \tau$ for all $\alpha \in L$ and $A \in \tau$.

Definition 2.7. [4,6] An operator $\mathbf{T} : L^X \to L$ is called an *Alexandrov fuzzy topology* on X iff it satisfies the following conditions, for all $A, A_i \in L^X$, and $\alpha \in L$,

(T1) $\mathbf{T}(\alpha_X) = \top$,

(T2) $\mathbf{T}(\bigwedge_{i\in\Gamma} A_i) \geq \bigwedge_{i\in\Gamma} \mathbf{T}(A_i)$ and $\mathbf{T}(\bigvee_{i\in\Gamma} A_i) \geq \bigwedge_{i\in\Gamma} \mathbf{T}(A_i)$,

(T3) $\mathbf{T}(\alpha \odot A) \geq \mathbf{T}(A)$,

(T4) $\mathbf{T}(\alpha \to A) \geq \mathbf{T}(A)$.

Theorem 2.8. [6] *Let $\mathscr{H}$ be an upper approximation operator. Define $\mathbf{T}_{\mathscr{H}} : L^X \to L$ as*

$$\mathbf{T}_{\mathscr{H}}(A) = \bigwedge_{x\in X} (\mathscr{H}(A)(x) \to A(x)).$$

Then we have the following properties.

(1) *$\mathbf{T}_{\mathscr{H}}$ is an Alexander fuzzy topology on X.*

(2) *$\mathbf{T}_{\mathscr{H}}(A) = \bigwedge_{x,y\in X} (\mathscr{H}(\top_x)(y) \to (A(x) \to A(y))$ such that $\mathbf{T}_{\mathscr{H}}(A) \geq \bigwedge_{x\neq y\in X} \mathscr{H}^*(\top_x)(y)$.*

(3) *$\mathbf{T}_{\mathscr{H}}(\mathscr{H}(\top_x)) = \top$.*

(4) *If $\mathscr{H}^{-1}$ is an upper approximation operator such that $\mathscr{H}^{-1}(\top_x)(y) = \mathscr{H}(\top_y)(x)$ for all $x, y \in X$. Define $\mathbf{T}^*_{\mathscr{H}}(A) = \mathbf{T}_{\mathscr{H}}(A^*)$. Then $\mathbf{T}^*_{\mathscr{H}} = \mathbf{T}_{\mathscr{H}^{-1}}$ is an Alexander fuzzy topology.*

(5) *$\mathbf{T}_{\mathscr{H}}(\mathscr{H}^{-1*}(\top_x)) = \mathbf{T}_{\mathscr{H}^{-1}}(\mathscr{H}^*(\top_x)) = \top$.*

3. Relationships between Alexandrov (fuzzy) topologies and upper approximation operators

Theorem 3.1. *Let $\mathscr{H}, \mathscr{H}^{-1} : L^X \to L^X$ be upper approximation operators such that $\mathscr{H}^{-1}(\top_x)(y) = \mathscr{H}(\top_y)(x)$ for all $x, y \in X$. Then the following properties hold.*

(1) $\tau_{\mathscr{H}} = \{A \in L^X \mid \mathscr{H}(A) = A\}$ *is an Alexandrov topology on* X *such that* $\tau_{\mathscr{H}} = \{\mathscr{H}(A) \mid A \in L^X\}$.

(2) *For each* $A \in L^X$, $\mathscr{H}^{-1}(A) = A$ *iff* $\mathscr{H}(A^*) = A^*$. *Moreover,* $\tau_{\mathscr{H}^{-1}} = \tau^*_{\mathscr{H}} = \{A^* \in L^X \mid \mathscr{H}(A) = A\}$.

(3) *Define* $R_{\mathscr{H}} : X \times X \to L$ *as* $R_{\mathscr{H}}(x,y) = \mathscr{H}(\top_x)(y)$. *Then* $R_{\mathscr{H}}$ *is a fuzzy preorder such* $\mathscr{H}_{R_{\mathscr{H}}} = \mathscr{H}$ *and* $R_{\tau_{\mathscr{H}}} = R_{\mathscr{H}}$.

(4) $R_{\mathscr{H}^{-1}}(x,y) = R_{\mathscr{H}}(y,x) = \mathscr{H}(\top_y)(x)$, $\mathscr{H}_{R_{\mathscr{H}^{-1}}} = \mathscr{H}^{-1}$ *and* $R_{\tau^*_{\mathscr{H}}} = R_{\tau_{\mathscr{H}^{-1}}} = R_{\mathscr{H}^{-1}}$.

Proof. (1) (O1) Since $\alpha_X \le \mathscr{H}(\alpha_X)$ and $\mathscr{H}(\alpha_X) = \mathscr{H}(\alpha \odot \top) = \alpha \odot \top = \alpha_X$, then $\alpha_X \in \tau_{\mathscr{H}}$.

(O2) For $A_i \in \tau_{\mathscr{H}}$ for each $i \in \Gamma$, by (H3), $\bigvee_{i\in\Gamma} A_i \in \tau_{\mathscr{H}}$. Since $\bigwedge_{i\in\Gamma} A_i \le \mathscr{H}(\bigwedge_{i\in\Gamma} A_i) \le \bigwedge_{i\in\Gamma} \mathscr{H}(A_i) = \bigwedge_{i\in\Gamma} A_i$, Thus, $\bigwedge_{i\in\Gamma} A_i \in \tau_{\mathscr{H}}$.

(O3) For $A \in \tau_{\mathscr{H}}$, by (H2), $\alpha \odot A \in \tau_{\mathscr{H}}$.

(O4) For $A \in \tau_{\mathscr{H}}$, since $\alpha \odot \mathscr{H}(\alpha \to A) = \mathscr{H}(\alpha \odot (\alpha \to A)) \le \mathscr{H}(A)$, $\mathscr{H}(\alpha \to A) \le \alpha \to \mathscr{H}(A) = \alpha \to A$. Then $\alpha \to A \in \tau_{\mathscr{H}}$. Hence $\tau_{\mathscr{H}}$ is an Alexandrov topology on X. Let $A \in \tau_{\mathscr{H}}$. Then $A = \mathscr{H}(A) \in \{\mathscr{H}(A) \mid A \in L^X\}$. Let $\mathscr{H}(A) \in \{\mathscr{H}(A) \mid A \in L^X\}$. Since $\mathscr{H}(\mathscr{H}(A)) = \mathscr{H}(A)$, $\mathscr{H}(A) \in \tau_{\mathscr{H}}$.

(2)

$$\begin{aligned}
&\mathscr{H}(A^*) = A^* \\
&\text{iff } \mathscr{H}(A^*)(y) = \textstyle\bigvee_{x\in X}(A^*(x) \odot \mathscr{H}(\top_x)(y)) \le A^*(y) \\
&\text{iff } A(y) \le \textstyle\bigwedge_{x\in X}(\mathscr{H}(\top_x)(y) \to A(x)) \\
&\text{iff } \textstyle\bigvee_{y\in X} A(y) \odot \mathscr{H}^{-1}(\top_y)(x) \le A(x) \\
&\text{iff } \mathscr{H}^{-1}(A)(x) \le A(x) \\
&\text{iff } \mathscr{H}^{-1}(A) = A.
\end{aligned}$$

(3) Since $R_{\mathscr{H}}(x,x) = \mathscr{H}(\top_x)(x) \ge \top_x(x) = \top$ and

$$\begin{aligned}
&\textstyle\bigvee_{y\in X}(R_{\mathscr{H}}(x,y) \odot R_{\mathscr{H}}(y,z)) = \bigvee_{y\in X}(\mathscr{H}(\top_x)(y) \odot \mathscr{H}(\top_y)(z)) \\
&= \mathscr{H}(\textstyle\bigvee_{y\in X}(\mathscr{H}(\top_x)(y) \odot \top_y)(z)) = \mathscr{H}(\mathscr{H}(\top_x))(z) \\
&\le \mathscr{H}(\top_x)(z) = R_{\mathscr{H}}(x,z),
\end{aligned}$$

then $R_{\mathscr{H}}$ is a fuzzy preorder. Moreover,

$$\begin{aligned}\mathscr{H}_{R_{\mathscr{H}}}(A)(y) &= \bigvee_{x\in X}(A(x)\odot R_{\mathscr{H}}(x,y)))\\ &= \bigvee_{x\in X}(A(x)\odot \mathscr{H}(\top_x)(y))\\ &= \mathscr{H}(\bigvee_{x\in X}(A(x)\odot \top_x))(y) = \mathscr{H}(A)(y),\end{aligned}$$

$$\begin{aligned}R_{\tau_{\mathscr{H}}}(x,y) &= \bigwedge_{A\in\tau_{\mathscr{H}}}(A(x)\to A(y))\\ &\le \bigwedge_{z\in X}(\mathscr{H}(\top_z)(x)\to \mathscr{H}(\top_z)(y))\\ &\le \mathscr{H}(\top_x)(x)\to \mathscr{H}(\top_x)(y))\\ &\le (\top_x)(x)\to \mathscr{H}(\top_x)(y) = \mathscr{H}(\top_x)(y),\end{aligned}$$

$$\begin{aligned}R_{\tau_{\mathscr{H}}}(x,y) &= \bigwedge_{A\in\tau_{\mathscr{H}}}(A(x)\to A(y))\\ &= \bigwedge_{A\in L^X}(\mathscr{H}(A)(x)\to \mathscr{H}(A)(y))\\ &= \bigwedge_{A\in L^X}(\bigvee_{z\in X}(A(z)\odot \mathscr{H}(\top_z)(x))\to \bigvee_{z\in X}(A(z)\odot \mathscr{H}(\top_z)(y)))\\ &\ge \bigwedge_{z\in X}(\mathscr{H}(\top_z)(x)\to \mathscr{H}(\top_z)(y))\\ &\ge \mathscr{H}(\top_x)(y).\end{aligned}$$

Hence $R_{\tau_{\mathscr{H}}}(x,y) = \mathscr{H}(\top_x)(y) = R_{\mathscr{H}}(x,y)$.

(4) It is similarly proved as (3).

Theorem 3.2. *Let τ be Alexandrov topology on X. Then the following properties hold.*

(1) *Define $\mathscr{H}_\tau : L^X \to L^X$ as follows:*

$$\mathscr{H}_\tau(A) = \bigwedge\{B \mid A \le B,\ B\in\tau\}.$$

Then $\mathscr{H}_\tau$ is an L-upper approximation operator such that $\tau_{\mathscr{H}_\tau} = \tau$, $\mathscr{H}_{\tau_{\mathscr{H}}} = \mathscr{H}$.

(2) *Define $R_\tau : X\times X \to L$ as*

$$R_\tau(x,y) = \bigwedge_{A\in\tau}(A(x)\to A(y))$$

Then R_τ is a fuzzy preorder such that $\tau = \tau_{\mathscr{H}_{R_\tau}}$. Moreover, $\mathscr{H}_\tau = \mathscr{H}_{R_\tau}$.

(3) *$R_{\tau^*} = R_\tau^{-1}$, $\tau^* = \tau_{\mathscr{H}_{R_{\tau^*}}} = \tau_{\mathscr{H}_{R_\tau^{-1}}}$ and $\mathscr{H}_{\tau^*} = \mathscr{H}_{R_\tau^{-1}}$.*

Proof. (1) We show $\mathscr{H}_\tau(A) = \bigwedge\{B \mid A\le B,\ B\in\tau\}$ is an L-upper approximation operator.

(H1) We have $\alpha \odot \mathscr{H}_\tau(A) \leq \mathscr{H}_\tau(\alpha \odot A)$ from:

$$\begin{aligned}
&\alpha \to \mathscr{H}_\tau(\alpha \odot A)\\
&= \alpha \to \bigwedge\{B \mid \alpha \odot A \leq B,\ B \in \tau\}\\
&= \bigwedge\{\alpha \to B \mid A \leq \alpha \to B,\ \alpha \to B \in \tau\}\\
&\geq \mathscr{H}_\tau(A).
\end{aligned}$$

Since $\alpha \odot A \leq \alpha \odot \mathscr{H}_\tau(A)$ and $\alpha \odot \mathscr{H}_\tau(A) \in \tau$, then $\mathscr{H}_\tau(\alpha \odot A) \leq \alpha \odot \mathscr{H}_\tau(A)$. Hence $\mathscr{H}_\tau(\alpha \odot A) = \alpha \odot \mathscr{H}_\tau(A)$.

(H2) Since $\mathscr{H}_\tau(A) \leq \mathscr{H}_\tau(B)$ for $A \leq B$, we have $\bigvee_{i\in\Gamma} \mathscr{H}_\tau(A_i) \leq \mathscr{H}_\tau(\bigvee_{i\in\Gamma} A_i)$. Since $\bigvee_{i\in\Gamma} A_i \leq \bigvee_{i\in\Gamma} \mathscr{H}_\tau(A_i) \in \tau$, then

$$\mathscr{H}_\tau(\bigvee_{i\in\Gamma} A_i) \leq \mathscr{H}_\tau(\bigvee_{i\in\Gamma} \mathscr{H}_\tau(A_i)) = \bigvee_{i\in\Gamma} \mathscr{H}_\tau(A_i).$$

(H3) It follows from the definition.

(H4) Since $\mathscr{H}_\tau(A) \in \tau$, we have $\mathscr{H}_\tau(\mathscr{H}_\tau(A)) = \mathscr{H}_\tau(A)$.

Let $A \in \tau_{\mathscr{H}_\tau}$. Then $A = \mathscr{H}_\tau(A) \in \tau$. Hence $\tau_{\mathscr{H}_\tau} \subset \tau$.

Let $A \in \tau$. Then $\mathscr{H}_\tau(A) = A$. So, $A \in \tau_{\mathscr{H}_\tau}$. Hence $\tau \subset \tau_{\mathscr{H}_\tau}$.

Since $\mathscr{H}_{\tau_{\mathscr{H}}}(A) = \bigwedge\{B \mid A \leq B,\ B \in \tau_{\mathscr{H}}\}$ and $A \leq \mathscr{H}(\mathscr{H}(A)) = \mathscr{H}(A)$, we have $\mathscr{H}_{\tau_{\mathscr{H}}}(A) \leq \mathscr{H}(A)$. For $B_i \in \tau_{\mathscr{H}}$, since $\mathscr{H}(\bigwedge_{i\in\Gamma} B_i) \leq \bigwedge_{i\in\Gamma} \mathscr{H}(B_i) = \bigwedge_{i\in\Gamma} B_i$, then $\mathscr{H}(\mathscr{H}_{\tau_{\mathscr{H}}}(A)) = \mathscr{H}_{\tau_{\mathscr{H}}}(A)$. So, $\mathscr{H}(A) \leq \mathscr{H}_{\tau_{\mathscr{H}}}(A)$.

(2) We easily show that R_τ is a fuzzy preorder.

Let $A \in \tau$. Since $\mathscr{H}_{R_\tau}(A)(y) = \bigvee_{x\in X}(A(x) \odot R_\tau(x,y)) = \bigvee_{x\in X}(A(x) \odot \bigwedge_{B\in\tau}(B(x) \to B(y)) \leq \bigvee_{x\in X}(A(x) \odot (A(x) \to A(y)) \leq A(y)$, then $\mathscr{H}_{R_\tau}(A) = A$. So, $\tau \subset \tau_{\mathscr{H}_{R_\tau}}$

Let $A = \mathscr{H}_{R_\tau}(A)$. Then $A = \mathscr{H}_{R_\tau}(A) = \bigvee_{x\in X}(A(x) \odot \bigwedge_{B\in\tau}(B(x) \to B)) \in \tau$. So, $\tau_{\mathscr{H}_{R_\tau}} \subset \tau$.

Since $A \leq \mathscr{H}_{R_\tau}(A) \in \tau$, then $\mathscr{H}_\tau(A) \leq \mathscr{H}_{R_\tau}(A)$. Since $\mathscr{H}_{R_\tau}(A)(y) = \bigvee_{x\in X}(A(x) \odot \bigwedge_{B\in\tau}(B(x) \to B(y)) \leq \bigvee_{x\in X}(\mathscr{H}_\tau(A)(x) \odot (\mathscr{H}_\tau(A)(x) \to \mathscr{H}_\tau(A)(y)) \leq \mathscr{H}_\tau(A)(y)$, then $\mathscr{H}_{R_\tau}(A) \leq \mathscr{H}_\tau(A)$.

(3) $R_{\tau^*}(x,y) = \bigwedge_{A\in\tau^*}(A(x) \to A(y)) = \bigwedge_{A\in\tau}(A^*(x) \to A^*(y)) = \bigwedge_{A\in\tau}(A(y) \to A(x)) = R_\tau^{-1}(x,y)$. Other cases are similarly proved as (3).

Theorem 3.3. *Let $\mathscr{H}_X, \mathscr{H}_X^{-1} : L^X \to L^X$ be upper approximation operators such that $\mathscr{H}_X^{-1}(\top_x)(y) = \mathscr{H}_X(\top_y)(x)$ for all $x, y \in X$. Let $\mathscr{H}_Y, \mathscr{H}_Y^{-1} : L^Y \to L^Y$ be upper approximation operators such* *that* $\mathscr{H}_Y^{-1}(\top_a)(b) = \mathscr{H}_Y(\top_b)(a)$ *for all* $a, b \in Y$. *Let* $f : (X, \mathscr{H}_X) \to (Y, \mathscr{H}_Y)$ *be a map. Then the following statements are equivalent.*

(1) $\mathscr{H}_X(\top_x) \le f^{-1}(\mathscr{H}_Y(\top_{f(x)}))$ *for all* $x \in X$.

(2) $\mathscr{H}_X^{-1}(\top_x)) \le f^{-1}(\mathscr{H}_Y^{-1}(\top_{f(x)}))$ *for all* $x \in X$.

(3) $R_{\mathscr{H}_X}(x,y) \le R_{\mathscr{H}_Y}(f(x), f(y))$ *for all* $x, y \in X$.

(4) $R_{\mathscr{H}_X^{-1}}(x,y) \le R_{\mathscr{H}_Y^{-1}}(f(x), f(y))$ *for all* $x, y \in X$.

(5) $f(\mathscr{H}_X(A)) \le \mathscr{H}_Y(f(A))$ *for all* $A \in L^X$.

(6) $f(\mathscr{H}_X^{-1}(A)) \le \mathscr{H}_Y^{-1}(f(A))$ *for all* $A \in L^X$.

(7) $\mathscr{H}_X(f^{-1}(B)) \le f^{-1}(\mathscr{H}_Y(B))$ *for all* $B \in L^Y$.

(8) $\mathscr{H}_X^{-1}(f^{-1}(B)) \le f^{-1}(\mathscr{H}_Y^{-1}(B))$ *for all* $B \in L^Y$.

(9) $f^{-1}(B) \in \tau_{\mathscr{H}_X}$ *for all* $B \in \tau_{\mathscr{H}_Y}$.

(10) $f^{-1}(B) \in \tau_{\mathscr{H}_X^{-1}}$ *for all* $B \in \tau_{\mathscr{H}_Y^{-1}}$.

(11) $\mathbf{T}_{\mathscr{H}_X}(f^{-1}(B)) \ge \mathbf{T}_{\mathscr{H}_Y}(B)$ *for all* $B \in L^Y$.

(12) $\mathbf{T}_{\mathscr{H}_X^{-1}}(f^{-1}(B)) \ge \mathbf{T}_{\mathscr{H}_Y^{-1}}(B)$ *for all* $B \in L^Y$.

Proof (1) $\Leftrightarrow$ (3) From Theorem 3.1 (3), it follows from:

$$R_{\mathscr{H}_X}(x,y) = \mathscr{H}_X(\top_x)(y) \le R_{\mathscr{H}_Y}(f(x), f(y)) = f^{-1}(\mathscr{H}_Y(\top_{f(x)}))(y).$$

(1) $\Rightarrow$ (5)

$$\begin{aligned}
\mathscr{H}_Y(f(A))(y) &= \mathscr{H}_Y(\textstyle\bigvee_{x\in X} f(A)(f(x)) \odot \top_{f(x)})(y) \\
&= \textstyle\bigvee_{x\in X}(A(x) \odot \mathscr{H}_Y(\top_{f(x)})(y) \\
&\ge \textstyle\bigvee_{x\in X}(A(x) \odot f(\mathscr{H}_X(\top_x))(y) \\
&= \textstyle\bigvee_{x\in X}(A(x) \odot \bigvee_{z\in f^{-1}(\{y\})} \mathscr{H}_X(\top_x)(z)) \\
&= \textstyle\bigvee_{z\in f^{-1}(\{y\})}(\bigvee_{x\in X}(A(x) \odot \mathscr{H}_X(\top_x)(z)) \\
&= \textstyle\bigvee_{z\in f^{-1}(\{y\})} \mathscr{H}_X(\bigvee_{x\in X}(A(x) \odot \top_x))(z) \\
&= \textstyle\bigvee_{z\in f^{-1}(\{y\})} \mathscr{H}_X(A)(z) = f(\mathscr{H}_X(A))(y).
\end{aligned}$$

(5) $\Rightarrow$ (7) By (5), put $A = f^{-1}(B)$. Then

$$\begin{aligned}
&f(\mathscr{H}_X(f^{-1}(B))) \le \mathscr{H}_Y(f(f^{-1}(B))) \le \mathscr{H}_Y(B) \\
&\text{iff} \quad \mathscr{H}_X(f^{-1}(B)) \le f^{-1}(\mathscr{H}_Y(B)).
\end{aligned}$$

(7) ⇒ (9) For $B \in \tau_{\mathscr{H}_Y}$, since $\mathscr{H}_Y(B) = B$, by (7), $\mathscr{H}_X(f^{-1}(B)) \leq f^{-1}(\mathscr{H}_Y(B)) = f^{-1}(B)$. So, $f^{-1}(B) \in \tau_{\mathscr{H}_X}$.

(9) ⇒ (5) Since $\mathscr{H}_Y = \mathscr{H}_{\tau_{\mathscr{H}_Y}}$ and $\mathscr{H}_X = \mathscr{H}_{\tau_{\mathscr{H}_X}}$, we have

$$\begin{aligned}
\mathscr{H}_Y(f(A)) &= \bigwedge\{B \mid f(A) \leq B,\ B \in \tau_{\mathscr{H}_Y}\} \\
&\geq \bigwedge\{B \mid A \leq f^{-1}(B),\ f^{-1}(B) \in \tau_{\mathscr{H}_X}\} \\
&\geq \bigwedge\{f(f^{-1}(B)) \mid A \leq f^{-1}(B),\ f^{-1}(B) \in \tau_{\mathscr{H}_X}\} \\
&\geq f\Big(\bigwedge\{f^{-1}(B) \mid A \leq f^{-1}(B),\ f^{-1}(B) \in \tau_{\mathscr{H}_X}\}\Big) \\
&\geq f(\mathscr{H}_X(A)).
\end{aligned}$$

(7) ⇒ (5)

$$\begin{aligned}
\mathbf{T}_{\mathscr{H}_X}(f^{-1}(B)) &= \bigwedge_{x\in X}(\mathscr{H}_X(f^{-1}(B))(x) \to f^{-1}(B)(x) \\
&\geq \bigwedge_{x\in X}(f^{-1}(\mathscr{H}_Y(B))(x) \to B(f(x)))) \\
&\geq \bigwedge_{y\in Y}(\mathscr{H}_Y(B)(y) \to B(y)) \\
&= \mathbf{T}_{\mathscr{H}_Y}(B).
\end{aligned}$$

(11) ⇒ (9)

For all $B \in \tau_Y$, since $\tau_Y = \tau_{\mathscr{H}_{R_{\tau_Y}}}$, then $\mathscr{H}_{R_{\tau_Y}}(B) = B$. Since $\mathbf{T}_{\mathscr{H}_{\tau_X}}(f^{-1}(B)) \geq \mathbf{T}_{\mathscr{H}_{\tau_Y}}(B) = \top$, then $\mathbf{T}_{\mathscr{H}_{\tau_X}}(f^{-1}(B)) = \top$. So, $f^{-1}(B) \in \tau_X$.

(11) ⇒ (1) Since $a \leq (a \to b) \to b$, we have

$$\begin{aligned}
\mathscr{H}_X(\top_x)(y) &\leq \bigwedge_{A\in L^X}((\mathscr{H}_X(\top_x)(y) \to (A(x) \to A(y)) \to (A(x) \to A(y))) \\
&\leq \bigwedge_{A\in L^X}(\bigwedge_{s,t}((\mathscr{H}_X(\top_s)(t) \to (A(s) \to A(t))) \to (A(x) \to A(y))) \\
&= \bigwedge_{A\in L^X}(\mathbf{T}_{\mathscr{H}_X}(A) \to (A(x) \to A(y))).
\end{aligned}$$

Since $\mathbf{T}_{\mathscr{H}_X}(\mathscr{H}_X(\top_x)) = \bigwedge_{y\in Y}(\mathscr{H}_X(\mathscr{H}_X(\top_x))(y) \to \mathscr{H}_X(\top_x)(y)) = \top$, we have

$$\begin{aligned}
&\mathscr{H}_X(\top_x)(y) \leq \bigwedge_{A\in L^X}(\mathbf{T}_{\mathscr{H}_X}(A) \to (A(x) \to A(y))) \\
&\leq \bigwedge_{z\in X}(\mathbf{T}_{\mathscr{H}_X}(\mathscr{H}_X(\top_z)) \to (\mathscr{H}_X(\top_z)(x) \to \mathscr{H}_X(\top_z)(y))) \\
&= \bigwedge_{z\in X}(\mathscr{H}_X(\top_z)(x) \to \mathscr{H}_X(\top_z)(y)) \leq \top_x(x) \to \mathscr{H}_X(\top_x)(y) = \mathscr{H}_X(\top_x)(y).
\end{aligned}$$

$$\begin{aligned}
&f^{-1}(\mathscr{H}_Y(\top_{f(x)}))(z) = \mathscr{H}_Y(\top_{f(x)})(f(z)) \\
&= \bigwedge_{B\in L^Y}(\mathbf{T}_{\mathscr{H}_Y}(B) \to (B(f(x)) \to B(f(z)))) \\
&\geq \bigwedge_{B\in L^Y}(\mathbf{T}_{\mathscr{H}_X}(f^{-1}(B)) \to (f^{-1}(B)(x) \to f^{-1}(B)(z))) \\
&\geq \bigwedge_{A\in L^X}(\mathbf{T}_{\mathscr{H}_X}(A) \to (A(x) \to A(z))) = \mathscr{H}_X(\top_x)(z).
\end{aligned}$$

Hence $\mathscr{H}_Y(\top_{f(x)}) \geq f(f^{-1}(\mathscr{H}_Y(\top_{f(x)}))) \geq f(\mathscr{H}_X(\top_x))$.

(1) $\Leftrightarrow$ (2)

For all $x, z \in X$,

$$\begin{aligned} &\mathscr{H}_X(\top_x))(z) \leq f^{-1}(\mathscr{H}_Y(\top_{f(x)}))(z) = \mathscr{H}_Y(\top_{f(x)}))(f(z)) \\ \text{iff} \quad &\mathscr{H}_X^{-1}(\top_z))(x) \leq \mathscr{H}_Y^{-1}(\top_{f(z)}))(f(x)) = f^{-1}(\mathscr{H}_Y^{-1}(\top_{f(z)}))(x). \end{aligned}$$

Other cases are similarly proved.

Theorem 3.4. *Let τ_X and τ_Y be Alexandrov topologies. Let $f : (X, \tau_X) \to (Y, \tau_Y)$ be a map. Then the following statements are equivalent.*

(1) $f^{-1}(B) \in \tau_X$ *for all* $B \in \tau_Y$.

(2) $f^{-1}(B) \in \tau_X^*$ *for all* $B \in \tau_Y^*$.

(3) $R_{\tau_X}(x,y) \leq R_{\tau_Y}(f(x), f(y))$ *for all* $x, y \in X$.

(4) $R_{\tau_X^*}(x,y) = R_{\tau_X}^{-1}(y,x) \leq R_{\tau_Y^*}(f(x), f(y)) = R_{\tau_Y^*}^{-1}(f(y), f(x))$ *for all* $x, y \in X$.

(5) $f(\mathscr{H}_{\tau_X}(A)) \leq \mathscr{H}_{\tau_Y}(f(A))$ *for all* $A \in L^X$.

(6) $f(\mathscr{H}_{\tau_X}^{-1}(A)) \leq \mathscr{H}_{\tau_Y}^{-1}(f(A))$ *for all* $A \in L^X$.

(7) $\mathscr{H}_{\tau_X}(f^{-1}(B)) \leq f^{-1}(\mathscr{H}_{\tau_Y}(B))$ *for all* $B \in L^Y$.

(8) $\mathscr{H}_{\tau_X}^{-1}(f^{-1}(B)) \leq f^{-1}(\mathscr{H}_{\tau_Y}^{-1}(B))$ *for all* $B \in L^Y$.

(9) $\mathbf{T}_{\mathscr{H}_{\tau_X}}(f^{-1}(B)) \geq \mathbf{T}_{\mathscr{H}_{\tau_Y}}(B)$ *for all* $B \in L^Y$.

(10) $\mathbf{T}_{\mathscr{H}_{\tau_X}^{-1}}(f^{-1}(B)) \geq \mathbf{T}_{\mathscr{H}_{\tau_Y}^{-1}}(B)$ *for all* $B \in L^Y$.

Proof (1) $\Rightarrow$ (3)

$$\begin{aligned} R_{\tau_Y}(f(x), f(y)) &= \bigwedge_{B \in \tau_Y}(B(f(x)) \to B(f(y))) \\ &= \bigwedge_{B \in \tau_Y}(f^{-1}(B)(x) \to f^{-1}(B)(y)) \\ &\geq \bigwedge_{A \in \tau_X}(A(x) \to A(y)) = R_{\tau_X}(x,y). \end{aligned}$$

(3) $\Rightarrow$ (5) Since $\mathscr{H}_{R_{\tau_Y}} = \mathscr{H}_{\tau_Y}$ and $\mathscr{H}_{R_{\tau_X}} = \mathscr{H}_{\tau_X}$ from Theorem 3.2(2), we have

$$\begin{aligned} \mathscr{H}_{R_{\tau_Y}}(f(A))(f(x)) &= \bigvee_{w \in Y}(f(A)(w) \odot R_{\tau_Y}(w, f(x))) \\ &\geq \bigvee_{z \in X}(f(A)(f(z)) \odot R_{\tau_Y}(f(z), f(x))) \\ &\geq \bigvee_{z \in X}(A(z) \odot R_{\tau_X}(z,x)) = \mathscr{H}_{R_{\tau_X}}(A)(x). \end{aligned}$$

(5) $\Rightarrow$ (7) and (7) $\Rightarrow$ (9) are similarly proved as (5) $\Rightarrow$ (7) and (7) $\Rightarrow$ (9), respectively, in Theorem 3.3.

(9) $\Rightarrow$ (1) For all $B \in \tau_Y$, since $\tau_Y = \tau_{\mathscr{H}_{R_{\tau_Y}}}$ form Theorem 3.2(2), then $\mathscr{H}_{R_{\tau_Y}}(B) = B$. Since $\mathbf{T}_{\mathscr{H}_{\tau_X}}(f^{-1}(B)) \geq \mathbf{T}_{\mathscr{H}_{\tau_Y}}(B) = \top$, then $\mathbf{T}_{\mathscr{H}_{\tau_X}}(f^{-1}(B)) = \top$. So, $f^{-1}(B) \in \tau_X$.

Other cases are similarly proved.

Theorem 3.5. *Let R_X and R_Y be fuzzy preordered sets. Then the following statements are equivalent.*

(1) $R_X(x,y) \leq R_Y(f(x), f(y))$ *for all* $x,y \in X$.

(2) $R_X^{-1}(x,y) \leq R_Y^{-1}(f(x), f(y))$ *for all* $x,y \in X$ *where* $R_X^{-1}(x,y) = R_X(y,x)$.

(3) $f(\mathscr{H}_{R_X}(A)) \leq \mathscr{H}_{R_Y}(f(A))$ *for all* $A \in L^X$ *where* $\mathscr{H}_{R_X}(A)(y) = \bigvee_{x\in X}(A(x) \odot R(x,y))$.

(4) $f(\mathscr{H}_{R_X}^{-1}(A)) \leq \mathscr{H}_{R_Y}^{-1}(f(A))$ *for all* $A \in L^X$ *where* $\mathscr{H}_{R_X}^{-1} = \mathscr{H}_{R_X^{-1}}$.

(5) $\mathscr{H}_{R_X}(f^{-1}(B)) \leq f^{-1}(\mathscr{H}_{R_Y}(B))$ *for all* $B \in L^Y$.

(6) $\mathscr{H}_{R_X}^{-1}(f^{-1}(B)) \leq f^{-1}(\mathscr{H}_{R_Y}^{-1}(B))$ *for all* $B \in L^Y$.

(7) $f^{-1}(B) \in \tau_{\mathscr{H}_{R_X}}$ *for all* $B \in \tau_{\mathscr{H}_{R_Y}}$.

(8) $f^{-1}(B) \in \tau_{\mathscr{H}_{R_X}^{-1}}$ *for all* $B \in \tau_{\mathscr{H}_{R_Y}^{-1}}$.

(9) $\mathbf{T}_{\mathscr{H}_{R_X}}(f^{-1}(B)) \geq \mathbf{T}_{\mathscr{H}_{R_Y}}(B)$ *for all* $B \in L^Y$.

(10) $\mathbf{T}_{\mathscr{H}_{R_X}^{-1}}(f^{-1}(B)) \geq \mathbf{T}_{\mathscr{H}_{R_X}^{-1}}(B)$ *for all* $B \in L^Y$.

Proof (1) $\Rightarrow$ (3)

$$\begin{aligned}\mathscr{H}_{R_Y}(f(A))(f(x)) &= \textstyle\bigvee_{w\in Y}(f(A)(w) \odot R_Y(w, f(x)))\\ &\geq \textstyle\bigvee_{z\in X}(f(A)(f(z)) \odot R_Y(f(z), f(x)))\\ &\geq \textstyle\bigvee_{z\in X}(A(z) \odot R_X(z,x)) = \mathscr{H}_{R_X}(A)(x).\end{aligned}$$

(5) $\Rightarrow$ (7), (7) $\Rightarrow$ (9) and (9) $\Rightarrow$ (11) are similarly proved as (3) $\Rightarrow$ (5), (5) $\Rightarrow$ (7) and (7) $\Rightarrow$ (9), respectively, in Theorem 3.3.

(7) $\Rightarrow$ (3) Put $\mathscr{H}_X(A) = \bigwedge\{B_i \mid A \leq B_i, B_i \in \tau_{\mathscr{H}_{R_X}}\}$. Since $A \leq \mathscr{H}_{R_X}(A) = \mathscr{H}_{R_X}(\mathscr{H}_{R_X}(A))$, then $\mathscr{H}_X(A) \leq \mathscr{H}_{R_X}(A)$.

For $A \leq B_i, B_i \in \tau_{\mathscr{H}_{R_X}}$, since $\mathscr{H}_{R_X}(B_i) = B_i$, $\mathscr{H}_{R_X}(\bigwedge B_i) \leq \bigwedge \mathscr{H}_{R_X}(B_i) = \bigwedge B_i$. Hence

$$\mathscr{H}_{R_X}(A) \leq \mathscr{H}_{R_X}(\mathscr{H}_X(A)) = \mathscr{H}_{R_X}(\bigwedge B_i) = \bigwedge B_i = \mathscr{H}_X(A)$$

Thus $\mathscr{H}_X(A) = \mathscr{H}_{R_X}(A)$.

$$\begin{aligned}\mathscr{H}_{R_Y}(f(A)) &= \bigwedge\{B \mid f(A) \le B,\ B \in \tau_{\mathscr{H}_{R_Y}}\}\\ &\ge \bigwedge\{B \mid A \le f^{-1}(B),\ f^{-1}(B) \in \tau_{\mathscr{H}_{R_X}}\}\\ &\ge \bigwedge\{f(f^{-1}(B)) \mid A \le f^{-1}(B),\ f^{-1}(B) \in \tau_{\mathscr{H}_{R_X}}\}\\ &\ge f\Big(\bigwedge\{f^{-1}(B) \mid A \le f^{-1}(B),\ f^{-1}(B) \in \tau_{\mathscr{H}_{R_X}}\}\Big)\\ &\ge f(\mathscr{H}_{R_Y}(A)).\end{aligned}$$

(9) $\Rightarrow$ (1) Since $a \le (a \to b) \to b$, we have

$$\begin{aligned}R_X(x,y) &\le \textstyle\bigwedge_{A\in L^X}((R_X(x,y) \to (A(x) \to A(y))) \to (A(x) \to A(y)))\\ &\le \textstyle\bigwedge_{A\in L^X}(\bigwedge_{s,t}((R_X(s,t) \to (A(s) \to A(t))) \to (A(x) \to A(y)))\\ &= \textstyle\bigwedge_{A\in L^X}(\mathbf{T}_{\mathscr{H}_{R_X}}(A) \to (A(x) \to A(y))).\end{aligned}$$

Since $\mathscr{H}_{R_X}(R_x)(y) = \bigvee_{z\in X}(R_x(z) \odot R(z,y)) = R(x,y)$ where $R_x(y) = R(x,y)$ for all $x,y \in X$,

$$\mathbf{T}_{\mathscr{H}_{R_X}}(R_x) = \bigwedge_{y\in Y}(\mathscr{H}_{R_X}(R_x)(y) \to R_x(y)) = \bigwedge_{y\in Y}(R_x(y) \to R_x(y)) = \top.$$

$$\begin{aligned}R_X(x,y) &\le \textstyle\bigwedge_{A\in L^X}(\mathbf{T}_{\mathscr{H}_{R_X}}(A) \to (A(x) \to A(y)))\\ &\le \textstyle\bigwedge_{z\in X}(\mathbf{T}_{\mathscr{H}_{R_X}}(R_z) \to (R_z(x) \to R_z(y)))\\ &= \textstyle\bigwedge_{z\in X}(R_z(x) \to R_z(y)) \le \top_x(x) \to R_x(y) = R_X(x,y).\end{aligned}$$

Hence $R_X(x,y) \le \bigwedge_{A\in L^X}(\mathbf{T}_{\mathscr{H}_{R_X}}(A) \to (A(x) \to A(y)))$. Thus,

$$\begin{aligned}R_Y(f(x),f(z)) &= \textstyle\bigwedge_{B\in L^Y}(\mathbf{T}_{\mathscr{H}_{R_Y}}(B) \to (B(f(x)) \to B(f(z))))\\ &\ge \textstyle\bigwedge_{B\in L^Y}(\mathbf{T}_{\mathscr{H}_{R_X}}(f^{-1}(B)) \to (f^{-1}(B)(x) \to f^{-1}(B)(z)))\\ &\ge \textstyle\bigwedge_{A\in L^X}(\mathbf{T}_{\mathscr{H}_{R_X}}(A) \to (A(x) \to A(z))) = R_X(x,z).\end{aligned}$$

Other cases are similarly proved.

Theorem 3.6. *For $B \in L^Y$, we define*

$$R_Y(x,y) = B(x) \to B(y)$$

$$R_X(a,b) = f^{-1}(B)(a) \to f^{-1}(B)(b).$$

Then the following properties hold.

(1) *R_X and R_Y are fuzzy preordered sets such that $R_X(x,y) = R_Y(f(x),f(y))$ for all $x,y \in X$.*

(2) *$\mathscr{H}_{R_Y}(C) = \bigvee_{y\in Y}(C(y) \odot (B(y) \to B))$ is an upper approximation operator on Y.*

(3) $\mathscr{H}_{R_X}(A) = \bigvee_{x\in X}(A(x)\odot(f^{-1}(B)(x)\to f^{-1}(B)))$ *is an upper approximation operator on* X.

(4)

$$\begin{aligned}\tau_{\mathscr{H}_{R_Y}} &= \{\textstyle\bigvee_{y\in Y}(C(y)\odot(B(x)\to B)) \mid C\in L^Y\}\\ &= \{\mathscr{H}_{R_Y}(C)\mid C\in L^Y\}\\ &= \{C\in L^Y \mid C=\mathscr{H}_{R_Y}(C)\}\end{aligned}$$

(5)

$$\begin{aligned}\tau_{\mathscr{H}_{R_X}} &= \{\textstyle\bigvee_{x\in X}(A(x)\odot(f^{-1}(B)(x)\to f^{-1}(B))) \mid A\in L^X\}\\ &= \{\mathscr{H}_{R_X}(A)\mid A\in L^X\}\\ &= \{A\in L^X \mid A=\mathscr{H}_{R_X}(A)\}\end{aligned}$$

(6) $R_Y = R_{\tau_{\mathscr{H}_{R_Y}}}$ *and* $R_X = R_{\tau_{\mathscr{H}_{R_X}}}$.

(7)

$$\begin{aligned}\mathbf{T}_{\mathscr{H}_{R_Y}}(C) &= \textstyle\bigwedge_{x,y\in Y}((B(x)\to B(y))\to(C(x)\to C(y)))\\ \mathbf{T}_{\mathscr{H}_{R_X}}(D) &= \textstyle\bigwedge_{x,y\in X}((f^{-1}(B)(x)\to f^{-1}(B)(y))\to(D(x)\to D(y)))\end{aligned}$$

Proof. (1) $R_Y(f(x),f(y)) = B(f(x))\to B(f(y)) = f^{-1}(B)(x)\to f^{-1}(B)(y) = R_X(x,y)$, for all $x,y\in X$.

(2) and (3) are easily proved as Example 2.5.

(4) and (5). Since $\mathscr{H}_{R_Y}(C) = \bigvee_{y\in Y}(C(y)\odot(B(y)\to B))$ and $\mathscr{H}_{R_X}(A) = \bigvee_{x\in Y}(A(x)\odot(f^{-1}(B)(x)\to f^{-1}(B)))$, by Theorem 3.1(2), the results hold.

(6)

$$\begin{aligned}&R_{\tau_{\mathscr{H}_{R_Y}}}(x,y) = \textstyle\bigwedge_{A\in\tau_{\mathscr{H}_{R_Y}}}(A(x)\to A(y))\\ &= \textstyle\bigwedge_{A\in\tau_{\mathscr{H}_{R_Y}}}(\bigvee_{z\in X}(A(z)\odot(B(z)\to B(x)))\to\bigvee_{z\in X}(A(z)\odot(B(z)\to B(y))))\\ &\geq \textstyle\bigwedge_{A\in\tau_{\mathscr{H}_{R_Y}}}((A(z)\odot(B(z)\to B(x)))\to(A(z)\odot(B(z)\to B(y))))\\ &\geq \textstyle\bigwedge_{A\in\tau_{\mathscr{H}_{R_Y}}}((B(z)\to B(x)))\to(B(z)\to B(y)))\\ &\geq B(x)\to B(y).\end{aligned}$$

Since $B(x)\odot(B(x)\to B)\leq B$ and $B\leq\mathscr{H}_{R_Y}(B)$, then $B=\mathscr{H}_{R_Y}(B)$;i.e. $B\in\tau_{\mathscr{H}_{R_Y}}$. So, $R_{\tau_{\mathscr{H}_{R_Y}}}(x,y)\leq B(x)\to B(y)$. Thus, $R_{\tau_{\mathscr{H}_{R_Y}}}(x,y) = B(x)\to B(y)$.

(7)

$$\begin{aligned}\mathbf{T}_{\mathscr{H}_{R_Y}}(C) &= \bigwedge_{y\in X}(\mathscr{H}_{R_Y}(C)(y)\to C(y))\\ &= \bigwedge_{y\in X}(\bigvee_{x\in X}(C(x)\odot(B(x)\to B(y))\to C(y))\\ &= \bigwedge_{x,y\in X}((B(x)\to B(y))\to(C(x)\to C(y)).\end{aligned}$$

Example 3.7. Let $(L=[0,1],\odot,\to,^*)$ be a complete residuated lattice with a strong negation defined by

$$x\odot y=(x+y-1)\vee 0,\ \ x\to y=(1-x+y)\wedge 1,\ x^*=1-x.$$

Let $X=\{a,b,c,d\}$ and $Y=\{x,y,z\}$ be a set. Define a map $f:X\to Y$ as

$$f(a)=f(b)=x,\ f(c)=y,\ f(d)=z.$$

(1) We define fuzzy preorders R_X and R_Y as follows

$$R_X=\begin{pmatrix}1 & 0.8 & 0.7 & 0.5\\ 0.5 & 1 & 0.6 & 0.7\\ 0.4 & 0.8 & 1 & 0.6\\ 0.7 & 0.8 & 0.9 & 1\end{pmatrix}\ R_Y=\begin{pmatrix}1 & 0.8 & 0.7\\ 0.8 & 1 & 0.7\\ 0.8 & 0..9 & 1\end{pmatrix}.$$

By Theorem 3.1(3), we obtain $\mathscr{H}_{R_X}(A)(y)=\bigvee_{x\in X}(A(x)\odot R_X(x,y))$. For $B=(0.3,0.7,0.4)^t$, $\mathscr{H}_{R_Y}(B)=(0.5,0.7,0.4)^t$. Then $B\notin\tau_{\mathscr{H}_{R_Y}},\mathscr{H}_{R_Y}(B)\in\tau_{\mathscr{H}_{R_Y}}$. Since $R_X(a,b)\le R_Y(f(a),f(b))$, by Theorem 3.5(7) , $f^{-1}(\mathscr{H}_{R_Y}(B))=(0.5,0.5,0.7,0.4)^t\in\tau_{\mathscr{H}_{R_X}}$.

$$\mathscr{H}_{R_X}(f^{-1}(B))=(0.3,0.5,0.7,0.4)^t\le f^{-1}(\mathscr{H}_{R_Y}(B))=(0.5,0.5,0.7,0.4)^t.$$

$$\begin{aligned}\mathbf{T}_Y(B) &= \bigwedge_{y\in Y}(\mathscr{H}_{R_Y}(B)(y)\to B(y))=0.8\\ \mathbf{T}_X(f^{-1}(B)) &= \bigwedge_{x\in X}(\mathscr{H}_{R_X}(f^{-1}(B))(x)\to f^{-1}(B)(x))=0.8\end{aligned}$$

$\mathscr{H}_{R_Y^{-1}}(B)=(0.5,0.7,0.6)^t$. Then $B\notin\tau_{\mathscr{H}_{R_Y^{-1}}},\mathscr{H}_{R_Y^{-1}}(B)\in\tau_{\mathscr{H}_{R_Y^{-1}}}$. Since $R_X(a,b)\le R_Y(f(a),f(b))$, by Theorem 3.5(8) , $f^{-1}(\mathscr{H}_{R_Y^{-1}}(B))=(0.5,0.5,0.7,0.6)^t\in\tau_{\mathscr{H}_{R_X^{-1}}}$.

$$\mathscr{H}_{R_X^{-1}}(f^{-1}(B))=(0.4,0.3,0.7,0.4)^t\le f^{-1}(\mathscr{H}_{R_Y^{-1}}(B))=(0.5,0.5,0.7,0.6)^t.$$

$$\begin{aligned}\mathbf{T}_Y^{-1}(B) &= \bigwedge_{y\in Y}(\mathscr{H}_{R_Y^{-1}}(B)(y)\to B(y))=0.8\\ \mathbf{T}_X^{-1}(f^{-1}(B)) &= \bigwedge_{x\in X}(\mathscr{H}_{R_X^{-1}}(f^{-1}(B))(x)\to f^{-1}(B)(x))=0.9.\end{aligned}$$

(2) For $B=(0.3,0.7,0.4)^t$ and $f^{-1}(B)=(0.3,0.3,0.7,0.4)^t$, $R_Y(x,y)=B(x)\to B(y)$ and $R_Y(x,y)=f^{-1}(B)(x)\to f^{-1}(B)(y)$ as follows:

$$R_X=\begin{pmatrix}1&1&1&1\\1&1&1&1\\0.6&0.6&1&0.7\\0.9&0.9&1&1\end{pmatrix}\quad R_Y=\begin{pmatrix}1&1&1\\0.6&1&0.7\\0.9&1&1\end{pmatrix}$$

$$\tau_{\mathscr{H}_{R_Y}}=\{\bigvee_{y\in Y}(C(y)\odot(B(y)\to B))\mid C\in L^Y\}=\{\mathscr{H}_{R_Y}(C)\mid C\in L^Y\}$$

For $A=(0.3,0.5,0.7,0.4)^t$ and $f(A)=(0.5,0.7,0.4)^t$,

$$f(\mathscr{H}_{R_X}(A))=(0.5,0.7,0.5)^t=\mathscr{H}_{R_Y}(f(A)).$$

For $C=(0.8,0.2,0.6)^t$ and $f^{-1}(C)=(0.8,0.8,0.2,0.6)^t$,

$$\mathscr{H}_{R_X}(f^{-1}(C))=(0.8,0.8,0.8,0.8)^t=f^{-1}(\mathscr{H}_{R_Y}(C))$$

$$\mathscr{H}_{R_X^{-1}}(f^{-1}(C))=(0.8,0.8,0.4,0.7)^t=f^{-1}(\mathscr{H}_{R_Y^{-1}}(C)).$$

$$\begin{aligned}\mathbf{T}_Y(C)&=\textstyle\bigwedge_{y\in Y}(\mathscr{H}_{R_Y}(C)(y)\to C(y))=0.4\\ \mathbf{T}_X(f^{-1}(C))&=\textstyle\bigwedge_{x\in X}(\mathscr{H}_{R_X}(f^{-1}(C))(x)\to f^{-1}(C)(x))=0.4.\end{aligned}$$

$$\begin{aligned}\mathbf{T}_Y^{-1}(C)&=\textstyle\bigwedge_{y\in Y}(\mathscr{H}_{R_Y^{-1}}(C)(y)\to C(y))=0.8\\ \mathbf{T}_X^{-1}(f^{-1}(C))&=\textstyle\bigwedge_{x\in X}(\mathscr{H}_{R_X^{-1}}(f^{-1}(C))(x)\to f^{-1}(C)(x))=0.8.\end{aligned}$$

Conflict of Interests

The author declares that there is no conflict of interests.

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