Available online at http://scik.org
J. Math. Comput. Sci. 4 (2014), No. 3, 574-586
ISSN: 1927-5307

ORDERINGS AND PREORDERINGS ON MODULES

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Abstract. The purpose of this paper is to investigate some relevant properties of semireal modules. By introducing the notion of preorderings on modules over commutative rings, we discuss the interplay between semireality, preorderings and orderings on modules. In particular, we obtain that an $R-$module M is semireal if and only if M possesses a preordering. We also give some necessary and sufficient conditions for a preordering to be an ordering on modules.

Keywords: semireal module; ordering; preordering.

2010 AMS Subject Classification: 13J30,16W80.

1. Introduction

In Artin's solution to the Hilbert's 17th problem, the notion of orderings on fields played an important role. The notion of orderings on fields was first studied by Artin and Schreier [1,2]. Later, the notion of orderings was generalized to the category of commutative rings by Coste and Roy [4]; see also [5]. Recently, it was further generalized to the category of modules by Zeng[9]. It's well known that the notion of preorderings plays an important role in the study of real algebra in the categories of fields and commutative rings. Naturally, such a question

Received May 14, 2014

arises: can we introduce preorderings on modules over commutative rings? This paper is aimed to answer this question.

Throughout this paper,"ring" means "commutative ring with identity 1", and all modules are unitary. For two subsets A and B of a set S, denote by $A \setminus B$ the complement of B in A, i.e. $A \setminus B = \{x \in S | x \in A$ but $x \notin B\}$. Let M be an $R-$module. For nonempty subsets S, T of R, set $S+T = \{s+t | s \in S$ and $t \in T\}$. In particular, if $S = \{s\}$, we write $s+T$ instead of $S+T$. For nonempty subsets A, B of M, set $(A : B) = \{r \in R | rb \in A$ for all $b \in B\}$ and $A+B = \{a+b | a \in A$ and $b \in B\}$. In particular, if $B = \{b\}$, we write $(A : b)$ instead of $(A : B)$ and $A+b(or\ b+A)$ instead of $A+B$. For a nonempty subset S of R and a nonempty subset A of M, set $SA = \{sa | s \in S, a \in A\}$. In particular, if $S = \{s\}$, we write sA instead of SA. A proper submodule $\mathscr{P}$ of M is called prime if $ax \in \mathscr{P}$ with $a \in R$ and $x \in M$ implies either $a \in (\mathscr{P} : M)$ or $x \in \mathscr{P}$. For properties of modules and prime submodules, we refer the reader to [6,7] and the references therein.

Conventionally, we use $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{R}$ for the set of all positive integers, the set of integers and the field of real numbers, respectively.

2. Orderings

Let R be a ring. Then we may obtain the two multiplicative subsets of R as follows:

$$T_R := \{\sum_{i=1}^{n} a_i^2 \mid n \in \mathbb{N}, \text{ and } a_i \in R \text{ for } i = 1, \cdots, n\};$$

$$1+T_R := \{1+t | t \in T_R\}.$$

Definition 2.1. [9] Let M be an $R-$module. M is said to be semireal, if there exists an element e in M such that $(1+t)e \neq 0$ for every $t \in T_R$. In this case, such an element e is said to be a semireal element.

M is said to be real, if $(\sum_{i=1}^{n} a_i^2)x = 0$, where $x \in M$ and $a_i \in R$ for $i = 1, \cdots, n$, implies $a_i x = 0$ for $i = 1, \cdots, n$.

Definition 2.2. [9] A subset Q of M is called an ordering of M if the following conditions are satisfied: (1)$Q+Q \subseteq Q$; (2)$Q \cup -Q = M$; (3)$Q \cap -Q$ is a prime submodule of M; and (4)$(Q:Q)$ is an ordering of ring R.

Lemma 2.3. *Let M be an $R-$module and Q an ordering of M. If $x \in R$, then $x \notin (Q:Q)$ if and only if $Q \cap -Q = \{q \in Q \mid xq \in Q\}$.*

Proof. "The sufficiency" Suppose that $Q \cap -Q = \{q \in Q \mid xq \in Q\}$. If $x \in (Q:Q)$, then $Q \subseteq \{q \in Q \mid xq \in Q\} = Q \cap -Q$. Thus, $M = Q \cup -Q \subseteq Q \cap -Q$, and so $M = Q \cap -Q$. This contradicts the fact that $Q \cap -Q$ is a proper submodule of M(since $Q \cap -Q$ is a prime submodule of M). Hence, $x \notin (Q:Q)$.

"The necessity" Suppose that $x \notin (Q:Q)$. Set $Q_1 = \{q \in Q \mid xq \in Q\}$. Since $(Q:Q)$ is an ordering of R,$x \in -(Q:Q)$. Thus, $xQ_1 \subseteq xQ \subseteq -Q$. On the other hand, by the construction of Q_1,$xQ_1 \subseteq Q$. Thus, $xQ_1 \subseteq Q \cap -Q$. Observe that $x \notin ((Q \cap -Q):M)$(since $x \notin (Q:Q)$). By the primity of $Q \cap -Q$, we have $Q_1 \subseteq Q \cap -Q$. Conversely, since $x(Q \cap -Q) \subseteq x(-Q) \subseteq Q$, we have $Q \cap -Q \subseteq Q_1$. Therefore, $Q \cap -Q = Q_1 = \{q \in Q \mid xq \in Q\}$. This completes the proof.

Theorem 2.4. *Let M be an $R-$module and Q a subset of M. If Q satisfies: $Q+Q \subseteq Q$, $Q \cup -Q = M$, and $(Q:Q)$ is an ordering of R, then $Q \cap -Q$ is a prime submodule of M if and only if for any $x \notin (Q:Q)$, $Q \cap -Q = \{q \in Q \mid xq \in Q\}$.*

Proof. The necessity follows immediately from Lemma 2.3. It's enough to prove the sufficiency. Suppose that $Q \cap -Q = \{q \in Q \mid xq \in Q\}$ for any $x \notin (Q:Q)$. We shall first show that the subset $Q \cap -Q$ is a submodule of M. Obviously, $Q \cap -Q$ is a subgroup of M. Suppose that there is an $x \in R$ such that $x(Q \cap -Q) \nsubseteq Q \cap -Q$. Then $x(Q \cap -Q) \nsubseteq Q$ (if otherwise, $x(Q \cap -Q) = x((Q \cap -Q) \cap -(Q \cap -Q)) = x(Q \cap -Q) \cap -x(Q \cap -Q) \subseteq Q \cap -Q$, a contradiction). Thus $xQ \nsubseteq Q$, and so $x \notin (Q:Q)$. Since $(Q:Q)$ is an ordering of R, $x \in -(Q:Q)$. Thus, $x(Q \cap -Q) \subseteq x(-Q) \subseteq Q$, a contradiction. It follows that $x(Q \cap -Q) \subseteq (Q \cap -Q)$ for every $x \in R$. Therefore, $Q \cap -Q$ is a submodule of M.

Next, we shall show the submodule $Q \cap -Q$ is a prime submodule. Since $(Q:Q)$ is an ordering of R, $-1 \notin (Q:Q)$. Thus, $-Q \nsubseteq Q$, and so $Q \cap -Q \subsetneqq Q \cup -Q = M$. Hence, $Q \cap -Q$ is a proper submodule of M.

Let $xq \in Q \cap -Q$, where $x \in R$ and $q \in M$. Since $M = Q \cup -Q$, we have either $q \in Q$ or $q \in -Q$. Without loss of generality, we may assume $q \in Q$. Assume $x \notin (Q \cap -Q : M) = (Q \cap -Q : Q \cup -Q)$. Then, there are two cases to consider:

Case 1. $x \notin (Q : Q)$. In this case, by supposition, $Q \cap -Q = \{q \in Q \mid xq \in Q\}$. Thus, $q \in \{q \in Q \mid xq \in Q\} = Q \cap -Q$, Since $xq \in Q \cap -Q \subseteq Q$.

Case 2. $x \in (Q : Q)$. In this case, we have $x \notin (Q : -Q)$, since $x \notin (Q \cap -Q : M)$ and $(Q \cap -Q : M) = (Q \cap -Q : Q \cup -Q) = (Q : Q) \cap (Q : -Q)$. Thus, $-x \notin (Q : Q)$. Hence, by supposition, $Q \cap -Q = \{q \in Q \mid -xq \in Q\}$. Since $-xq \in -(Q \cap -Q) \subseteq Q$, we have $q \in \{q \in Q \mid -xq \in Q\} = Q \cap -Q$.

Therefore, $Q \cap -Q$ is a prime submodule of M. This completes the proof.

By Theorem 2.4, we may obtain an equivalent definition of orderings on modules over commutative rings as follows:

Definition 2.5. *Let M be an $R-$module. A subset Q of M is called an ordering of M if the following conditions are satisfied: (1)$Q + Q \subseteq Q$; (2)$Q \cup -Q = M$; (3)for any $x \notin (Q : Q)$, $Q \cap -Q = \{q \in Q \mid xq \in Q\}$; and (4)$(Q : Q)$ is an ordering of ring R.*

Proposition 2.6. *Let A be a submodule of an $R-$module M and Q an ordering of M. If $A \nsubseteq Q \cap -Q$, then $Q \cap A$ is an ordering of A.*

Proof. Obviously, $Q \cap A$ satisfies the conditions (1) and (2) in Definition 2.5. For the check of the condition (4), it suffices to prove $(Q : Q) = (Q \cap A : Q \cap A)$. Clearly, $(Q : Q) \subseteq (Q \cap A : Q \cap A)$. Conversely, suppose that there is $x \in (Q \cap A : Q \cap A)$, but $x \notin (Q : Q)$. Then $x(Q \cap A) \subseteq Q \cap A \subseteq Q$. Thus $Q \cap A \subseteq \{q \in Q \mid xq \in Q\}$. By Lemma 2.3, $Q \cap A \subseteq Q \cap -Q$. Notice that $-Q \cap A = -(Q \cap Q) \subseteq Q \cap -Q$. Hence, $A = M \cap A = (Q \cup -Q) \cap A = (Q \cap A) \cap (-Q \cap A) \subseteq Q \cap -Q$, this contradicts the assumption $A \nsubseteq Q \cap -Q$. Thereby, we have $(Q : Q) = (Q \cap A : Q \cap A)$. Now, we check $Q \cap A$ satisfies condition (3). If $x \notin (Q \cap -Q : Q \cap -Q) = (Q : Q)$. Then $(Q \cap A) \cap -(Q \cap A) = (Q \bigcap -Q) \bigcap A = \{q \in Q | xq \in Q\} \bigcap A = \{q \in Q \bigcap A | xq \in Q \bigcap A\}$. By definition 2.5, the proof is completed.

3. Preorderings

The notion of preorderings play an important role in the study of real rings . In the category of commutative rings, preordering was defined as follows: Let R be a ring. A subset T of R is called a preordering if $T+T\subseteq T$, $T\cdot T\subseteq T$, $R^2\subseteq T$ and $-1\notin T$.

In this section, we introduce the notion of preorderings in the category of modules and then establish some results about preorderings and orderings on modules.

Definition 3.1. *Let M be an $R-$module. A subset Q of M is called a preordering of M if the following conditions are satisfied: (1) $Q+Q\subseteq Q$; (2) $(Q:Q)$ is a preordering of ring R.*

Remarks 1. (1) If Q is a preordering of an $R-$module M, so is $-Q$.

(2) For a preordering Q of a ring R, Q is obviously a preordering of R as an $R-$module. However, as an $R-$module , a preordering Q of R need not be a preordering of the ring R. Even neither Q nor $-Q$ would be. For example, set $Q:=2\mathbb{N}\cup\{0\}$. We may check that Q is a preordering of $\mathbb{Z}$ as a $\mathbb{Z}$-module. Note that $1^2\notin -Q\cup Q$. Therefore, neither Q nor $-Q$ will be a preordering of $\mathbb{Z}$.

It's easy to check the following Lemma:

Lemma 3.2. *Let M be an R-module and Q a subset of M. Then, Q is a preordering if and only if Q satisfies the following conditions: (1)$Q+Q\subseteq Q$; (2) for all $x\in R$, $x^2\in(Q:Q)$; and (3)$-1\notin(Q:Q)$.*

Theorem 3.3. *An $R-$module M is semireal if and only if M possesses a preordering.*

Proof. "The necessity" Suppose M is semireal. Then, for any $t\in T_R$, there is an $e\in M$ such that $(1+t)e\neq 0$. Set $Q=\{te\mid t\in T_R\}$. We claim that Q is a preordering of M. By Lemma 3.2, it is enough to check conditions (1)-(3) in Lemma 3.2. (1)For $q_1,q_2\in Q$, there exist $t_1,t_2\in T_R$ such that $q_i=t_ie, i=1,2$. Thus, $q_1+q_2=t_1e+t_2e=(t_1+t_2)e\in Q$. It follows that $Q+Q\subseteq Q$. (2)For $x\in R$ and $q\in Q$ where $q=te$, $t\in T_R$, we have $x^2q=x^2(te)=(x^2t)e\in Q$. Hence, $x^2\in(Q:Q)$; (3)If $-1\in(Q:Q)$, then $-e=-1\cdot e\in -1\cdot Q\subseteq Q$. Thus, $-e=te$ for some $t\in T_R$. Hence, $(1+t)e=0$, a contradiction. It is follows that $-1\notin(Q:Q)$.

"The sufficiency" Let Q be a preordering of M. Suppose M is not a semireal module. Then, for any $q\in Q$, there exists a $t=\sum_{i=1}^n {a_i}^2\in T_R$, where $n\in\mathbb{N}, a_i\in R,\ i=1,\ \cdots,\ n$, such that $(1+t)q=0$. Thus, $-q=tq=(\sum_{i=1}^n {a_i}^2)q=\sum_{i=1}^n {a_i}^2q\in Q+\cdots+Q\subseteq Q$, this implies $-Q\subseteq Q$, and so $-1 \in (Q:Q)$. This contradicts condition (3) in Lemma 3.2. Therefore, M is a semireal module. This completes the proof.

Remark 2. $Q = \{te \mid t \in T_R\}$ is called the induced preordering by the semireal element e. Clearly, it's the minimal preordering (minimal with respect to set inclusion) containing e.

According to [9], there are modules which are semireal but not necessarily possess orderings. This also means, by Theorem 3.3, that there are modules which possess preorderings but not possess orderings . But when the considered module is finitely generated, we have

Proposition 3.4. *Let M be a finitely generated $R-$module. Then the following conditions are equivalent:*

(1)M is semireal;

(2)M possesses a preordering;

(3)M possesses an ordering.

Proof. This follows immediately from Theorem 3.3 and Proposition 3.5 in [9].

Proposition 3.5. *Let $\varphi : M \to M'$ be an $R-$module homomorphism, and let Q' be a preordering of M' with $Q' \subseteq \varphi(M)$, and denote $Q = \varphi^{-1}(Q')$. Then Q is a preordering of M.*

Proof. For $q_1,\ q_2 \in Q$, we have $q_1+q_2 \in \varphi^{-1}(\varphi(q_1+q_2)) = \varphi^{-1}(\varphi(q_1)+\varphi(q_2)) \subseteq \varphi^{-1}(Q') = Q$. Thus, $Q+Q \subseteq Q$. For any $x \in R$ and $q \in Q$. Since $\varphi(x^2q) = \varphi(x)^2\varphi(q) \in Q'$, $x^2q \in \varphi^{-1}(Q') = Q$, i.e. $x^2 \in (Q:Q)$. Since $-1 \notin (Q':Q')$, there is a $q' \in Q'$, but $-q' \notin Q'$. Then, there is a $q \in Q$ such that $\varphi(q) = q'$. Observe that $\varphi(-q) = -q' \notin Q'$. Hence, $-q \notin \varphi^{-1}(Q') = Q$. Thus, $-1 \notin (Q:Q)$. By Lemma 3.2, the proof is completed.

Proposition 3.6. *Let $M_1 \xrightarrow{\phi_1} M \xrightarrow{\phi_2} M_2$ be an exact sequence of R-module homomorphisms. If Q is a preordering of M, then either $\phi_1^{-1}(Q)$ is a preordering of M_1 or $\phi_2(Q)$ is a preordering of M_2.*

Proof. Assume that $\phi_1^{-1}(Q)$ is not a preordering of M_1. Notice that $\phi_1^{-1}(Q)+\phi_1^{-1}(Q) \subseteq \phi_1^{-1}(Q)$ and $x^2\phi_1^{-1}(Q) \subseteq \phi_1^{-1}(Q)$ for any $x \in R$. Thus, by Lemma 3.2, $-1 \in (\phi_1^{-1}(Q) : \phi_1^{-1}(Q))$. Now, we may assert that $\phi_2(Q)$ is a preordering of M_2. Clearly, we have $\phi_2(Q)+\phi_2(Q) \subseteq \phi_2(Q)$ and $x^2\phi_2(Q) \subseteq \phi_2(Q)$ for any $x \in R$. Suppose $-1 \in (\phi_2(Q) : \phi_2(Q))$. Then, for any $q \in Q, -\phi_2(q) \in$

$\phi_2(Q)$. Thus, there exists a $q_1 \in Q$, such that $-\phi_2(q) = \phi_2(q_1)$, and so $\phi_2(q+q_1) = 0$. Hence, $q+q_1 \in Ker\phi_2$. By the exactness of the sequence, we have $q+q_1 \in Im\phi_1$. Thereby, there exists $m_1 \in M_1$, such that $\phi_1(m_1) = q+q_1$. Thus, $m_1 \in \phi_1^{-1}(q+q_1) \subseteq \phi_1^{-1}(Q) \subseteq -\phi_1^{-1}(Q)$, since $-1 \in (\phi_1^{-1}(Q) : \phi_1^{-1}(Q))$ we got above. Thus, $\phi_1(-m_1) \in Q$, i.e. $-(q+q_1) \in Q$. Hence, $q \in -q_1 - Q \subseteq -Q$. This implies $-1 \in (Q : Q)$, a contradiction. It follows that $\phi_2(Q)$ is a preordering of M_2. This completes the proof.

Clearly, by definitions, an ordering is always being a preordering. However, the converse is not necessary true. Then such a question naturally arises: Under what conditions that might be true. In the sequel, we seek some sufficient and necessary conditions for a preordering to be an ordering.

Lemma 3.7. [5] *Let T be a preordering of ring R. Then, T is an ordering if and only if, for $x, y \in R$, $xy \in -T$ implies either $x \in T$ or $y \in T$.*

Lemma 3.8. *Let M be an $R-$module and Q a subset of M. Then, (1) implies (2)*

(1) For $x \in R$ and $m \in M$, $xm \in -Q$ implies either $x \in (Q : Q)$ or $m \in Q$;

(2) For $x, y \in R$, $xy \in -(Q : Q)$ implies either $x \in (Q : Q)$ or $y \in (Q : Q)$.

Proof. Suppose that (1) holds. If $xy \in -(Q : Q)$ and $x \notin (Q : Q)$, then $x(yQ) = (xy)Q \subseteq -Q$. By supposition, we have $yQ \subseteq Q$, and so $y \in (Q : Q)$. This completes the proof.

Theorem 3.9. *Let M be an $R-$module and Q a preordering of M. Then, Q is an ordering if and only if the following conditions are satisfied:*

(1) For $x \in R$ and $m \in M$, $xm \in -Q$ implies either $x \in (Q : Q)$ or $m \in Q$;

(2) For $p, q \in M$, $p+q \in Q \cap -Q$ implies either $p \in Q$ or $q \in Q$.

Proof. ”The necessity” (1)Suppose that $xm \in -Q$ where $x \in R$ and $m \in M$, but $x \notin (Q : Q)$ and $m \notin Q$. Then, $-m \in Q$ and $x(-m) \in Q$. By Lemma 2.3, $-m \in Q \cap -Q$. Thus, $m \in Q$, a contradiction. (2)Assume $p+q \in Q \cap -Q$. If $p \notin Q$, then $-p \in Q$. Thus, $q \in -p + Q \cap -Q \subseteq Q + Q \subseteq Q$. This implies $p \in Q$ or $q \in Q$.

”The sufficiency” Since assumption (1) holds, by Lemma 3.8 and by Lemma 3.7, $(Q : Q)$ is an ordering of R. If $Q \cup -Q \neq M$, let $m \in M \setminus (Q \cup -Q)$. Since $m + (-m) = 0 \in Q \cap -Q$ and assumption(2) holds, $m \in Q$ or $-m \in Q$, a contradiction. Thus, $Q \cup -Q = M$. For proving that Q

is an ordering of M, by Definition 2.5, it's now sufficient to show that $Q\cap -Q=\{q\in Q\mid xq\in Q\}$ for any $x\notin (Q:Q)$. Assume $x\notin (Q:Q)$. Then $x\in -(Q:Q)$, since $(Q:Q)$ is an ordering of R. Thus, $x(Q\cap -Q)\subseteq x(-Q)\subseteq Q$. Hence, we have $Q\cap -Q\subseteq \{q\in Q\mid xq\in Q\}$. On the other hand, for $q\in \{q\in Q\mid xq\in Q\}, x(-q)\in -Q$. Thus, by assumption (1), either $x\in (Q:Q)$ or $-q\in Q$. Since $x\notin (Q:Q)$, we have $q\in -Q$. It follows that $q\in Q\cap -Q$. Therefore, we have $Q\cap -Q=\{q\in Q\mid xq\in Q\}$. This completes the proof.

Remarks 3. (1) According to the proof above, condition (2) can be replaced by (2) $Q\cup -Q=M$.

(2) Theorem 3.9 can be considered as a generalization of Lemma 3.7. In Lemma 3.7 , the condition $R=T\cup -T$ (in the proof of sufficiency, cf.[5]) can be deduced from other conditions. However, in Theorem 3.9, condition (2) is indispensable. For example, let x be an indeterminant element on $\mathbb{R}$. Then, $\mathbb{R}[x]$ can naturally be an $\mathbb{R}-$module. It's easy to check that $\mathbb{R}^+$ is a preordering of $\mathbb{R}-$module $\mathbb{R}[x]$ and satisfies the condition (1), but $\mathbb{R}^+$ is not an ordering of $\mathbb{R}[x]$. This implies condition (2) is indispensable.

Corollary 3.10. *Let M be an $R-$module and Q a preordering of M. Then, Q is an ordering if and only if, for any $m\notin Q$, $(-Q:m)=(Q:Q)$.*

Proof. "The necessity " Let $m\notin Q$. If $r\in (-Q:m)$, then $rm\in -Q$. Thus, by Theorem 3.9, $r\in (Q:Q)$. Hence, $(-Q:m)\subseteq (Q:Q)$. On the other hand, observe that $m\in -Q$, we have $(Q:Q)m\subseteq (Q:Q)(-Q)\subseteq -Q$. Thus, $(Q:Q)\subseteq (-Q:m)$. It follows that $(-Q:m)=(Q:Q)$.

"The sufficiency"(1)Suppose that for any $m\notin Q$, $(-Q:m)=(Q:Q)$. Let $rm\in -Q$ and $m\notin Q$. Then, $r\in (-Q:m)$ and so $r\in (Q:Q)$ by supposition. (2)For any $m\in R$, if $m\notin Q$, then $(-Q:m)=(Q:Q)$. Observe that $1\in (Q:Q)$. Thus $m=1\cdot m\in (Q:Q)m\in -Q$. Hence $M=Q\cup -Q$. By Theorem 3.9 and Remarks 3(1), the proof is completed.

Proposition 3.11. *Let $M_1\xrightarrow{\phi_1} M\xrightarrow{\phi_2} M_2\longrightarrow 0$ be an exact sequence of R-module homomorphisms. If Q is an ordering of M, then either $\phi_1^{-1}(Q)$ is an ordering of M_1 or $\phi_2(Q)$ is an ordering of M_2.*

Proof. Assume that $\phi_1^{-1}(Q)$ is not an ordering of M_1. Then, by Lemma 3 in [9], we have $\phi_1(M_1)\subseteq Q$. Further, $\phi_1(M_1)\subseteq Q\cap -Q$. i.e. $Im\phi_1\subseteq Q\cap -Q$. In this case, since $\phi_1^{-1}(Q)=M_1, \phi_1^{-1}(Q)$ is not a preordering of M_1. Hence, by Proposition 3.6, $\phi_2(Q)$ is a preordering of M_2. Since ϕ_2 is surjective, we have $\phi_2(Q) \cup -\phi_2(Q) = M_2$. Let $xm_2 \in -\phi_2(Q)$, where $x \in R, m_2 \in M_2$. There exist an $m \in M$ and a $q \in Q$, such that $\phi_2(m) = m_2$ and $x\phi_2(m) = -\phi_2(q)$, i.e. $\phi_2(xm+q) = 0$. Thus, $xm+q \in Ker\phi_2 = Im\phi_1 \subseteq Q \cap -Q$, since the sequence is exact and $Im\phi_1 \subseteq Q \cap -Q$ we got above. Hence, $xm \in -q+Q \cap -Q \subseteq -Q$. Thereby, we have either $x \in (Q:Q)$ or $m \in Q$ by Theorem 3.9. Thus, either $x \in (\phi_2(Q):\phi_2(Q))$ or $m_2 = \phi_2(m) \in \phi_2(Q)$, since $(Q:Q) \subseteq (\phi_2(Q):\phi_2(Q))$. Applying Theorem 3.9 again, we obtain that $\phi_2(Q)$ is an ordering of M_2. This completes the proof.

Proposition 3.12. *Let M be an $R-$module, Q an ordering of M, and S a multiplicative subset of R. Set $S^{-1}Q = \{\frac{q}{s} \mid q \in Q \text{ and } s \in S\}$. If $S \subseteq (Q:Q) \setminus -(Q:Q)$, then $S^{-1}Q$ is an ordering of $S^{-1}R-$module $S^{-1}M$.*

Proof. It's easy to check that $S^{-1}Q + S^{-1}Q \subseteq S^{-1}Q$ and $(\frac{r}{s})^2 \in (S^{-1}Q : S^{-1}Q)$ for $\frac{r}{s} \in S^{-1}R$ where $r \in R$, $s \in S$, since $S \subseteq (Q:Q)$. Suppose $-\frac{1}{1} \in (S^{-1}Q : S^{-1}Q)$. Pick out a $q \in Q \setminus -Q$ and an $s \in S$. Then, $-\frac{q}{s} = -\frac{1}{1} \cdot \frac{q}{s} \in S^{-1}Q$. Thus, there exist $q' \in Q$, $s' \in S$, such that $-\frac{q}{s} = \frac{q'}{s'}$. Further, we have $t(s'q+sq') = 0$ for some $t \in S$. Hence, $ts'q = -tsq' \in Q \bigcap -Q$. Note that $q \in Q \setminus -Q$. Then $ts' \in (Q \bigcap -Q : M) \subseteq -(Q:Q)$, since $Q \bigcap -Q$ is a prime module of M. On the other hand, $ts' \in S \subseteq (Q:Q) \setminus -(Q:Q)$, a contradiction. It follows that $-\frac{1}{1} \notin (S^{-1}Q : S^{-1}Q)$. Therefore, $S^{-1}Q$ is a preordering of $S^{-1}R-$module $S^{-1}M$.

Since $Q \bigcup -Q = M$, we have $S^{-1}Q \bigcup -S^{-1}Q = S^{-1}M$. Assume $x'm' \in -S^{-1}Q$ for $x' = \frac{x}{s_1}$, $m' = \frac{m}{s_2}$ where $x \in R, m \in M$ and s_1, $s_2 \in S$. Then ,there exist a $q \in Q$ and an $s_3 \in S$ such that $\frac{x}{s_1}\frac{m}{s_2} = -\frac{q}{s_3}$. Thus , we have $t(s_3xm + s_1s_2q) = 0$ for some $t \in S$. In this case, we may assert that $xm \in -Q$. Indeed, if not, then $xm \in Q \setminus -Q$. Thus, $(ts_3)(xm) = -ts_1s_2q \in Q \bigcap -Q$. Thereby, we have $xm \in Q \bigcap -Q$, since $Q \bigcap -Q$ is a prime submodule of M and $ts_3 \in S \subseteq (Q:Q) \setminus -(Q:Q)$. Thus, $xm \in -Q$, a contradiction. It follows that $xm \in -Q$. By Theorem 3.9, we have either $x \in (Q:Q)$ or $m \in Q$. Further, either $x' = \frac{x}{s_1} \in S^{-1}(Q:Q)$ or $m' \in S^{-1}Q$. Observe that $S^{-1}(Q:Q) \subseteq (S^{-1}Q : S^{-1}Q)$. Thus, we have $x' \in (S^{-1}Q : S^{-1}Q)$ or $m' \in S^{-1}Q$. Therefore, by Theorem 3.9, $S^{-1}Q$ is an ordering of $S^{-1}R-$module $S^{-1}M$. The proof is completed.

We can find many multiplicative subsets Ses of R, such that S satisfies $S \subseteq (Q:Q) \setminus -(Q:Q)$. For example, for an ordering Q of M, set $S = 1 + supp((Q:Q))$, where $supp((Q:Q))$ is the support of ordering $(Q:Q)$ of R. Then such an S satisfies the condition. More generally, we have

Corollary 3.13. *Let M be an $R-$module, Q an ordering of M, and S a multiplicative subset of R. Set $T=\{1+\sum_{i=1}^{n} p_i {s_i}^2 | n\in\mathbb{N}, p_i\in(Q:Q), s_i\in S, i=1,\cdots,n\}$. Then $T^{-1}Q$ is an ordering of $T^{-1}R-$module $T^{-1}M$.*

Proof. Obviously, T is a multiplicative subset of R. Since $(T-1)Q=\left(\sum_{i=1}^{n} p_i {s_i}^2\right)Q\subseteq Q$, we have $T\subseteq 1+(Q:Q)$. Observe that $1+(Q:Q)\subseteq(Q:Q)\setminus -(Q:Q)$. Therefore, by Proposition 3.12, the proof is completed.

4. $\mathscr{N}(M)$

Definition 4.1. Let M be an $R-$module. Define $\mathscr{N}(M)$ be the set of elements which aren't semireal in M, i.e. $\mathscr{N}(M)=\{v\in M \mid \text{there is a } t\in T_R \text{ such that } (1+t)v=0\}$.

Lemma 4.2. *Let M be an $R-$module. Then*

(1)$\mathscr{N}(M)$ is a submodule of M;

(2)If Q is a preordering of M, so is $Q^\star=Q+\mathscr{N}(M)$;

(3)If Q is a preordering of M, then $(Q\cup -Q)\cap\mathscr{N}(M)\subseteq Q\cap -Q$. In particular, when Q is an ordering of M, $\mathscr{N}(M)\subseteq Q\cap -Q$.

Proof. (1)For $v_1,v_2\in\mathscr{N}(M)$, there exist $t_1,t_2\in T_R$ such that $(1+t_i)v_i=0, i=1,2$. Thus, $(1+t_1+t_2+t_1t_2)(v_1-v_2)=(1+t_1)(1+t_2)(v_1-v_2)=0$. Since $t+t_1+tt_1\in T_R$, $v_1-v_2\in\mathscr{N}(M)$. For any $x\in R, v\in\mathscr{N}(M)$, there is a $t\in T_R$ such that $(1+t)v=0$. Thus, $(1+t)(xv)=x(1+t)v=0$. Hence, $xv\in\mathscr{N}(M)$. Therefore, $\mathscr{N}(M)$ is a submodule of M.

(2)It's easy to check that $Q^\star+Q^\star\subseteq Q^\star$ and $x^2\in(Q^\star:Q^\star)$. Since Q is a preordering, there exists a $q\in Q$, but $-q\notin Q$. We claim that $q\in Q^\star$, but $-q\notin Q^\star$. Indeed, if not, then there exist a $q'\in Q$ and a $v\in\mathscr{N}(M)$ such that $-q=q'+v$. Further, there is a $t\in T_R$, such that $(1+t)(q+q')=(1+t)(-v)=0$. Thus, $-q=(1+t)q'+tq\in Q$, a contradiction. Thereby, $-1\notin(Q^\star:Q^\star)$. By Lemma 3.2, $Q^\star$ is a preordering of M.

(3)Assume $m \in (Q \cup -Q) \cap \mathscr{N}(M)$. Then, there exists a $t \in T_R$, such that $(1+t)m = 0$. Thus, $m = -tm \in Q \cap -Q$, whether $m \in Q$ or $m \in -Q$. Therefore, $(Q \cup -Q) \cap \mathscr{N}(M) \subseteq Q \cap -Q$. This completes the proof.

Proposition 4.3. *Set $\mathscr{Y}_{\mathscr{N}}(M) = \{Q \mid Q$ is a preordering of M with $\mathscr{N}(M) \subseteq Q\}$. If $\mathscr{Y}_{\mathscr{N}}(M)$ is not empty, then $\mathscr{N}(M) = \bigcap_{Q \in \mathscr{Y}_{\mathscr{N}}(M)} Q$.*

Proof. Obviously, $\mathscr{N}(M) \subseteq \bigcap_{Q \in \mathscr{Y}_{\mathscr{N}}(M)} Q$. Suppose that there is an $e \in \bigcap_{Q \in \mathscr{Y}_{\mathscr{N}}(M)} Q \setminus \mathscr{N}(M)$. By definition of $\mathscr{N}(M)$, e is a semireal element. By Remark 2, e induces a preordering $Q = \{te \mid t \in T_R\}$. Set $Q^\star = Q + \mathscr{N}(M)$. Then, by Lemma 4.2(2), $Q^\star$ is a preordering of M. Thus, $Q^\star \in \mathscr{Y}_{\mathscr{N}}(M)$, since $\mathscr{N}(M) \subseteq Q^\star$. Note that $\mathscr{N}(M) \subseteq -Q^\star$ and $-Q^\star$ is a preordering of M by the Remarks 1(1). Thus $-Q^\star \in \mathscr{Y}_{\mathscr{N}}(M)$. Hence $e \in \bigcap_{Q \in \mathscr{Y}_{\mathscr{N}}(M)} Q \subseteq -Q^\star$. On the other hand, from the proof of Lemma 4.2(2), we have $-e \notin Q^\star$, a contradiction. It follows that $\bigcap_{Q \in \mathscr{Y}_{\mathscr{N}}(M)} Q \subseteq \mathscr{N}(M)$. Therefore, $\mathscr{N}(M) = \bigcap_{Q \in \mathscr{Y}_{\mathscr{N}}(M)} Q$. This completes the proof.

Corollary 4.4. *Set $\mathscr{Y}(M) = \{Q \mid Q$ is a preordering of $M\}$. If $\mathscr{Y}(M)$ is not empty, then $\mathscr{N}(M) = \bigcap_{Q \in \mathscr{Y}(M)} (Q + \mathscr{N}(M))$.*

Proof. Let $\mathscr{Y}_{\mathscr{N}}(M)$ be the same as above. It's easy to check that $\mathscr{Y}_{\mathscr{N}}(M) = \{Q + \mathscr{N}(M) \mid Q \in \mathscr{Y}(M)\}$. By Proposition 4.3, the proof is completed.

Lemma 4.5. *Let Q be a subset of $R-$module M. Set $\overline{Q} = Q/\mathscr{N}(M) = \{\bar{q} = q + \mathscr{N}(M) \mid q \in Q\}$. Then we have*

(1)If Q is a preordering of $R-$module M, then $\overline{Q}$ is a preordering of R-module $M/\mathscr{N}(M)$.

(2)If Q is an ordering of $R-$module M, then $\overline{Q}$ is an ordering of $R-$module $M/\mathscr{N}(M)$.

Proof. (1)Consider the exact sequence $\mathscr{N}(M) \xrightarrow{i} M \xrightarrow{\phi} M/\mathscr{N}(M)$, where i is the identity homomorphism and ϕ is the canonical homomorphism. Since $\mathscr{N}(M)$ is not semireal, $i^{-1}(Q)$ is not a preordering of $\mathscr{N}(M)$. By Proposition 3.6, $\overline{Q} = \phi(Q)$ is a preordering of $M/\mathscr{N}(M)$.

(2)Consider the exact sequence $\mathscr{N}(M) \xrightarrow{i} M \xrightarrow{\phi} M/\mathscr{N}(M) \longrightarrow 0$, where i is the identity homomorphism and ϕ is the canonical homomorphism. Since $\mathscr{N}(M)$ is not semireal, $i^{-1}(Q)$ is not an ordering of $\mathscr{N}(M)$. By Proposition 3.11, $\overline{Q} = \phi(Q)$ is an ordering of $M/\mathscr{N}(M)$. This completes the proof.

Note that every nonzero element of $M/\mathscr{N}(M)$ is semireal. For proving this, suppose that $\bar{m}$ is not semireal in $M/\mathscr{N}(M)$. Then there is a $t \in T_R$ such that $(1+t)\bar{m} = \bar{0}$, and so $(1+t)m \in \mathscr{N}(M)$. Hence, for some $t_1 \in T_R, (1+t_1)(1+t)m = 0$, i.e. $(1+t+t_1+tt_1)m = 0$. Thus, $m \in \mathscr{N}(M)$. It follows that $\bar{m} = 0$.

Theorem 4.6. *Let M be an $R-$module. Then M possesses an ordering if and only if $M/\mathscr{N}(M)$ possesses an ordering. Furthermore, there is an one-to-one correspondence between the set of all orderings on $R-$module M and the set of all orderings of $R-$module $M/\mathscr{N}(M)$.*

Proof. It suffices to prove the later conclusion. For this end, let $\mathscr{X}(M)$ (resp. $\mathscr{X}(M/\mathscr{N}(M))$) be the set of all orderings on M (resp. $M/\mathscr{N}(M)$)), and let $\varphi : \mathscr{X}(M) \to \mathscr{X}(M/\mathscr{N}(M))$ be defined as follows:

$$\varphi(Q) = \overline{Q} = \{\bar{q} = q + \mathscr{N}(M) \mid q \in Q\},\ Q \in \mathscr{X}(M).$$

By Lemma 4.5(2), $\overline{Q}$ is an ordering of $R-$module $M/\mathscr{N}(M)$, and so φ is a map from $\mathscr{X}(M)$ to $\mathscr{X}(M/\mathscr{N}(M))$. Now, we shall first show that the mapping is injective. For this purpose, let $\varphi(Q_1) = \varphi(Q_2)$, where $Q_1, Q_2 \in \mathscr{X}(M)$. Then $\overline{Q_1} = \overline{Q_2}$. We claim that $Q_1 = Q_2$. If otherwise, without loss of generality, we may assume that $Q_1 \nsubseteq Q_2$. Let $q \in Q_1 \setminus Q_2$. Then, $\bar{q} \in \overline{Q_1} = \overline{Q_2}$. Thus, there exists a $q_2 \in Q_2$, such that $q - q_2 \in \mathscr{N}(M)$. By Lemma 4.2(3), $\mathscr{N}(M) \subseteq Q_2 \cap -Q_2$. Hence, $q \in q_2 + \mathscr{N}(M) \subseteq Q_2 + Q_2 \cap -Q_2 \subseteq Q_2$, a contradiction. Now, it remains to show the map is surjective. For this, let $\overline{Q}$ be an ordering of $M/\mathscr{N}(M)$. Denote by ϕ the canonical module homomorphism from M to $M/\mathscr{N}(M)$. Observe that $\phi(M) = M/\mathscr{N}(M) \nsubseteq \bar{Q}$. Then, by Lemma 3 in [9], $\phi^{-1}(\overline{Q})$ is an ordering of M. On the other hand , notice that $\varphi(\phi^{-1}(\overline{Q})) = \overline{Q}$. It follows that the map φ is surjective. This completes the proof.

Conflict of Interests

The author declares that there is no conflict of interests.

Acknowledgements

This work was supported by the youth foundation of Hainan University (qnjj1246) and the National Natural Science Foundation of China (10971044).

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