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ON LIE SYMMETRY OF THE ABEL EQUATION

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Abstract: In this paper, we review the dimension of the Lie symmetry of linear and nonlinear second order ordinary differential equations using the increasing of order approach to the well known Abel equation of first and second kinds. We report that the Lie point symmetry of the realized second order equations from the Abel equation using increasing of order is at least two and at most eight.

Keywords: symmetries; Abel; invariant; ordinary differential equations; linear; Lie; nonlinear; mechanics.

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1. Introduction

It is well known in the literature (Olver [1], [2]) that Lie point symmetry is a formidable tool for solving differential equations. Application of symmetry approach stimulates new research into development of new methods for constructing solutions of ordinary differential equations (ODEs) in the closed form. Symmetries of the first order ODEs are infinites so the direct application of Lie method is complicated in the general case. Indirect application of Lie method is to obtain a second order ODE which is related to the respective first order ODE by a change of variable; this induced equation of higher order admits nontrivial Lie symmetry (that generates nonlocal

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symmetry for the original equation). Abraham-Shrauner [3],[4],[5],[6] introduced this ad-hoc method popularly known as reduction or increasing of order. These further produced the concept of hidden symmetries in the form of loss or gain of point symmetries and later unveil the best choice of Lie symmetry for reducing or increasing of order of differential equations. Starting from some integrable Abel equation, Boyko [7],[8],[9] obtained new integrable cases of the Abel equation. It has been established however that the existence of solution of differential equations stem from the existence of infinite nonlocal symmetries of the differential equation even though there is no known algorithm for tracking them (Leach and Andropoulos [10]). The paper reviews the Lie point symmetry of the Abel equation, Lie point symmetry of linear and nonlinear second order differential equations in section 2. The main results are in section 3 where we utilized the point transformation to obtain Lie point symmetries of some nonlinear equations and conclusion.

2. Preliminaries

2.1 Abel equations

The Abel equations of the first and second kind are respectively

$$y' = a_4(x)y^3 + a_3(x)y^2 + a_2(x)y + a_1(x) \quad , \tag{2.1}$$

$$y'(a_5(x)y + a_0(x)) = a_4(x)y^3 + a_3(x)y^2 + a_2(x)y + a_1(x) \tag{2.2}$$

where $y = y(x)$, $y' = \dfrac{dy}{dx}$; $a_i(x), i = 0,1,...,5$ are arbitrary smooth functions and $a_i(x), i = 1,...4$ are not identically zero simultaneously. Boyko [9] showed that the equations (2.1) and (2.2) are related via change of variable $y = 1/v(x) - a_0$.

By a change of variable $x = x(t)$, $\dot{x} = \dfrac{dx}{dt}$, $\ddot{x} = \dfrac{d^2x}{dt^2}$, equations (2.1) and (2.2) transformed to second order equations respectively

$$\ddot{x} = a_4(x)\dot{x}^4 + a_3(x)\dot{x}^3 + a_2(x)\dot{x}^2 + a_1(x)\dot{x}, \tag{2.3}$$

$$\ddot{x}(\dot{x} + a_0(x)) = a_4(x)\dot{x}^4 + a_3(x)\dot{x}^3 + a_2(x)\dot{x}^2 + a_1(x)\dot{x}, \tag{2.4}$$

while the change of variable $\dot{x} = y(x)$ reduces (2.3) and (2.4) to (2.1) and (2.2) respectively.

Boyko [7], [9], [8] shown the normal Lie point symmetry $X_1 = \partial_t$ which corresponds to invariance of (2.3) and 2.4) with respect to translation variable t which induced the Abel equations (2.1) and (2.2). They further inferred that when (2.3) and (2.4) admit two-dimensional Lie algebras, then (2.3) and (2.4) are integrable in the framework of Lie method; and in this way the exact solution of (2.1) and (2.2) were obtained. The analysis of (2.4) was on the understanding that (2.1) – (2.4) are interconnected and their work was based on the assumption that (2.4) admits a two-dimensional Lie algebra

$$X_1 = \partial_t \text{ and } X_1 = \xi(t,x)\partial_t + \eta(t,x)\partial_x . \tag{2.5}$$

From which the following realizations of two-dimensional Lie algebras were obtained:

$$\begin{aligned}
&X_1 = \partial_t, \quad X_2 = \xi(x)\partial_t, \quad \xi(x) \neq const.;\\
&X_1 = \partial_t, \quad X_2 = \xi(x)\partial_t + \eta(x)\partial_x, \quad \xi(x) \neq const. \text{ or } \xi(x) \equiv 0, \quad \eta(x) \neq 0;\\
&X_1 = \partial_t, \quad X_2 = (t + \xi(x))\partial_t, \quad \xi(x) \neq const. \text{ or } \quad \xi(x) \equiv 0;\\
&X_1 = \partial_t, \quad X_2 = (t + \xi(x))\partial_t + \eta(x)\partial_x \quad , \quad \xi(x) \neq const. \text{ or } \quad , \ \eta(x) \neq 0, \xi(x) \equiv 0;\\
&X_1 = \partial_t, \quad X_2 = e^t \xi(x)\partial_t \quad , \quad \xi(x) \neq 0 \quad ;\\
&X_1 = \partial_t, \quad X_2 = e^t(\xi(x)\partial_t + \eta(x)\partial_x), \quad \eta(x) \neq 0 .
\end{aligned} \tag{2.6}$$

From (2.6), (2.4) admits the following canonical realizations respectively (Boyko [7], [9]):

$$\begin{aligned}
&X_1 = \partial_t, \quad X_2 = x\partial_t;\\
&X_1 = \partial_t, \quad X_2 = \partial_x;\\
&X_1 = \partial_t, \quad X_2 = t\partial_t;\\
&X_1 = \partial_t, \quad X_2 = t\partial_t + x\partial_x;\\
&X_1 = \partial_t, \quad X_2 = e^t\partial_t \quad ;
\end{aligned} \tag{2.7}$$

$$X_1 = \partial_t, \quad X_2 = e^t(\partial_t + \partial_x).$$

In accordance with (2.7) the following integrable cases for (2.4) were obtained:

i) $\ddot{x} = \alpha(x)\dot{x}^3$,

ii) $\ddot{x}(\dot{x} + e) = d\dot{x}^4 + c\dot{x}^3 + b\dot{x}^2 + a\dot{x}$,

iii) $\ddot{x} = \alpha(x)\dot{x}^2$,

iv) $x\ddot{x}(\dot{x} + e) = d\dot{x}^4 + c\dot{x}^3 + b\dot{x}^2 + a\dot{x}$,

v) $\ddot{x}(\dot{x} + \beta(x)) = \alpha(x)\dot{x}^3 + (1 - \alpha(x)\beta(x))\dot{x}^2 - \beta(x)\dot{x}$, (2.8)

vi) 1. $a_0(x) = 0$:

$$\ddot{x} = de^x\dot{x}^3 + (c - 3de^x)\dot{x}^2 + (be^{-x} - (2c+1) - 3de^x)\dot{x} + ((x+1) - de^x - be^{-x} + ae^{-2x}),$$

2. $a_0(x) \neq 0$:

$$\ddot{x}(\dot{x} + \alpha(x)) = -\dot{x}^3 + (1 - \alpha(x))\dot{x}^2 + \alpha(x)\dot{x},$$

where $\alpha(x)$ and $\beta(x)$ are arbitrary smooth functions, while a, b, c, d, e are constants.

2.2 **Infinitesimal Lie symmetry generator of second order differential equations**

In this section we summarize the Lie symmetry generators of Linear and nonlinear second order differential equations as in the literature below. The general second order equation is

$$\ddot{x} = f_N(t, x, \dot{x}). \tag{2.9}$$

With the infinitesimal Lie point transformations

$$\bar{t} = t + \lambda\xi(t, x), \quad \bar{x} = x + \lambda\eta(t, x);\ .0 \le \lambda \le 1.$$

The Lie point symmetry generator is

$$V = \xi(t, x)\partial_t + \eta(t, x)\partial_x \quad .$$

And the second prolongation of the symmetry generator is

$$V^{[2]} = \xi\partial_t + \eta\partial_x + \eta'\partial_{\dot{x}} + \eta''\partial_{\ddot{x}}, \tag{2.10}$$

where $$\eta' = \eta_t + (\eta_x - \xi_t)\dot{x} - \xi_x \dot{x}^2 ,$$

and

$$\eta'' = \eta_{xx} + (2\eta_{tx} - \xi_{xx})\dot{x} + (\eta_{xx} - 2\xi_{tx})\dot{x}^2 - \xi_{xx}\dot{x}^3 + (\eta_x - 2\xi_t - 3\xi_x \dot{x})\ddot{x} .$$

The invariant action of (2.10) on (2.9)

$$V^{[2]}(\ddot{x} - f_N(t, x, \dot{x})) = 0 \quad , \tag{2.11}$$

generates the over determined partial differential equation,

$$\eta_{tt} + (2\eta_{xt} - \xi_{tt})\dot{x} + (\eta_{xx} - 2\xi_{xt})\dot{x}^2 - \xi_{tt}\dot{x}^3 + ((\eta_x - 2\xi_t) - 3\xi_x \dot{x}) f_N - \xi f_{Nt} - \eta f_{Nx}$$

$$+ (\eta_t + (\eta_x - \xi_t)\dot{x} - \xi_x \dot{x}^2) f_{N\dot{x}} \equiv 0 \quad . \tag{2.12}$$

Note that Lie [11] had shown that (2.9) can be reduced to free fall particle equation ($\ddot{u} = 0$) if it is at most cubic in the first derivative via point change of variables and that certain auxiliary system is compatible with a compatibility condition, see Boyko [7],[9] for details.

For the purpose of nonlinear mechanics, it is general enough to assume a positive power series in $\dot{x}$.

i.e $$\ddot{x} = \sum_{k=0}^{N} \alpha_k (t, x)\dot{x}^k , N = 0,1,2,... \tag{2.13}$$

The invariant property of (2.10) on (2.13) when $N = 3$, produced over determined partial differential equation which separates into following PDEs:

$$\begin{aligned} &\xi_{xx} = -\alpha_2 \xi_x + \alpha_3(\xi_t - 2\eta_x) - \alpha_{3,t}\xi - \alpha_{3,x}\eta , \\ &\eta_{xx} - 2\xi_{xt} - \alpha_1 \eta_x = \alpha_2 \eta_x + \alpha_{2,t}\xi + \alpha_{2,x}\eta + 3\alpha_3 \eta_t , \\ &2\eta_{xt} - \xi_{tt} - 3\alpha_0 \xi_x - \alpha_1 \xi_t - \alpha_{1,t}\xi - \alpha_{1,x}\eta = 2\alpha_2 \eta_t , \\ &\eta_{tt} - \alpha_0 (2\xi_t - \eta_x) - \alpha_{0,t}\xi - \alpha_{0,x}\eta - \alpha_1 \eta_t = 0 , \end{aligned} \tag{2.14}$$

where α_i may be constant and eventually zero.

The general solution of (2.14) can be formally written as a superposition of linearly independent basis solutions $\xi_k(t, x)$ and $\eta_k(t, x)$ for $k = 1,2,..., r \le 8$; where $V_k = \xi_k(t, x)\partial_t + \eta_k(t, x)\partial_x$ is the infinitesimal generator of the Lie symmetry group, and $[V_i, V_j] = f_{ij}^c V_c$ is the algebraic structure (Aguirre and Krause [12],[13]).

3. Main results

3.1 **Equations and their symmetries**

In this section we x-ray the literature and summarized the results pertaining to the admittance of Lie point symmetry of linear and nonlinear second order differential equations below.

i. When $\alpha_0, \alpha_1, \alpha_2, \alpha_3$ are all zero constant, (2.13) becomes the well known equation of the free fall particle which admits the well known eight Lie point symmetries (Aguirre and Krause [13]).

ii. The equation $\ddot{x} = \varsigma x\dot{x} + \beta x^3$ where ς, β are constants admits only two Lie point symmetries. However if $\varsigma^2 = 9\beta$ the equation admits eight Lie point symmetries. i.e $\ddot{x} = 3x\dot{x} + x^3$ admits eight Lie symmetries (Sarlet, Mahomed and Leach [14]).

iii. The equation $\ddot{x} = f(\dot{x})$, i.e $\ddot{x} = g\dot{x}^3 + a\dot{x}^2 + b\dot{x} + c$; $g \neq 0$, a, b, c are constants is linearizable and so admits eight Lie symmetries. Also the equations $\ddot{x} = \dot{x}^2 - \omega^2$, and $\ddot{x} = \dot{x}^2 + \omega^2$ admit eight Lie point symmetries (Sarlet, Mahomed and Leach [14]).

iv. The equation $\ddot{x} = (t - x)\dot{x}^3$ admits eight Lie symmetries (Stephani [15]).

v. The equation $\ddot{x} = \alpha_0(x) + \alpha_1(x)\dot{x}$, in the case of the Van der Pol oscillator $\ddot{x} + \omega^2 x - \gamma(1 - x^2)\dot{x} = 0$ admits only the point symmetry of time translation invariance.

Remark: we observed that when $\alpha = \alpha(t)$ or $\alpha = \alpha(x)$, (2.13) admits less than eight linearly independent Lie symmetries.

3.2 **Symmetries of some equations**

In the following, we compute the symmetries of some linearizable equations by the method of change of variables.

The change of variables $(X,T)=(x,e^{-t})$, $(x,t)=(X,-\ln T)$ transformed the equation $\ddot{x}=\dot{x}$ to a free fall particle equation, so that it admits the following eight Lie point symmetries:

$$V_1=\partial_x,\ V_2=x\partial_x,\ V_3=e^{-t}\partial_x\ ,\ V_4=-e^{t}\partial_t,\ V_5=\partial_t,\ V_6=-xe^{-t}\partial_t,$$
$$V_7=e^{-t}(x\partial_x-\partial_t),\ V_8=x(x\partial_x-\partial_t) \tag{3.1}$$

The equation $\ddot{x}=A\dot{x}^2$ where A is a constant is linearizable with the change of variables $(X,T)=(xe^{-t},t)$, $(x,t)=(Xe^{T},T)$ and we obtain the following eight Lie point symmetries:

$$V_1=e^{t}\partial_x,\ V_2=x\partial_x,\ V_3=te^{t}\partial_x\ ,\ V_4=x\partial_x+\partial_t,\ V_5=tx\partial_x+t\partial_t,$$
$$V_6=x^2e^{-t}\partial_x+xe^{-t}\partial_t,$$
$$V_7=t((x+tx)\partial_x+t\partial_t),\ V_8=xe^{-t}((x+tx)\partial_x+t\partial_t). \tag{3.2}$$

The equation $\ddot{x}=\dot{x}^3$ is linearizable with change of variables $(X,T)=(x,-\frac{1}{2}t^2+c_1t+c_2)$, $(x,t)=(X,T)$ and we obtain the following eight Lie point symmetries:

$$V_1=\partial_t,\ V_2=x\partial_x,\ V_3=x\partial_t\ ,\ V_4=\partial_x,\ V_5=(-\tfrac{1}{2}t^2+c_1t+c_2)\partial_x,$$
$$V_6=(-\tfrac{1}{2}t^2+c_1t+c_2)\partial_t,$$
$$V_7=x((-\tfrac{1}{2}t^2+c_1t+c_2)\partial_t+x\partial_x),$$
$$V_8=(-\tfrac{1}{2}t^2+c_1t+c_2)^2\partial_t+x(-\tfrac{1}{2}t^2+c_1t+c_2)\partial_x; \tag{3.3}$$

where c_1, c_2 are constants.

3.3 **Conclusion**

We notice that the integrable cases of (2.4) fell within the purview of the linearizable nonlinear differential equations except the well known Van der Pol oscillator equation. The equations ii) and iv) in (2.8) admit eight Lie point symmetries while equations i) and iii) in (2.8) admit Lie point symmetries of dimensions r, $2 \le r < 8$. In general (2.4) admits Lie point symmetries of dimension r, $2 \le r \le 8$ depending on the specific form of $a_k(x)$; $k = 0,1,...,4$; which are not identically zero simultaneously. From the theory of increasing of order approach for solving for Lie symmetry of differential equations, it is well known that change of variables to the original variables produced symmetries of the original equation which are mostly nonlocal type. Following from the fact that the translation invariant ∂_t is preserved up to the point symmetry of the Abel equations (1.1) and (1.2) it is noted that the dimension of the Lie point symmetries of the Abel equation of first and second kind is more than one while the nonlocal symmetries are infinitely many except the Van der Pol oscillator equation which does not have active point transformation (which transforms one solution into another by means of continuous adjustment of a symmetry group parameter).

Conflict of Interests

The author declares that there is no conflict of interests.

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