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NONNEGATIVE SOLUTIONS IN BOUNDARY VALUE PROBLEMS

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Abstract. In this paper, we study the solutions to the two point boundary value problem:

$$-u''(x) = \lambda f(u(x)) \quad ; \qquad x \in (-1,1),$$

$$u(-1) = 0 = u(1),$$

where $\lambda > 0$ is a positive parameter and f is a smooth function. We obtain the exact number of positive solutions.

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1. Introduction

Here we consider the autonomous two point boundary value problem

$$-u''(x) = \lambda f(u(x)) \quad ; \qquad x \in (-1,1), \tag{1}$$

$$u(-1) = 0 = u(1), \tag{2}$$

where λ is a positive parameter and f is a smooth function. We define g by $g(t) = f(t)/t$ and F by $F(t) = \int_0^t f(s)ds$ and F_ϵ by $F_\epsilon(t) = \int_\epsilon^{\epsilon+t} f(s)ds$ for any $\epsilon > 0$. Let $f''(t) > 0$ for

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all $t > 0$.

We analyze in detail the nonnegative solutions to (1),(2). In this paper we employ the Quadrature Method in [1,2]. Our results on positive solutions are in contrast to the case of semipositone (see [1,2]) where $f'' < 0$ guaranteed existence and multiplicity. Our methods are based on building a Quadrature Method for such explosive solutions.

We will discuss the Quadrature Method in section 2, the statements and discussion of the main results and proofs of main results in section 3, and finally the discussion of the complete bifurcation curve of nonnegative solutions for the special case $f(u) = e^{-u}$, in section 4.

2. Quadrature Method

First, note that any solution $u(x)$ of (1),(2) is symmetric about any point $x_0 \in (-1, 1)$ such that $u'(x_0) = 0$. That is, u(x) must achieve its maximum at $x = 0$. Multiplying (1) by $u'(x)$ and integrating, we obtain

$$-[u'(x)]^2/2 = \lambda F(u(x)) + c. \tag{3}$$

Since positive solutions are known to be symmetric with respect to $x = 0$ and $u'(x) > 0$ for $x \in (-1, 0)$ we have $\rho := \sup_{x\in(-1,1)} u(x) = u(0)$. Taking $x = 0$ in (3) implies that

$$u'(x) = \sqrt{2\lambda[F(\rho) - F(u)]} \quad ; \qquad x \in [-1, 0]. \tag{4}$$

Now integrating (4) over [-1,x], we obtain

$$\int_0^{u(x)} \frac{du}{\sqrt{F(\rho) - F(u)}} = \sqrt{2\lambda}(x + 1) \quad ; \qquad x \in [-1, 0], \tag{5}$$

which in turn implies that

$$\sqrt{\lambda} = \frac{1}{\sqrt{2}} \int_0^{\rho} \frac{du}{\sqrt{F(\rho) - F(u)}} := G(\rho), \tag{6}$$

by taking $x = 0$ in (5). Hence for any $\lambda > 0$ if there exists a $\rho \in (0, +\infty)$ with $G(\rho) = \sqrt{\lambda}$, then (1),(2) has a positive solution $u(x)$ given by (5) satisfying $\sup\{u(x)|x \in (-1, 1)\} =$

$u(0) = \rho$. In fact, $G(\rho)$ is a continuous function which is differentiable over $(0, +\infty)$ with

$$\frac{d}{d\rho}G(\rho) = \frac{1}{\sqrt{2}} \int_0^1 \frac{H(\rho) - H(\rho v)}{[F(\rho) - F(\rho v)]^{3/2}} dv, \tag{7}$$

where

$$H(t) = F(t) - (t/2)f(t). \tag{8}$$

For $\rho \in (0, +\infty)$, we recall from (6) that

$$G(\rho) = \frac{1}{\sqrt{2}} \frac{\rho}{\sqrt{F(\rho)}} \int_0^1 \frac{dv}{\sqrt{1 - [F(\rho v)/F(\rho)]}}. \tag{9}$$

3. Main results

Theorem 3.1. If $f(0) > 0$, $\lim_{t\to+\infty} f(t) = M$ where $0 \le M < f(0)$ and $f : [0, +\infty) \to R$ is monotonically decreasing, then (1),(2) has a unique positive solution for any $\lambda > 0$. Also, $\lim_{\lambda\to 0} \rho_\lambda = 0$, and $\lim_{\lambda\to+\infty} \rho_\lambda = +\infty$.

Proof. Firstly, note that from hypotheses we have $\lim_{t\to 0^+} g(t) = +\infty$, $\lim_{t\to+\infty} g(t) = 0$, and $g'(t) < 0$ for all $t > 0$. Hence $H'(t) > 0$ for all $t > 0$ and $H''(t) < 0$ for all $t > 0$. Also, we have $H(0) = 0$. Consequently $G'(\rho) > 0$ for any $\rho \in (0, +\infty)$. Next, let $L(v) := F(\rho v)/F(\rho)$. Hence $L(v) \ge v$ for $v \in [0, 1]$. Consequently from (9) we have

$$G(\rho) \ge (1/\sqrt{2})(\rho/\sqrt{F(\rho)}) \int_0^1 \frac{dv}{\sqrt{1 - v}} = \sqrt{2}(\rho/\sqrt{F(\rho)}).$$

But since $\lim_{t\to+\infty} f(t) = M$; $0 < M \le f(0)$, we have $\lim_{\rho\to+\infty} \rho^2/F(\rho) = \lim_{\rho\to+\infty} 2\rho/f(\rho) = +\infty$, and hence $\lim_{\rho\to+\infty} G(\rho) = +\infty$. Finally, it remains to prove that $\lim_{\rho\to 0^+} G(\rho) = 0$. Since $\lim_{t\to 0^+} f(t)/t = +\infty$, consequently we have

$$\lim_{\rho\to 0^+} G(\rho) = \lim_{\rho\to 0^+} (1/\sqrt{2})(\rho/\sqrt{F(\rho)}) \int_0^1 \frac{dv}{\sqrt{1 - v}} = \lim_{\rho\to 0^+} \sqrt{2}(\rho/\sqrt{F(\rho)}) = 0.$$

Hence Theorem 3.1 is proved.

Theorem 3.2. If $\lim_{t\to 0} f(t) = +\infty$, $\lim_{t\to+\infty} f(t) = M$ where $M = +\infty$,(or $0 < M < +\infty$), then there exists $\lambda_1 > 0$ with $0 < \lambda_1 < +\infty$ such that (1),(2) has no positive solutions for $\lambda \in (\lambda_1, +\infty)$. For $\lambda \in (0, \lambda_1)$ the problem (1),(2) has two positive solutions, and for $\lambda = \lambda_1$ the problem (1),(2) has exactly one positive solution. (or the problem (1),(2) has a unique positive solution for any $\lambda > 0$ and $\lim_{\lambda\to 0}\rho_\lambda = 0$, and $\lim_{\lambda\to\infty}\rho_\lambda = +\infty$).

Proof. To prove of Theorem 3.2, we shall need following Lemma.

If in the Quadrature Method taking F_ϵ instead of F, we obtain

$$G(\rho) = (1/\sqrt{2})(\rho/\sqrt{F_\epsilon(\rho-\epsilon)})\int_0^1 \frac{dv}{\sqrt{1-[F_\epsilon(\rho v-\epsilon)/F_\epsilon(\rho-\epsilon)]}}.$$

In what follows, we use this new G.

Lemma 3.3. If $g(t) = f(t)/t$, then $g(t)$ is monotonically decreasing, or $g(t)$ has a unique extremum point.

Proof. We know that

$$\lim_{t\to 0^+} g(t) = +\infty, \tag{10}$$

and

$$\lim_{t\to +\infty} g(t) = +\infty. \tag{11}$$

Thus there exists an $\eta_1 \in (0, +\infty)$ such that $g'(\eta_1) = 0$. We show that η_1 is unique zero of $g'(t)$. Suppose, on the contrary, that there exists an η_2 such that $g'(\eta_2) = 0$. Also, suppose that η_1 is first zero of $g'(t)$ and η_2 is second zero of $g'(t)$. Consequently from (10) we understand that η_1 is length of minimum point and η_2 is length of maximum point. In view of (11), there exists an $\eta_3 \in (\eta_2, +\infty)$ such that $g'(\eta_3) = 0$. Now, let $\phi(t) = g'(t)$. Thus we have

$$\lim_{t\to 0^+} \phi(t) = -\infty, \tag{12}$$

and

$$\lim_{t\to +\infty} \phi(t) = \alpha \quad ; \qquad 0 < \alpha \leq +\infty. \tag{13}$$

Also, we have

$$\phi(\eta_1) = \phi(\eta_2) = 0, \tag{14}$$

hence there exists $c_1 \in (\eta_1, \eta_2)$ such that $\phi'(c_1) = 0$. Implicitly, we have

$$\phi(\eta_2) = \phi(\eta_3) = 0, \tag{15}$$

hence there exists $c_2 \in (\eta_2, \eta_3)$ such that $\phi'(c_2) = 0$. In view of (12), (13), c_1, c_2 are length of maximum and minimum points of ϕ, respectively. Consequently from (14) and (15) we have $\phi(c_1) > 0$ and $\phi(c_2) < 0$. But we have $\phi'(t) = [f''(t) - 2\phi(t)]/t$, hence $f''(c_2) = 2\phi(c_2)$ which is a contradiction.

When $0 < M < +\infty$ it is clear to show that g is monotonically decreasing. Hence the Lemma 3.3 is proved.

Proof Of Theorem 3.2. Now, we show for $\lim_{t\to+\infty} f(t) = M$ where $M = +\infty$, we have $\lim_{\rho\to+\infty} G(\rho) = 0$, and $\lim_{\rho\to 0^+} G(\rho) = 0$, and $G(\rho)$ has a unique maximum point. Also, for $0 < M < +\infty$, we have $\lim_{\rho\to 0^+} G(\rho) = 0$, $\lim_{\rho\to+\infty} G(\rho) = +\infty$, and $G(\rho)$ is monotonically increasing. But in view of Lemma 3.3, in the first case $H(t)$ has a unique maximum point and $\lim_{t\to+\infty} H(t) = -\infty$. Consequently It remains to prove that $\lim_{\rho\to 0^+} G(\rho) = 0$, $\lim_{\rho\to+\infty} G(\rho) = 0$, and $G(\rho)$ has a unique maximum point. Since $H(0) = 0$ and Lemma 3.3 holds, we have $H(t) > 0$ for $t \in (0.\eta_1]$ which, in turn, implies that $G'(\rho) > 0$ for $\rho \le \eta_1$. Since $\lim_{t\to\infty} H(t) = -\infty$ we have $H(t) < 0$ for t large and hence $G'(\rho) < 0$ for ρ large. Let $L(v) := F(\rho v)/F(\rho)$. Then $L(v) \le v$ for $v \in [0, 1]$. With this (9) would yield

$$G(\rho) \le \frac{1}{\sqrt{2}} \frac{\rho}{\sqrt{F(\rho)}} \int_0^1 \frac{dv}{\sqrt{1-v}} = \sqrt{2} \frac{\rho}{\sqrt{F(\rho)}}. \tag{16}$$

But since $\lim_{t\to+\infty} f(t)/t = +\infty$, we have $\lim_{\rho\to+\infty} \rho^2/F(\rho) = \lim_{\rho\to+\infty} 2\rho/f(\rho) = 0$, and hence $\lim_{\rho\to+\infty} G(\rho) = 0$.

Finally, it remains to prove that $\lim_{\rho\to 0^+} G(\rho) = 0$, which follows by the similar arguments that we used in the proof of Theorem 3.1.

Also, in view of diagram $k(v) := H(\rho_0) - H(\rho_0 v)$ it is clear that $G'(\rho)$ has a unique zero and consequently $G(\rho)$ has a unique maximum point.

In the second case, that is, $0 < M < +\infty$, the proof is similar to the ones in Theorem 3.1.

Hence Theorem 3.2 is proved.

4. Examples

Example 4.1. *Consider the problem*

$$-u'' = \lambda e^{-u}$$

$$u(-1) = 0 = u(1).$$

This example for which $f(u) = e^{-u}$ *demonstrates Theorem 3.1 since* $f(0) = 1 > 0$, $\lim_{u\to+\infty} f(u)/u = 0$, *i.e.* $M = 0 < 1 = f(0)$, *and* $f(u) = e^{-u}$ *is decreasing for* $u > 0$. *Note that* $F(u) = -e^{-u} + 1$ *implies*

$$G(\rho) = \frac{1}{\sqrt{2}} \int_0^\rho \frac{du}{\sqrt{e^{-u} - e^{-\rho}}}.$$

Letting $w = e^{-u/2}$ *we obtain*

$$G(\rho) = -\sqrt{2} \int_{\sec^{-1}(e^{\rho/2})}^{0} \frac{e^{-\rho/2} \sec\theta \tan\theta d\theta}{e^{-\rho/2} \sec\theta \sqrt{e^{-\rho} \tan^2\theta}} = \sqrt{2} e^{\rho/2} \int_0^{\sec^{-1}(e^{\rho/2})} d\theta = \sqrt{2} e^{\rho/2} \sec^{-1}(e^{\rho/2}).$$

Hence, $\lim_{\rho\to 0^+} G(\rho) = 0$, *and* $\lim_{\rho\to+\infty} G(\rho) = +\infty$.

Consequently, this example shows truth of Theorem 3.1.

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