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INTERVAL FUZZY CONNECTIVES AND PAIRS OF IMPLICATIONS

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Abstract. In this paper, we construct interval fuzzy connectives from pairs of negations and implications. Moreover, we investigate their properties and give examples.

Keywords: pairs of (interval) negations; pairs of (interval) implications; interval pseudo t-norms; interval pseudo t-conorms.

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1. Introduction

Many researchers [5-8] introduced non-commutative algebraic structures. This concept is motivated not only by the logically interest, but also by the relation with some remarkable mathematical structures. In [5], pseudo t-norms were investigated in bounded lattices in a sense as non-commutative property. Bedregal and Takahashi [4] introduced interval fuzzy connectives as an extension for fuzzy connectives. This concept provides tools for approximate reasoning

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and decision making with a frame work to deal with uncertainty and incompleteness of information [1-3]. Kim [11,12] introduced pairs of (interval) negations and (interval) implications. which are induced by non-commutative property.

In this paper, we construct interval fuzzy connectives from pairs of negations and implications. Moreover, we investigate their properties and give examples.

2. Preliminaries

In this paper, we assume that $(L,\vee,\wedge,\bot,\top)$ is a bounded lattice with a bottom element $\bot$ and a top element $\top$.

Definition 2.1. [5,11,12] A map $T : L\times L \to L$ is called a *pseudo t-norm* if it satisfies the following conditions:

(T1) $T(x,T(y,z)) = T(T(x,y),z)$ for all $x,y,z \in L$,

(T2) If $y \le z$, $T(x,y) \le T(x,z)$ and $T(y,x) \le T(z,x)$,

(T3) $T(x,\top) = T(\top,x) = x$.

A pseudo t-norm is called a *t-norm* if $T(x,y) = T(y,x)$ for $x,y \in L$

A map $S : L\times L \to L$ is called a *pseudo t-conorm* if it satisfies the following conditions:

(S1) $S(x,S(y,z)) = S(S(x,y),z)$ for all $x,y,z \in L$,

(S2) If $y \le z$, $S(x,y) \le S(x,z)$ and $S(y,x) \le S(z,x)$,

(S3) $S(x,\bot) = S(\bot,x) = x$.

A pseudo t-conorm is called a *t-conorm* if $S(x,y) = S(y,x)$ for $x,y \in L$.

Definition 2.2. [11,12] A pair (n_1,n_2) with maps $n_i : L \to L$ is called a *pair of negations* if it satisfies the following conditions:

(N1) $n_i(\top) = \bot, n_i(\bot) = \top$ for all $i \in \{1,2\}$.

(N2) $n_i(x) \ge n_i(y)$ for $x \le y$ and $i \in \{1,2\}$.

(N3) $n_1(n_2(x)) = n_2(n_1(x)) = x$ for all $x \in X$.

Definition 2.3. [11,12] A pair (I_1,I_2) with maps $I_1,I_2 : L\times L \to L$ is called a *pair of implications* if it satisfies the following conditions:

(I1) $I_i(\top,\top) = I_i(\bot,\top) = I_i(\bot,\bot) = \top, I_i(\top,\bot) = \bot$ for all $i \in \{1,2\}$, (I2) If $x \leq y$, then $I_i(x,z) \geq I_i(y,z)$ for all $i \in \{1,2\}$,

(I3) $I_i(\top,x) = x$ for all $i \in \{1,2\}$,

(I4) $I_1(x,I_2(y,z)) = I_2(y,I_1(x,z))$ for all $x,y,z \in X$,

(I5) $I_1(I_2(x,\perp),\perp) = I_2(I_1(x,\perp),\perp) = x$.

Let $(L,\vee,\wedge,\top,\perp)$ be a bounded lattice. Let $L^{[2]} = \{[x_1,x_2] \mid x_1 \leq x_2, x_1,x_2 \in L\}$ where $[x_1,x_2] = \{x \in L \mid x_1 \leq x \leq x_2\}$. We define

$$[x_1,x_2] \leq [y_1,y_2], \text{ iff } x_1 \leq y_1,\ x_2 \leq y_2$$

$$[x_1,x_2] \subset [y_1,y_2], \text{ iff } y_1 \leq x_1 \leq x_2 \leq y_2$$

$$l([x_1,x_2]) = x_1,\ \ r([x_1,x_2]) = x_2.$$

Definition 2.4. [11,12] A map $\mathbf{T} : L^{[2]} \times L^{[2]} \to L^{[2]}$ is called a *interval pseudo t-norm* if it satisfies the following conditions: for all $[x_1,x_2],[y_1,y_2],[z_1,z_2] \in L^{[2]}$,

(IT1) $\mathbf{T}([x_1,x_2],\mathbf{T}([y_1,y_2],[z_1,z_2])) = \mathbf{T}(\mathbf{T}([x_1,x_2],[y_1,y_2]),[z_1,z_2])$,

(IT2) $\mathbf{T}([x_1,x_2],[\top,\top]) = \mathbf{T}([\top,\top],[x_1,x_2]) = [x_1,x_2]$.

(IT3) If $[x_1,x_2] \leq [z_1,z_2]$ and $[y_1,y_2] \leq [w_1,w_2]$, then $\mathbf{T}([x_1,x_2],[y_1,y_2]) \leq \mathbf{T}([z_1,z_2],[w_1,w_2])$.

(IT4) If $[x_1,x_2] \subset [z_1,z_2]$ and $[y_1,y_2] \subset [w_1,w_2]$, then $\mathbf{T}([x_1,x_2],[y_1,y_2]) \subset \mathbf{T}([z_1,z_2],[w_1,w_2])$.

A pseudo t-norm is called a *interval t-norm* if $\mathbf{T}([x_1,x_2],[y_1,y_2]) = \mathbf{T}([y_1,y_2],[x_1,x_2])$ for $[x_1,x_2],[y_1,y_2] \in L^{[2]}$

Definition 2.5. [11,12] A map $\mathbf{S} : L^{[2]} \times L^{[2]} \to L^{[2]}$ is called a *interval pseudo t-conorm* if it satisfies the following conditions: for all $[x_1,x_2],[y_1,y_2],[z_1,z_2] \in L^{[2]}$,

(IS1) $\mathbf{S}([x_1,x_2],\mathbf{S}([y_1,y_2],[z_1,z_2])) = \mathbf{S}(\mathbf{S}([x_1,x_2],[y_1,y_2]),[z_1,z_2])$,

(IS2) $\mathbf{S}([x_1,x_2],[\perp,\perp]) = \mathbf{S}([\perp,\perp],[x_1,x_2]) = [x_1,x_2]$.

(IS3) If $[x_1,x_2] \leq [z_1,z_2]$ and $[y_1,y_2] \leq [w_1,w_2]$, then $\mathbf{S}([x_1,x_2],[y_1,y_2]) \leq \mathbf{S}([z_1,z_2],[w_1,w_2])$.

(IS4) If $[x_1,x_2] \subset [z_1,z_2]$ and $[y_1,y_2] \subset [w_1,w_2]$, then $\mathbf{S}([x_1,x_2],[y_1,y_2]) \subset \mathbf{S}([z_1,z_2],[w_1,w_2])$.

An interval pseudo t-conorm is called an *interval t-conorm* if $\mathbf{S}([x_1,x_2],[y_1,y_2]) = \mathbf{S}([y_1,y_2],[x_1,x_2])$ for $[x_1,x_2],[y_1,y_2] \in L^{[2]}$

Remark 2.6. (1) If T is a pseudo t-norm, then $T([x_1,x_2],\perp) = \perp = T(\perp,x)$ because $T([x_1,x_2],\perp) \leq T(\perp,\top) = \perp$.

(2) If S is a pseudo t-conorm, then $S([x_1,x_2],\top) = \top = S(\top,x)$ because $S([x_1,x_2],\top) \geq S(\bot,\top) = \top$.

Definition 2.7. [11,12] A pair $(\mathbf{N}_1,\mathbf{N}_2)$ with maps $\mathbf{N}_i : L^{[2]} \to L^{[2]}$ is called a *a pair of interval negations* if it satisfies the following conditions:

(IN1) $\mathbf{N}_i([\top,\top]) = [\bot,\bot]$, $\mathbf{N}_i([\bot,\bot]) = [\top,\top]$ for all $i \in \{1,2\}$.

(IN2) If $[x_1,x_2] \leq [y_1,y_2]$, then $\mathbf{N}([y_1,y_2]) \leq \mathbf{N}_i([x_1,x_2])$ for all $i \in \{1,2\}$.

(IN3) If $[x_1,x_2] \subset [y_1,y_2]$, then $\mathbf{N}_i([x_1,x_2]) \subset \mathbf{N}_i([y_1,y_2])$ for all $i \in \{1,2\}$,

(IN4) $\mathbf{N}_1(\mathbf{N}_2([x_1,x_2])) = \mathbf{N}_2(\mathbf{N}_1([x_1,x_2])) = [x_1,x_2]$ for all $[x_1,x_2] \in L^{[2]}$.

Definition 2.8. [11,12] A pair $(\mathbf{I}_1,\mathbf{I}_2)$ with maps $\mathbf{I}_1,\mathbf{I}_2 : L^{[2]} \times L^{[2]} \to L^{[2]}$ is called a *pair of interval implications* if it satisfies the following conditions:

(II1) $\mathbf{I}_i([\top,\top],[\top,\top]) = \mathbf{I}_i([\bot,\bot],[\top,\top]) = \mathbf{I}_i([\bot,\bot],[\bot,\bot]) = [\top,\top]$, $\mathbf{I}_i([\top,\top],[\bot,\bot]) = [\bot,\bot]$ for all $i \in \{1,2\}$.

(II2) If $[x_1,x_2] \leq [y_1,y_2]$, then $\mathbf{I}_i([x_1,x_2],[z_1,z_2]) \geq \mathbf{I}_i([y_1,y_2],[z_1,z_2])$ for all $i \in \{1,2\}$.

(II3) If $[x_1,x_2] \subset [y_1,y_2]$, then $\mathbf{I}_i([x_1,x_2],[z_1,z_2]) \subset \mathbf{I}_i([y_1,y_2],[z_1,z_2])$ for all $i \in \{1,2\}$.

(II4) $\mathbf{I}_i([\top,\top],[x_1,x_2]) = [x_1,x_2]$ for all $i \in \{1,2\}$.

(II5) $\mathbf{I}_1([x_1,x_2],\mathbf{I}_2([y_1,y_2],[z_1,z_2])) = \mathbf{I}_2([y_1,y_2],\mathbf{I}_1([x_1,x_2],[z_1,z_2]))$
for all $[x_1,x_2],[y_1,y_2],[z_1,z_2] \in L^{[2]}$.

(II6) $\mathbf{I}_1(\mathbf{I}_2([x_1,x_2],[\bot,\bot]),[\bot,\bot]) = \mathbf{I}_2(\mathbf{I}_1([x_1,x_2],[\bot,\bot]),[\bot,\bot]) = [x_1,x_2]$.

Theorem 2.9. [11,12] *Let (n_1,n_2) be a pair of negations on L. Then we have the following properties.*

(1) *Define maps $I_i : L \times L \to L$ as*

$$I_1(x,y) = n_1(x) \vee y,\ I_2(x,y) = n_2(x) \vee y.$$

Then (I_1,I_2) is a pair of implications.

(2) *Define maps $\mathbf{N}_i : L^{[2]} \to L^{[2]}$ as*

$$\mathbf{N}_1([x_1,x_2]) = [n_1(x_2),n_1(x_1)],\mathbf{N}_2([x_1,x_2]) = [n_2(x_2),n_2(x_1)]$$

Then $(\mathbf{N}_1,\mathbf{N}_2)$ is a pair of interval negations such that

$$\underline{\mathbf{N}_i}(x) = \overline{\mathbf{N}_i}(x) = n_i(x),$$

$$\mathbf{N}_i([x_1,x_2]) = [\underline{\mathbf{N}_i}(x_2), \overline{\mathbf{N}_i}(x_1)].$$

(3) *For maps I_i in (1), we define maps $\mathbf{I}_i : L^{[2]} \times L^{[2]} \to L^{[2]}$ as*

$$\mathbf{I}_1([x_1,x_2],[y_1,y_2]) = [n_1(x_2) \vee y_1, n_1(x_1) \vee y_2],$$

$$\mathbf{I}_2([x_1,x_2],[y_1,y_2]) = [n_2(x_2) \vee y_1, n_2(x_1) \vee y_2].$$

Then $(\mathbf{I}_1, \mathbf{I}_2)$ is a pair of interval implications such that $\underline{\mathbf{I}_i}(x,y) = n_i(x) \vee y = \overline{\mathbf{I}_i}(x,y)$ and

$$\mathbf{I}_i([x_1,x_2],[y_1,y_2]) = [\underline{\mathbf{I}_i}(x_2,y_1), \overline{\mathbf{I}_i}(x_1,y_2)].$$

Theorem 2.10. [11,12] *Let $(L, \vee, \wedge, \top, \bot)$ be a bounded lattice and (I_1, I_2) an pair of implications on L. We define*

$$n_1(x) = I_1(x, \bot),\ n_2(x) = I_2(x, \bot).$$

(1) *(n_1, n_2) is a pair of negations.*

(2) *$I_1(n_2(y), n_2(x)) = I_2(x,y)$ and $I_2(n_1(y), n_1(x)) = I_1(x,y)$.*

(3)*If $y \leq z$, then $I_i(x,y) \leq I_i(x,z)$.*

(4) *For maps I_i in (1), we define maps $\mathbf{I}_i : L^{[2]} \times L^{[2]} \to L^{[2]}$ as*

$$\mathbf{I}_1([x_1,x_2],[y_1,y_2]) = [I_1(x_2,y_1), I_1(x_1,y_2)],$$

$$\mathbf{I}_2([x_1,x_2],[y_1,y_2]) = [I_2(x_2,y_1), I_2(x_1,y_2)].$$

Then $(\mathbf{I}_1, \mathbf{I}_2)$ is a pair of interval implications such that $\underline{\mathbf{I}_i}(x,y) = I_i(x,y) = \overline{\mathbf{I}_i}(x,y)$ and

$$\mathbf{I}_i([x_1,x_2],[y_1,y_2]) = [\underline{\mathbf{I}_i}(x_2,y_1), \overline{\mathbf{I}_i}(x_1,y_2)].$$

(5) *Define maps $\mathbf{N}_i : L^{[2]} \to L^{[2]}$ as*

$$\mathbf{N}_1([x_1,x_2]) = [I_1(x_2,\bot), I_1(x_1,\bot)],$$

$$\mathbf{N}_2([x_1,x_2]) = [I_2(x_2,\bot), I_2(x_1,\bot)].$$

Then $(\mathbf{N}_1, \mathbf{N}_2)$ is a pair of interval negations such that

$$\underline{\mathbf{N}_i}(x) = \overline{\mathbf{N}_i}(x) = I_i(x,\bot),$$

$$\mathbf{N}_i([x_1,x_2]) = [\underline{\mathbf{N}_i}(x_2), \overline{\mathbf{N}_i}(x_1)].$$

In this paper, sometimes composition function $f\circ g\circ h$ will be denoted by fgh.

3. Interval fuzzy connectives and pairs of implications

Theorem 3.1. *Let $(L,\vee,\wedge,\top,\bot)$ be a bounded lattice and (I_1,I_2) an pair of implications on L. We define $\mathbf{I}_i,\mathbf{T}_i,\mathbf{S}_i : L^{[2]}\times L^{[2]}\to L^{[2]}$ and $\mathbf{N}_i : L^{[2]}\to L^{[2]}$*

$$\mathbf{I}_1([x_1,x_2],[y_1,y_2]) = [I_1(x_2,y_1),I_1(x_1,y_2)],$$

$$\mathbf{I}_2([x_1,x_2],[y_1,y_2]) = [I_2(x_2,y_1),I_2(x_1,y_2)],$$

$$n_1(x) = I_1(x,\bot),\ \ n_2(x) = I_2(x,\bot),$$

$$\mathbf{N}_1([x_1,x_2]) = [n_1(x_2),n_1(x_1)],$$

$$\mathbf{N}_2([x_1,x_2]) = [n_2(x_2),n_2(x_1)],$$

$$\mathbf{T}_1([x_1,x_2],[y_1,y_2]) = \mathbf{N}_1(\mathbf{I}_2([x_1,x_2],\mathbf{N}_2([y_1,y_2]))),$$

$$\mathbf{T}_2([x_1,x_2],[y_1,y_2]) = \mathbf{N}_1(\mathbf{I}_2([y_1,y_2],\mathbf{N}_2([x_1,x_2]))),$$

$$\mathbf{T}_3([x_1,x_2],[y_1,y_2]) = \mathbf{N}_2(\mathbf{I}_1([x_1,x_2],\mathbf{N}_1([y_1,y_2]))),$$

$$\mathbf{T}_4([x_1,x_2],[y_1,y_2]) = \mathbf{N}_2(\mathbf{I}_1([y_1,y_2],\mathbf{N}_1([x_1,x_2]))),$$

$$\mathbf{S}_1([x_1,x_2],[y_1,y_2]) = \mathbf{I}_1(\mathbf{N}_2([x_1,x_2],[y_1,y_2])),$$

$$\mathbf{S}_2([x_1,x_2],[y_1,y_2]) = \mathbf{I}_1(\mathbf{N}_2([y_1,y_2],[x_1,x_2])),$$

$$\mathbf{S}_3([x_1,x_2],[y_1,y_2]) = \mathbf{I}_2(\mathbf{N}_1([x_1,x_2],[y_1,y_2])),$$

$$\mathbf{S}_4([x_1,x_2],[y_1,y_2]) = \mathbf{I}_2(\mathbf{N}_1([y_1,y_2],[x_1,x_2])).$$

Then The the following properties hold.

(1) $\mathbf{I}_1(\mathbf{N}_2([y_1,y_2]),\mathbf{N}_2([x_1,x_2])) = \mathbf{I}_2([x_1,x_2],[y_1,y_2])$ *and*

$\mathbf{I}_2(\mathbf{N}_1([y_1,y_2]),\mathbf{N}_1([x_1,x_2])) = \mathbf{I}_1([x_1,x_2],[y_1,y_2])$.

(2) *For each $i\in\{1,2,3,4\}$, $\mathbf{T}_i$ is an interval pseudo t-norms such that*

$$\mathbf{T}_1([x_1,x_2],[y_1,y_2]) = [n_1I_2(x_1,n_2(y_1)),n_1I_2(x_2,n_2(y_2))],$$

$$\mathbf{T}_2([x_1,x_2],[y_1,y_2]) = [n_1I_2(y_1,n_2(x_1)),n_1I_2(y_2,n_2(x_2))],$$

$$\mathbf{T}_3([x_1,x_2],[y_1,y_2]) = [n_2I_1(x_1,n_1(y_1)), n_2I_1(x_2,n_1(y_2))],$$

$$\mathbf{T}_4([x_1,x_2],[y_1,y_2]) = [n_2I_1(y_1,n_1(x_1)), n_2I_1(y_2,n_1(x_2))].$$

(3) *If* $n_1I_2(x,y) = n_2I_1(n_1(y),n_1(x))$*, then* $\mathbf{T}_1 = \mathbf{T}_4$ *and* $\mathbf{T}_2 = \mathbf{T}_3$*.*

(4)

$$\mathbf{I}_2(\mathbf{T}_1([x_1,x_2],[y_1,y_2]),[z_1,z_2]) = \mathbf{I}_2([x_1,x_2],\mathbf{I}_2([y_1,y_2],[z_1,z_2])),$$

$$\mathbf{I}_2(\mathbf{T}_2([x_1,x_2],[y_1,y_2]),[z_1,z_2]) = \mathbf{I}_2([y_1,y_2],\mathbf{I}_2([x_1,x_2],[z_1,z_2])),$$

$$\mathbf{I}_1(\mathbf{T}_3([x_1,x_2],[y_1,y_2]),[z_1,z_2]) = \mathbf{I}_1([x_1,x_2],\mathbf{I}_1([y_1,y_2],[z_1,z_2])),$$

$$\mathbf{I}_1(\mathbf{T}_4([x_1,x_2],[y_1,y_2]),[z_1,z_2]) = \mathbf{I}_1([y_1,y_2],\mathbf{I}_1([x_1,x_2],[z_1,z_2])).$$

(5) *If* $[x_1,x_2] \leq [y_1,y_2]$ *iff* $\mathbf{I}_1([x_1,x_2],[y_1,y_2]) = [\top,\top]$ *iff* $\mathbf{I}_2([x_1,x_2],[y_1,y_2]) = [\top,\top]$*, then*

$$\mathbf{T}_1([x_1,x_2],[y_1,y_2]) \leq [z_1,z_2] \textit{ iff } [x_1,x_2] \leq \mathbf{I}_2([y_1,y_2],[z_1,z_2])$$
$$\textit{iff } [y_1,y_2] \leq \mathbf{I}_1([x_1,x_2],[z_1,z_2]) \textit{ iff } \mathbf{T}_4([x_1,x_2],[y_1,y_2]) \leq [z_1,z_2],$$

$$\mathbf{T}_3([x_1,x_2],[y_1,y_2]) \leq [z_1,z_2] \textit{ iff } [x_1,x_2] \leq \mathbf{I}_1([y_1,y_2],[z_1,z_2])$$
$$\textit{iff } [y_1,y_2] \leq \mathbf{I}_2([x_1,x_2],[z_1,z_2]) \textit{ iff } \mathbf{T}_2([x_1,x_2],[y_1,y_2]) \leq [z_1,z_2].$$

Moreover, $\mathbf{T}_1([x_1,x_2],[y_1,y_2]) = \mathbf{T}_4([x_1,x_2],[y_1,y_2])$ *and*

$\mathbf{T}_2([x_1,x_2],[y_1,y_2]) = \mathbf{T}_3([x_1,x_2],[y_1,y_2])$.

(6) *For each* $i \in \{1,2,3,4\}$, $\mathbf{S}_i$ *is an interval pseudo t-norms such that*

$$\mathbf{S}_1([x_1,x_2],[y_1,y_2]) = [I_1(n_2(x_1),y_1), I_1(n_2(x_2),y_2)],$$

$$\mathbf{S}_2([x_1,x_2],[y_1,y_2]) = [I_1(n_2(y_1),x_1), I_1(n_2(y_2),x_2)],$$

$$\mathbf{S}_3([x_1,x_2],[y_1,y_2]) = [I_2(n_1(x_1),y_1), I_2(n_1(x_2),y_2)],$$

$$\mathbf{S}_4([x_1,x_2],[y_1,y_2]) = [I_2(n_1(y_1),x_1), I_2(n_1(y_2),x_2)].$$

Moreover,

$$\mathbf{S}_1([x_1,x_2],[y_1,y_2]) = \mathbf{S}_4([x_1,x_2],[y_1,y_2]),$$

$$\mathbf{S}_2([x_1,x_2],[y_1,y_2]) = \mathbf{S}_3([x_1,x_2],[y_1,y_2]).$$

(7)

$$\mathbf{T}_1([x_1,x_2],[y_1,y_2]) = \mathbf{N}_1\mathbf{S}_3(\mathbf{N}_2([x_1,x_2]),\mathbf{N}_2([y_1,y_2])),$$

$$\mathbf{T}_2([x_1,x_2],[y_1,y_2]) = \mathbf{N}_1\mathbf{S}_4(\mathbf{N}_2([x_1,x_2]),\mathbf{N}_2([y_1,y_2])),$$

$$\mathbf{T}_3([x_1,x_2],[y_1,y_2]) = \mathbf{N}_2\mathbf{S}_1(\mathbf{N}_1([x_1,x_2]),\mathbf{N}_1([y_1,y_2])),$$

$$\mathbf{T}_4([x_1,x_2],[y_1,y_2]) = \mathbf{N}_2\mathbf{S}_2(\mathbf{N}_1([x_1,x_2]),\mathbf{N}_1([y_1,y_2])).$$

Proof. (1) $\mathbf{I}_1(\mathbf{N}_2([y_1,y_2]),\mathbf{N}_2([x_1,x_2]) = \mathbf{I}_2([x_1,x_2],[y_1,y_2])$ from:

$$\begin{aligned}
&\mathbf{I}_1(\mathbf{N}_2([y_1,y_2]),\mathbf{N}_2([x_1,x_2])\\
&= \mathbf{I}_1(\mathbf{I}_2([y_1,y_2],[\bot,\bot]),\mathbf{I}_2([x_1,x_2],[\bot,\bot]),\\
&= \mathbf{I}_2([x_1,x_2],\mathbf{I}_1(\mathbf{I}_2([y_1,y_2],[\bot,\bot]),[\bot,\bot])\\
&= \mathbf{I}_2([x_1,x_2],[y_1,y_2]).
\end{aligned}$$

Other case is similarly proved.

(2) (IT1)

$$\begin{aligned}
&\mathbf{T}_1(\mathbf{T}_1([x_1,x_2],[y_1,y_2]),[z_1,z_2])\\
&= \mathbf{N}_1(\mathbf{I}_2(\mathbf{T}_1([x_1,x_2],[y_1,y_2]),\mathbf{N}_2([z_1,z_2])))\\
&= \mathbf{N}_1(\mathbf{I}_2(\mathbf{N}_1(\mathbf{I}_2([x_1,x_2],\mathbf{N}_2([y_1,y_2]))),\mathbf{N}_2([z_1,z_2])))\\
&= \mathbf{N}_1(\mathbf{I}_1(\mathbf{N}_2(\mathbf{N}_2([z_1,z_2])),\mathbf{I}_2([x_1,x_2],\mathbf{N}_2([y_1,y_2]))))\\
&= \mathbf{N}_1(\mathbf{I}_2([x_1,x_2],\mathbf{I}_1(\mathbf{N}_2(\mathbf{N}_2([z_1,z_2])),\mathbf{N}_2([y_1,y_2]))))\\
&= \mathbf{N}_1(\mathbf{I}_2([x_1,x_2],\mathbf{I}_2([y_1,y_2],\mathbf{N}_2([z_1,z_2]))))\\
&= \mathbf{N}_1(\mathbf{I}_2([x_1,x_2],\mathbf{N}_2(\mathbf{T}_1([y_1,y_2],[z_1,z_2]))))\\
&= \mathbf{T}_1([x_1,x_2],\mathbf{T}_1([y_1,y_2],[z_1,z_2])).
\end{aligned}$$

(IT2) If $[y_1,y_2] \le [z_1,z_2]$, then $\mathbf{I}_2([y_1,y_2],\mathbf{N}_2([x_1,x_2]))) \ge \mathbf{I}_2([z_1,z_2],\mathbf{N}_2([x_1,x_2]))$, $\mathbf{I}_2([x_1,x_2],\mathbf{N}_2([y_1,y_2]))) \ge \mathbf{I}_2([x_1,x_2],\mathbf{N}_2([z_1,z_2]))$. Thus $\mathbf{T}_1([y_1,y_2],[x_1,x_2]) \le \mathbf{T}_1([z_1,z_2],[x_1,x_2])$, $\mathbf{T}_1([x_1,x_2],[y_1,y_2]) \le \mathbf{T}_1([x_1,x_2],[z_1,z_2])$.

(IT3) If $[x_1,x_2] \subset [z_1,z_2]$ and $[y_1,y_2] \subset [w_1,w_2]$, then $\mathbf{I}_2([x_1,x_2],\mathbf{N}_2([y_1,y_2]))) \subset \mathbf{I}_2([z_1,z_2],\mathbf{N}_2([w_1,w_2]))$. Thus $\mathbf{T}_{12}([x_1,x_2],[y_1,y_2]) \subset \mathbf{T}_{12}([x_1,x_2],[z_1,z_2])$.

(IT4)

$$\begin{aligned}
\mathbf{T}_1([x_1,x_2],[\top,\top]) &= \mathbf{N}_1(\mathbf{I}_2([x_1,x_2],[\bot,\bot]))\\
&= \mathbf{N}_1([n_2(x_2),n_2(x_1)])\\
&= [n_1(n_2(x_1)),n_1(n_2(x_2))] = [x_1,x_2].
\end{aligned}$$

$$\begin{aligned}
\mathbf{T}_1([\top,\top],[x_1,x_2]) &= \mathbf{N}_1(\mathbf{I}_2([\top,\top],\mathbf{N}_2([x_1,x_2])))\\
&= \mathbf{N}_1\mathbf{I}_2([\top,\top],[n_2(x_2),n_2(x_1)])\\
&= \mathbf{N}_1([I_2(\top,n_2(x_2)),I_2(\top,n_2(x_1))])\\
&= \mathbf{N}_1([n_2(x_2),n_2(x_1)])\\
&= [n_1(n_2(x_1)),n_1(n_2(x_2))] = [x_1,x_2].
\end{aligned}$$

Hence $\mathbf{T}_1$ is an interval pseudo t-norm. Moreover,

$$\begin{aligned}
\mathbf{T}_1([x_1,x_2],[y_1,y_2])) &= \mathbf{N}_1(\mathbf{I}_2([x_1,x_2],\mathbf{N}_2([y_1,y_2])))\\
&= \mathbf{N}_1(\mathbf{I}_2([x_1,x_2],[n_2(y_2),n_2(y_1)]))\\
&= \mathbf{N}_1(I_2(x_2,n_2(y_2)),I_2(x_1,n_2(y_1)))\\
&= [n_1(I_2(x_1,n_2(y_1))),n_1(I_2(x_2,n_2(y_2)))].
\end{aligned}$$

Similarly, $\mathbf{T}_4$ is an interval pseudo t-norm from:

$$\begin{aligned}
&\mathbf{T}_4(\mathbf{T}_4([x_1,x_2],[y_1,y_2]),[z_1,z_2])\\
&= \mathbf{N}_2(\mathbf{I}_1([z_1,z_2],\mathbf{N}_1(\mathbf{T}_4([x_1,x_2],[y_1,y_2]))))\\
&= \mathbf{N}_2(\mathbf{I}_1([z_1,z_2],\mathbf{I}_1([y_1,y_2],\mathbf{N}_1([x_1,x_2]))))\\
&= \mathbf{N}_2(\mathbf{I}_1([z_1,z_2],\mathbf{I}_2(\mathbf{N}_1(\mathbf{N}_1([x_1,x_2])),\mathbf{N}_1([y_1,y_2]))))\\
&= \mathbf{N}_2(\mathbf{I}_2(\mathbf{N}_1(\mathbf{N}_1([x_1,x_2])),\mathbf{I}_1([z_1,z_2],\mathbf{N}_1([y_1,y_2]))))\\
&= \mathbf{N}_2(\mathbf{I}_2(\mathbf{N}_1(\mathbf{N}_1([x_1,x_2])),\mathbf{I}_1(\mathbf{T}_4([y_1,y_2],[z_1,z_2]))))\\
&= \mathbf{N}_2(\mathbf{I}_1(\mathbf{T}_4([y_1,y_2],[z_1,z_2]),\mathbf{N}_1([x_1,x_2])))\\
&= \mathbf{T}_4([x_1,x_2],\mathbf{T}_4([y_1,y_2],[z_1,z_2]))
\end{aligned}$$

Other cases are similarly proved.

(3) If $n_1I_2(x,y) = n_2I_1(n_1(y),n_1(x))$, then

$$n_1I_2(x,n_2(z)) = n_2I_1(n_1(n_2(z)),n_1(x)) = n_2I_1(z,n_1(x)).$$

By (2), $\mathbf{T}_1 = \mathbf{T}_4$ and $\mathbf{T}_2 = \mathbf{T}_3$.

(4)

$$\begin{aligned}
&\mathbf{I}_2(\mathbf{T}_1([x_1,x_2],[y_1,y_2]),[z_1,z_2])\\
&=\mathbf{I}_2(\mathbf{N}_1(\mathbf{I}_2([x_1,x_2],\mathbf{N}_2([y_1,y_2])),[z_1,z_2]))\\
&=\mathbf{I}_1(\mathbf{N}_2([z_1,z_2]),\mathbf{N}_2(\mathbf{N}_1(\mathbf{I}_2([x_1,x_2],\mathbf{N}_2([y_1,y_2])))))\\
&=\mathbf{I}_1(n_2([z_1,z_2]),\mathbf{I}_2([x_1,x_2],\mathbf{N}_2([y_1,y_2])))\\
&=\mathbf{I}_2([x_1,x_2],\mathbf{I}_1(\mathbf{N}_2(z),\mathbf{N}_2([y_1,y_2])))\\
&=\mathbf{I}_2([x_1,x_2],\mathbf{I}_2([y_1,y_2],[z_1,z_2]))
\end{aligned}$$

$$\begin{aligned}
&\mathbf{I}_2(\mathbf{T}_2([x_1,x_2],[y_1,y_2]),[z_1,z_2])\\
&=\mathbf{I}_2(\mathbf{N}_1(\mathbf{I}_2([y_1,y_2],\mathbf{N}_2([x_1,x_2])),[z_1,z_2]))\\
&=\mathbf{I}_1(\mathbf{N}_2([z_1,z_2]),\mathbf{I}_2([y_1,y_2],\mathbf{N}_2([x_1,x_2]))))\\
&=\mathbf{I}_2([y_1,y_2],\mathbf{I}_1(\mathbf{N}_2([z_1,z_2]),\mathbf{N}_2([x_1,x_2]))))\\
&=\mathbf{I}_2([y_1,y_2],\mathbf{I}_2([x_1,x_2],[z_1,z_2])).
\end{aligned}$$

$$\begin{aligned}
&\mathbf{I}_1(\mathbf{T}_3([x_1,x_2],[y_1,y_2]),[z_1,z_2])\\
&=\mathbf{I}_1(\mathbf{N}_2(\mathbf{I}_1([x_1,x_2],\mathbf{N}_1([y_1,y_2])),[z_1,z_2]))\\
&=\mathbf{I}_2(\mathbf{N}_1([z_1,z_2]),\mathbf{N}_1(\mathbf{N}_2(\mathbf{I}_1([x_1,x_2],\mathbf{N}_1([y_1,y_2])))))\\
&=\mathbf{I}_2(\mathbf{N}_1([z_1,z_2]),\mathbf{I}_1([x_1,x_2],\mathbf{N}_1([y_1,y_2])))\\
&=\mathbf{I}_1([x_1,x_2],\mathbf{I}_2(\mathbf{N}_1([z_1,z_2]),\mathbf{N}_1([y_1,y_2])))\\
&=\mathbf{I}_1([x_1,x_2],\mathbf{I}_1([y_1,y_2],[z_1,z_2]))
\end{aligned}$$

$$\begin{aligned}
&\mathbf{I}_1(\mathbf{T}_4([x_1,x_2],[y_1,y_2]),[z_1,z_2])\\
&=\mathbf{I}_1(\mathbf{N}_2(\mathbf{I}_1([y_1,y_2],\mathbf{N}_1([x_1,x_2])),[z_1,z_2]))\\
&=\mathbf{I}_2(\mathbf{N}_1([z_1,z_2]),\mathbf{N}_1(\mathbf{N}_2(\mathbf{I}_1([y_1,y_2],\mathbf{N}_1([x_1,x_2])))))\\
&=\mathbf{I}_2(\mathbf{N}_1([z_1,z_2]),\mathbf{I}_1([y_1,y_2],\mathbf{N}_1([x_1,x_2])))\\
&=\mathbf{I}_1([y_1,y_2],\mathbf{I}_2(\mathbf{N}_1([z_1,z_2]),\mathbf{N}_1([x_1,x_2])))\\
&=\mathbf{I}_1([y_1,y_2],\mathbf{I}_1([x_1,x_2],[z_1,z_2]))
\end{aligned}$$

(5)

$$
\begin{aligned}
&\mathbf{T}_4([x_1,x_2],[y_1,y_2]) \leq [z_1,z_2]\\
&\text{ iff } [\top,\top] = \mathbf{I}_1(\mathbf{T}_4([x_1,x_2],[y_1,y_2]),[z_1,z_2]) = \mathbf{I}_1([y_1,y_2],\mathbf{I}_1([x_1,x_2],[z_1,z_2]))\\
&\text{ iff } [y_1,y_2] \leq \mathbf{I}_1([x_1,x_2],[z_1,z_2]).\\
&\text{ iff } [\top,\top] = I_2([y_1,y_2],\mathbf{I}_1([x_1,x_2],[z_1,z_2])) = I_1([x_1,x_2],\mathbf{I}_2([y_1,y_2],[z_1,z_2])).\\
&\text{ iff } [x_1,x_2] \leq \mathbf{I}_2([y_1,y_2],[z_1,z_2])\\
&\text{ iff } [\top,\top] = \mathbf{I}_2(\mathbf{T}_1([x_1,x_2],[y_1,y_2]),[z_1,z_2]) = \mathbf{I}_2([x_1,x_2],\mathbf{I}_2([y_1,y_2],[z_1,z_2]))\\
&\text{ iff } \mathbf{T}_1([x_1,x_2],[y_1,y_2]) \leq [z_1,z_2].
\end{aligned}
$$

Since $\mathbf{T}_1([x_1,x_2],[y_1,y_2]) \leq [z_1,z_2]$ iff $\mathbf{T}_4([x_1,x_2],[y_1,y_2]) \leq [z_1,z_2]$, then $\mathbf{T}_1([x_1,x_2],[y_1,y_2]) = \mathbf{T}_4([x_1,x_2],[y_1,y_2])$.

$$
\begin{aligned}
&\mathbf{T}_2([x_1,x_2],[y_1,y_2]) \leq [z_1,z_2]\\
&\text{ iff } [\top,\top] = \mathbf{I}_2(\mathbf{T}_2([x_1,x_2],[y_1,y_2]),[z_1,z_2]) = \mathbf{I}_2([y_1,y_2],\mathbf{I}_2([x_1,x_2],[z_1,z_2]))\\
&\text{ iff } [y_1,y_2] \leq \mathbf{I}_2([x_1,x_2],[z_1,z_2]).\\
&\text{ iff } [\top,\top] = I_1([y_1,y_2],\mathbf{I}_2([x_1,x_2],[z_1,z_2])) = I_2([x_1,x_2],\mathbf{I}_1([y_1,y_2],[z_1,z_2])).\\
&\text{ iff } [x_1,x_2] \leq \mathbf{I}_1([y_1,y_2],[z_1,z_2])\\
&\text{ iff } [\top,\top] = \mathbf{I}_1(\mathbf{T}_3([x_1,x_2],[y_1,y_2]),[z_1,z_2]) = \mathbf{I}_1([x_1,x_2],\mathbf{I}_1([y_1,y_2],[z_1,z_2]))\\
&\text{ iff } \mathbf{T}_3([x_1,x_2],[y_1,y_2]) \leq [z_1,z_2].
\end{aligned}
$$

Since $\mathbf{T}_2([x_1,x_2],[y_1,y_2]) \leq [z_1,z_2]$ iff $\mathbf{T}_3([x_1,x_2],[y_1,y_2]) \leq [z_1,z_2]$, then $\mathbf{T}_2([x_1,x_2],[y_1,y_2]) = \mathbf{T}_3([x_1,x_2],[y_1,y_2])$.

(6)

$$
\begin{aligned}
&\mathbf{S}_2(\mathbf{S}_2([x_1,x_2],[y_1,y_2]),[z_1,z_2]) = \mathbf{I}_1(\mathbf{N}_2([z_1,z_2]),\mathbf{S}_2([x_1,x_2],[y_1,y_2]))\\
&= \mathbf{I}_1(\mathbf{N}_2([z_1,z_2]),\mathbf{I}_1(\mathbf{N}_2([y_1,y_2]),[x_1,x_2])) = \mathbf{I}_1(\mathbf{N}_2([z_1,z_2]),\mathbf{I}_2(\mathbf{N}_1([x_1,x_2]),[y_1,y_2]))\\
&= \mathbf{I}_2(\mathbf{N}_1([x_1,x_2]),\mathbf{I}_1(\mathbf{N}_2([z_1,z_2]),[y_1,y_2])) = \mathbf{I}_2(\mathbf{N}_1([x_1,x_2]),\mathbf{S}_2([y_1,y_2],[z_1,z_2]))\\
&= \mathbf{I}_1(\mathbf{N}_2(\mathbf{S}_2([y_1,y_2],[z_1,z_2]))),[x_1,x_2]) = \mathbf{S}_2([x_1,x_2],\mathbf{S}_2([y_1,y_2],[z_1,z_2])).
\end{aligned}
$$

(IS2) If $[y_1,y_2] \leq [z_1,z_2]$ and $[x_1,x_2] \leq [w_1,w_2]$, then

$\mathbf{I}_1(\mathbf{N}_2([y_1,y_2]),[x_1,x_2])) \leq \mathbf{I}_1(\mathbf{N}_2([z_1,z_2]),[w_1,w_2])$. Thus

$\mathbf{S}_2([x_1,x_2],[y_1,y_2]) \leq \mathbf{S}_2([w_1,w_2],[z_1,z_2])$.

(IS3) If $[x_1,x_2] \subset [w_1,w_2]$ and $[y_1,y_2] \subset [z_1,z_2]$, then

$\mathbf{I}_1(\mathbf{N}_2([y_1,y_2]),[x_1,x_2])) \subset \mathbf{I}_1(\mathbf{N}_2([z_1,z_2]),[w_1,w_2])$. Thus

$\mathbf{S}_2([x_1,x_2],[y_1,y_2]) \subset \mathbf{S}_2([w_1,w_2],[z_1,z_2])$.

(IS3)

$$\begin{aligned}\mathbf{S}_2([x_1,x_2],[\bot,\bot]) &= \mathbf{I}_1(\mathbf{N}_1([\bot,\bot]),[x_1,x_2]) = \mathbf{I}_1([\top,\top],[x_1,x_2]) \\ &= [I_1(\top,x_1),I_1(\top,x_1)] = [x_1,x_2].\end{aligned}$$

$$\begin{aligned}\mathbf{S}_2([\bot,\bot],[x_1,x_2]) &= \mathbf{I}_1(\mathbf{N}_2([x_1,x_2]),[\bot,\bot]) \\ &= \mathbf{I}_1([n_2(x_2),n_2(x_1)],[\bot,\bot]) \\ &= [I_1(n_2(x_1),\bot),I_1(n_2(x_2),\bot)] \\ &= [n_1(n_2(x_1)),n_1(n_2(x_2))] = [x_1,x_2].\end{aligned}$$

Moreover,

$$\begin{aligned}\mathbf{S}_2([x_1,x_2],[y_1,y_2]) &= \mathbf{I}_1(\mathbf{N}_2([y_1,y_2]),[x_1,x_2]) \\ &= \mathbf{I}_1([n_2(y_2),n_2(y_1)],[x_1,x_2]) \\ &= [I_1(n_2(y_1),x_1),I_1(n_2(y_2),x_2)]\end{aligned}$$

$$\begin{aligned}&\mathbf{S}_2([x_1,x_2],[y_1,y_2]) = \mathbf{I}_1(\mathbf{N}_2([y_1,y_2]),[x_1,x_2]) \\ &= \mathbf{I}_2(\mathbf{N}_1([x_1,x_2]),[y_1,y_2]) = \mathbf{S}_3([x_1,x_2],[y_1,y_2]).\end{aligned}$$

Other cases are similarly proved.

Example 3.2. Put $L = \{(x,y) \in R^2 \mid (\frac{1}{2},1) \le (x,y) \le (1,0)\}$ with a bottom element $(\frac{1}{2},1)$ and a top element $(1,0)$ where

$$(x_1,y_1) \le (x_2,y_2) \Leftrightarrow x_1 < x_2 \text{ or } x_1 = x_2, y_1 \le y_2.$$

(1) Define $I_1, I_2 : L \times L \to L$ as follows:

$$\begin{aligned}I_1((x_1,y_1),(x_2,y_2)) &= (\tfrac{x_2}{x_1},\tfrac{y_2-y_1}{x_1}) \wedge (1,0) \\ I_2((x_1,y_1),(x_2,y_2)) &= (\tfrac{x_2}{x_1},y_2-\tfrac{x_2y_1}{x_1}) \wedge (1,0).\end{aligned}$$

Then it satisfies (I1)-(I3) and (I4) from:

$$\begin{aligned}I_1((x_1,y_1),I_2((x_2,y_2),(x_3,y_3))) &= I_1((x_1,y_1),(\tfrac{x_3}{x_2},y_3-\tfrac{x_3y_2}{x_2}) \wedge (1,0)) \\ &= (\tfrac{x_3}{x_1x_2},\tfrac{x_2y_3-x_3y_2-x_2y_1}{x_1x_2}) \wedge (1,0) \\ I_2((x_2,y_2),I_1((x_1,y_1),(x_3,y_3))) &= I_2((x_2,y_2),(\tfrac{x_3}{x_1},\tfrac{y_3-y_1}{x_1}) \wedge (1,0)) \\ &= (\tfrac{x_3}{x_1x_2},\tfrac{x_2y_3-x_3y_2-x_2y_1}{x_1x_2}) \wedge (1,0).\end{aligned}$$

(I5) $I_2(I_1((x_1,y_1),(\frac{1}{2},1)),(\frac{1}{2},1)) = (x_1,y_1) = I_1(I_2((x_1,y_1),(\frac{1}{2},1)),(\frac{1}{2},1))$ from

$$\begin{aligned} I_1((x_1,y_1),(\tfrac{1}{2},1)) &= (\tfrac{1}{2x_1},\tfrac{1-y_1}{x_1}) = n_1(x_1,y_1) \\ I_2((x_1,y_1),(\tfrac{1}{2},1)) &= (\tfrac{1}{2x_1},1-\tfrac{y_1}{2x_1}) = n_2(x_1,y_1). \end{aligned}$$

Hence (I_1,I_2) an pair of implications .

(2) By Theorem 3.1, we obtain: $\mathbf{I}_1,\mathbf{I}_2 : L^{[2]} \times L^{[2]} \to L^{[2]}$ as follows:

$$\begin{aligned} &\mathbf{I}_1([(x_1,y_1),(x_2,y_2)],[(z_1,w_1),(z_2,w_2)]) \\ &= [I_1((x_2,y_2),(z_1,w_1),I_1((x_1,y_1),(z_2,w_2))] \\ &= [(\tfrac{z_1}{x_2},\tfrac{w_1-y_2}{x_2}) \wedge (1,0),(\tfrac{z_2}{x_1},\tfrac{w_2-y_1}{x_1}) \wedge (1,0)] \\ &\mathbf{I}_2([(x_1,y_1),(x_2,y_2)],[(z_1,w_1),(z_2,w_2)]) \\ &= [I_2((x_2,y_2),(z_1,w_1),I_2((x_1,y_1),(z_2,w_2))] \\ &= [(\tfrac{z_1}{x_2},w_1-\tfrac{z_1y_2}{x_2}) \wedge (1,0),(\tfrac{z_2}{x_1},w_2-\tfrac{z_2y_1}{x_1}) \wedge (1,0)] \end{aligned}$$

(3) $\mathbf{N}_1,\mathbf{N}_2 : L^{[2]} \to L^{[2]}$ as follows:

$$\begin{aligned} \mathbf{N}_1([(x_1,y_1),(x_2,y_2)]) &= [n_1(x_2,y_2),n_1(x_1,y_1)] \\ &= [(\tfrac{1}{2x_2},\tfrac{1-y_2}{x_2}),(\tfrac{1}{2x_1},\tfrac{1-y_1}{x_1})] \\ \mathbf{N}_2([(x_1,y_1),(x_2,y_2)]) &= [n_2(x_2,y_2),n_2(x_1,y_1)] \\ &= [(\tfrac{1}{2x_2},1-\tfrac{y_2}{2x_2}),(\tfrac{1}{2x_1},1-\tfrac{y_1}{2x_1})]. \end{aligned}$$

(4)

$$\begin{aligned} &\mathbf{T}_1([(x_1,y_1),(x_2,y_2)],[(z_1,w_1),(z_2,w_2)]) \\ &= \mathbf{N}_1(\mathbf{I}_2([(x_1,y_1),(x_2,y_2)],\mathbf{N}_2([(z_1,w_1),(z_2,w_2)]))) \\ &= \mathbf{N}_1(\mathbf{I}_2([(x_1,y_1),(x_2,y_2)],[(\tfrac{1}{2z_2},1-\tfrac{w_2}{2z_2}),(\tfrac{1}{2z_1},1-\tfrac{w_1}{2z_1})]))) \\ &= \mathbf{N}_1([(\tfrac{1}{2x_2z_2},1-\tfrac{w_2}{2z_2}-\tfrac{y_2}{2x_2z_2}) \wedge (1,0),(\tfrac{1}{2x_1z_1},1-\tfrac{w_1}{2z_1}-\tfrac{y_1}{2x_1z_1}) \wedge (1,0)]) \\ &= [(x_1z_1,x_1w_1+y_1) \vee (\tfrac{1}{2},1)),(x_2z_2,x_2w_2+y_2) \vee (\tfrac{1}{2},1))] \\ &= [n_1I_2((x_1,y_1),n_2(z_1,w_1)),n_1I_2((x_2,y_2),n_2(z_2,w_2))]. \end{aligned}$$

$$\begin{aligned}
&\mathbf{T}_2([(x_1,y_1),(x_2,y_2)],[(z_1,w_1),(z_2,w_2)])\\
&=\mathbf{N}_1(\mathbf{I}_2([(z_1,w_1),(z_2,w_2)],\mathbf{N}_2([(x_1,y_1),(x_2,y_2)])))\\
&=\mathbf{N}_1(\mathbf{I}_2([(z_1,w_1),(z_2,w_2)],[(\tfrac{1}{2x_2},1-\tfrac{y_2}{2x_2}),(\tfrac{1}{2x_1},1-\tfrac{y_1}{2x_1})]))\\
&=\mathbf{N}_1([(\tfrac{1}{2x_2z_2},1-\tfrac{y_2}{2x_2}-\tfrac{w_2}{2z_2x_2})\wedge(1,0),(\tfrac{1}{2x_1z_1},1-\tfrac{y_1}{2x_1}-\tfrac{w_1}{2x_1z_1})\wedge(1,0)])\\
&=[(x_1z_1,z_1y_1+w_1)\vee(\tfrac{1}{2},1)),(x_2z_2,z_2y_2+w_2)\vee(\tfrac{1}{2},1))]\\
&=[n_1I_2((z_1,w_1),n_2(x_1,y_1)),n_1I_2((z_2,w_2),n_2(x_2,y_2))].
\end{aligned}$$

$$\begin{aligned}
&\mathbf{T}_3([(x_1,y_1),(x_2,y_2)],[(z_1,w_1),(z_2,w_2)])\\
&=\mathbf{N}_2(\mathbf{I}_1([(x_1,y_1),(x_2,y_2)],\mathbf{N}_1([(z_1,w_1),(z_2,w_2)])))\\
&=\mathbf{N}_2(\mathbf{I}_1([(x_1,y_1),(x_2,y_2)],[(\tfrac{1}{2z_2},\tfrac{1-w_2}{z_2}),(\tfrac{1}{2z_1},\tfrac{1-w_1}{z_1})]))\\
&=\mathbf{N}_2([(\tfrac{1}{2x_2z_2},\tfrac{1-w_2-y_2z_2}{x_2z_2})\wedge(1,0),(\tfrac{1}{2x_1z_1},\tfrac{1-w_1-y_1z_1}{x_1z_1})\wedge(1,0)])\\
&=[(x_1z_1,z_1y_1+w_1)\vee(\tfrac{1}{2},1)),(x_2z_2,z_2y_2+w_2)\vee(\tfrac{1}{2},1))]\\
&=[n_2I_1((x_1,y_1),n_1(z_1,w_1)),n_2I_1((x_2,y_2),n_1(z_2,w_2))].
\end{aligned}$$

$$\begin{aligned}
&\mathbf{T}_4([(x_1,y_1),(x_2,y_2)],[(z_1,w_1),(z_2,w_2)])\\
&=\mathbf{N}_2(\mathbf{I}_1([(z_1,w_1),(z_2,w_2)],\mathbf{N}_1([(x_1,y_1),(x_2,y_2)])))\\
&=\mathbf{N}_2(\mathbf{I}_1([(z_1,w_1),(z_2,w_2)],[(\tfrac{1}{2x_2},\tfrac{1-y_2}{x_2}),(\tfrac{1}{2x_1},\tfrac{1-y_1}{x_1})]))\\
&=\mathbf{N}_2([(\tfrac{1}{2x_2z_2},\tfrac{1-y_2-w_2x_2}{x_2z_2})\wedge(1,0),(\tfrac{1}{x_1z_1},\tfrac{1-y_1-w_1x_1}{x_1z_1})\wedge(1,0)])\\
&=[(x_1z_1,x_1w_1+y_1)\vee(\tfrac{1}{2},1)),(x_2z_2,x_2w_2+y_2)\vee(\tfrac{1}{2},1))]\\
&=[n_2I_1((z_1,w_1),n_1(x_1,y_1)),n_2I_1((z_2,w_2),n_1(x_2,y_2))].
\end{aligned}$$

(5)

$$\begin{aligned}
n_1I_2((x_1,y_1),(x_2,y_2)) &= n_1(\tfrac{x_1}{x_2},y_2-\tfrac{x_2y_1}{x_1})\wedge(1,0))\\
&=(\tfrac{x_1}{2x_1},\tfrac{x_1}{x_2}-\tfrac{x_1y_2}{x_2}+y_1)\vee(\tfrac{1}{2},1)\\
n_2I_1(n_1(x_2,y_2),n_1(x_1,y_1)) &= n_2(\tfrac{x_1}{x_2},2x_2(\tfrac{1-y_1}{x_1}-\tfrac{1-y_2}{x_2})\wedge(1,0))\\
&=(\tfrac{x_1}{2x_1},\tfrac{x_1}{x_2}-\tfrac{x_1y_2}{x_2}+y_1)\vee(\tfrac{1}{2},1).
\end{aligned}$$

Since $n_1I_2((x_1,y_1),(x_2,y_2))=n_2I_1(n_1(x_2,y_2),n_1(x_1,y_1))$, then $\mathbf{T}_1=\mathbf{T}_4$ and $\mathbf{T}_2=\mathbf{T}_3$.

(6) The converse of Theorem 3.1 (5) cannot be true because $\mathbf{T}_1=\mathbf{T}_4$ and $\mathbf{T}_2=\mathbf{T}_3$ but

$$[(\frac{2}{3},-1),(\frac{3}{4},0)]\leq[(\frac{2}{3},1),(\frac{3}{4},1)]$$

$$\mathbf{I}_1([(\frac{2}{3},-1),(\frac{3}{4},0)],[(\frac{2}{3},1),(\frac{3}{4},1)])=[(\frac{8}{9},\frac{4}{3}),(1,0)]\neq[(1,0),(1,0)].$$

(7)

$$\begin{aligned}
&\mathbf{S}_1([(x_1,y_1),(x_2,y_2)],[(z_1,w_1),(z_2,w_2)])\\
&=\mathbf{I}_1(\mathbf{N}_2([(x_1,y_1),(x_2,y_2)]),[(z_1,w_1),(z_2,w_2)])\\
&=\mathbf{I}_1([(\tfrac{1}{2x_2},1-\tfrac{y_2}{2x_2}),(\tfrac{1}{2x_1},1-\tfrac{y_1}{2x_1})],[(z_1,w_1),(z_2,w_2)])\\
&=[(2x_1z_1,2x_1w_1-2x_1+y_1)\wedge(1,0),(2x_2z_2,2x_2w_2-2x_2+y_2)\wedge(1,0)]\\
&=[I_1(n_2(x_1,y_1),(z_1,w_1)),I_1(n_2(x_2,y_2),(z_2,w_2))].
\end{aligned}$$

$$\begin{aligned}
&\mathbf{S}_2([(x_1,y_1),(x_2,y_2)],[(z_1,w_1),(z_2,w_2)])\\
&=\mathbf{I}_1([\mathbf{N}_2((z_1,w_1),(z_2,w_2))],[(x_1,y_1),(x_2,y_2)])\\
&=\mathbf{I}_1([(\tfrac{1}{2z_2},1-\tfrac{w_2}{2z_2}),(\tfrac{1}{2z_1},1-\tfrac{w_1}{2z_1})],[(x_1,y_1),(x_2,y_2)])\\
&=[(2x_1z_1,2z_1y_1-2z_1+w_1)\wedge(1,0),(2x_2z_2,2z_2y_2-2z_2+w_2)\wedge(1,0)]\\
&=[I_1(n_2(z_1,w_1),(x_1,y_1)),I_1(n_2(z_2,w_2),(x_2,y_2))].
\end{aligned}$$

$$\begin{aligned}
&\mathbf{S}_3([(x_1,y_1),(x_2,y_2)],[(z_1,w_1),(z_2,w_2)])\\
&=\mathbf{I}_2(\mathbf{N}_1([(x_1,y_1),(x_2,y_2)]),[(z_1,w_1),(z_2,w_2)])\\
&=\mathbf{I}_2([(\tfrac{1}{2x_2},\tfrac{1-y_2}{x_2}),(\tfrac{1}{2x_1},\tfrac{1-y_1}{x_1})],[(z_1,w_1),(z_2,w_2)])\\
&=[(2x_1z_1,2z_1y_1-2z_1+w_1)\wedge(1,0),(2x_2z_2,2z_2y_2-2z_2+w_2)\wedge(1,0)]\\
&=[I_2(n_1(x_1,y_1),(z_1,w_1)),I_2(n_1(x_2,y_2),(z_2,w_2))].
\end{aligned}$$

$$\begin{aligned}
&\mathbf{S}_4([(x_1,y_1),(x_2,y_2)],[(z_1,w_1),(z_2,w_2)])\\
&=\mathbf{I}_2(\mathbf{N}_1([(z_1,w_1),(z_2,w_2)]),[(x_1,y_1),(x_2,y_2)])\\
&=\mathbf{I}_2([(\tfrac{1}{2z_2},\tfrac{1-w_2}{z_2}),(\tfrac{1}{2z_1},\tfrac{1-w_1}{z_1})],[(x_1,y_1),(x_2,y_2)])\\
&=[(2x_1z_1,2x_1w_1-2x_1+y_1)\wedge(1,0),(2x_2z_2,2x_2w_2-2x_2+y_2)\wedge(1,0)]\\
&=[I_2(n_1(z_1,w_1),(x_1,y_1)),I_2(n_1(z_2,w_2),(x_2,y_2))].
\end{aligned}$$

Example 3.3. Put $L=\{(x,y)\in R^2 \mid (\frac{1}{2},1)\leq(x,y)\leq(1,0)\}$ with a bottom element $(\frac{1}{2},1)$ and a top element $(1,0)$ where

$$(x_1,y_1)\leq(x_2,y_2)\Leftrightarrow x_1<x_2 \text{ or } x_1=x_2,y_1\leq y_2.$$

(1) Define $I_1,I_2:L\times L\to L$ as follows:

$$\begin{aligned}
I_1((x_1,y_1),(x_2,y_2)) &=(\tfrac{x_2}{x_1},y_2-2x_2+\tfrac{2x_2-2x_2y_1}{x_1})\wedge(1,0)\\
I_2((x_1,y_1),(x_2,y_2)) &=(\tfrac{x_2}{x_1},1-\tfrac{y_1+2-2y_2}{2x_1})\wedge(1,0).
\end{aligned}$$

Then it satisfies (I1)-(I4) and (I5) from:

$$
\begin{aligned}
&I_1((x_1,y_1),I_2((x_2,y_2),(x_3,y_3))) = I_1((x_1,y_1),(\tfrac{x_3}{x_2},1-\tfrac{y_2+2-2y_3}{2x_2})\wedge(1,0))\\
&=(\tfrac{x_3}{x_1x_2},\tfrac{2x_1x_2-x_1y_2-2x_1+2x_1y_3-4x_3x_1+4x_3-4x_3y_1}{2x_1x_2})\wedge(1,0)\\
&I_2((x_2,y_2),I_1((x_1,y_1),(x_3,y_3))) = I_2((x_2,y_2),(\tfrac{x_3}{x_1},y_3-2x_3+\tfrac{2x_3-2x_3y_1}{x_1})\wedge(1,0))\\
&=(\tfrac{x_3}{x_1x_2},\tfrac{2x_1x_2-x_1y_2-2x_1+2x_1y_3-4x_3x_1+4x_3-4x_3y_1}{2x_1x_2})\wedge(1,0).
\end{aligned}
$$

Hence (I_1,I_2) an pair of implications .

(2) Put $n_1(x,y)=(\frac{1}{2x},\frac{1-y}{x})$, $n_2(x,y)=(\frac{1}{2x},1-\frac{y}{2x})$. Then

$$n_1(n_2([x_1,x_2],[y_1,y_2]))=([x_1,x_2],[y_1,y_2]),\ n_2(n_1([x_1,x_2],[y_1,y_2]))=([x_1,x_2],[y_1,y_2]).$$

$$
\begin{aligned}
I_1((x_1,y_1),(\tfrac{1}{2},1)) &= (\tfrac{1}{2x_1},\tfrac{1-y_1}{x_1}) = n_1(x_1,y_1)\\
I_2((x_1,y_1),(\tfrac{1}{2},1)) &= (\tfrac{1}{2x_1},1-\tfrac{y_1}{2x_1}) = n_2(x_1,y_1).
\end{aligned}
$$

Hence (n_1,n_2) is a pair of negations.

(3) By Theorem 3.1, we obtain: $\mathbf{I}_1,\mathbf{I}_2 : L^{[2]}\times L^{[2]}\to L^{[2]}$ as follows:

$$
\begin{aligned}
&\mathbf{I}_1([(x_1,y_1),(x_2,y_2)],[(z_1,w_1),(z_2,w_2)])\\
&=[I_1((x_2,y_2),(z_1,w_1),I_1((x_1,y_1),(z_2,w_2))]\\
&=[(\tfrac{z_1}{x_2},w_1-2z_1+\tfrac{2z_1-2z_1y_2}{x_2})\wedge(1,0),(\tfrac{z_2}{x_1},w_2-2z_2+\tfrac{2z_2-2z_2y_1}{x_1})\wedge(1,0)]\\
&\mathbf{I}_2([(x_1,y_1),(x_2,y_2)],[(z_1,w_1),(z_2,w_2)])\\
&=[I_2((x_2,y_2),(z_1,w_1),I_2((x_1,y_1),(z_2,w_2))]\\
&=[(\tfrac{z_1}{x_2},1-\tfrac{y_2+2-2w_1}{2x_2})\wedge(1,0),(\tfrac{z_2}{x_1},1-\tfrac{y_1+2-2w_2}{2x_1})\wedge(1,0)].
\end{aligned}
$$

(4)

$$
\begin{aligned}
&\mathbf{T}_1([(x_1,y_1),(x_2,y_2)],[(z_1,w_1),(z_2,w_2)])\\
&=\mathbf{N}_1(\mathbf{I}_2([(x_1,y_1),(x_2,y_2)],\mathbf{N}_2([(z_1,w_1),(z_2,w_2)])))\\
&=\mathbf{N}_1(\mathbf{I}_2([(x_1,y_1),(x_2,y_2)],[(\tfrac{1}{2z_2},1-\tfrac{w_2}{2z_2}),(\tfrac{1}{2z_1},1-\tfrac{w_1}{2z_1})]))\\
&=\mathbf{N}_1([(\tfrac{1}{2x_2z_2},1-\tfrac{y_2}{2x_2}-\tfrac{w_2}{2x_2z_2})\wedge(1,0),(\tfrac{1}{2x_1z_1},1-\tfrac{y_1}{2x_1}-\tfrac{w_1}{2x_1z_1})\wedge(1,0)])\\
&=[(x_1z_1,z_1y_1+w_1)\vee(\tfrac{1}{2},1)),(x_2z_2,z_2y_2+w_2)\vee(\tfrac{1}{2},1))]\\
&=[n_1I_2((x_1,y_1),n_2(z_1,w_1)),n_1I_2((x_2,y_2),n_2(z_2,w_2))].
\end{aligned}
$$

$$\begin{aligned}
&\mathbf{T}_2([(x_1,y_1),(x_2,y_2)],[(z_1,w_1),(z_2,w_2)])\\
&= \mathbf{N}_1(\mathbf{I}_2([(z_1,w_1),(z_2,w_2)],\mathbf{N}_2([(x_1,y_1),(x_2,y_2)])))\\
&= \mathbf{N}_1(\mathbf{I}_2([(z_1,w_1),(z_2,w_2)],[(\tfrac{1}{2x_2},1-\tfrac{y_2}{2x_2}),(\tfrac{1}{2x_1},1-\tfrac{y_1}{2x_1})])))\\
&= \mathbf{N}_1([(\tfrac{1}{2x_2z_2},1-\tfrac{w_2}{2z_2}-\tfrac{y_2}{2z_2x_2})\wedge(1,0),(\tfrac{1}{2x_1z_1},1-\tfrac{w_1}{2x_1}-\tfrac{y_1}{2x_1z_1})\wedge(1,0)])\\
&= [(x_1z_1,x_1w_1+y_1)\vee(\tfrac{1}{2},1)),(x_2z_2,x_2w_2+y_2)\vee(\tfrac{1}{2},1))]\\
&= [n_1I_2((z_1,w_1),n_2(x_1,y_1)),n_1I_2((z_2,w_2),n_2(x_2,y_2))].
\end{aligned}$$

$$\begin{aligned}
&\mathbf{T}_3([(x_1,y_1),(x_2,y_2)],[(z_1,w_1),(z_2,w_2)])\\
&= \mathbf{N}_2(\mathbf{I}_1([(x_1,y_1),(x_2,y_2)],\mathbf{N}_1([(z_1,w_1),(z_2,w_2)])))\\
&= \mathbf{N}_2(\mathbf{I}_1([(x_1,y_1),(x_2,y_2)],[(\tfrac{1}{2z_2},\tfrac{1-w_2}{z_2}),(\tfrac{1}{2z_1},\tfrac{1-w_1}{z_1})])))\\
&= \mathbf{N}_2([(\tfrac{1}{2x_2z_2},\tfrac{1-y_2-x_2w_2}{x_2z_2})\wedge(1,0),(\tfrac{1}{2x_1z_1},\tfrac{1-y_1-w_1x_1}{x_1z_1})\wedge(1,0)])\\
&= [(x_1z_1,x_1w_1+y_1)\vee(\tfrac{1}{2},1)),(x_2z_2,x_2w_2+y_2)\vee(\tfrac{1}{2},1))]\\
&= [n_2I_1((x_1,y_1),n_1(z_1,w_1)),n_2I_1((x_2,y_2),n_1(z_2,w_2))].
\end{aligned}$$

$$\begin{aligned}
&\mathbf{T}_4([(x_1,y_1),(x_2,y_2)],[(z_1,w_1),(z_2,w_2)])\\
&= \mathbf{N}_2(\mathbf{I}_1([(z_1,w_1),(z_2,w_2)],\mathbf{N}_1([(x_1,y_1),(x_2,y_2)])))\\
&= \mathbf{N}_2(\mathbf{I}_1([(z_1,w_1),(z_2,w_2)],[(\tfrac{1}{2x_2},\tfrac{1-y_2}{x_2}),(\tfrac{1}{2x_1},\tfrac{1-y_1}{x_1})])))\\
&= \mathbf{N}_2([(\tfrac{1}{2x_2z_2},\tfrac{1-w_2-y_2z_2}{x_2z_2})\wedge(1,0),(\tfrac{1}{x_1z_1},\tfrac{1-w_1-y_1z_1}{x_1z_1})\wedge(1,0)])\\
&= [(x_1z_1,z_1y_1+w_1)\vee(\tfrac{1}{2},1)),(x_2z_2,z_2y_2+w_2)\vee(\tfrac{1}{2},1))]\\
&= [n_2I_1((z_1,w_1),n_1(x_1,y_1)),n_2I_1((z_2,w_2),n_1(x_2,y_2))].
\end{aligned}$$

(5)

$$\begin{aligned}
n_1I_2((x_1,y_1),(x_2,y_2)) &= n_1(\tfrac{x_2}{x_1},1-\tfrac{y_1+2-2y_2}{2x_1})\wedge(1,0))\\
&= (\tfrac{x_1}{2x_2},\tfrac{y_1+2-2y_2}{2x_2})\vee(\tfrac{1}{2},1)\\
n_2I_1(n_1(x_2,y_2),n_1(x_1,y_1)) &= n_2(\tfrac{x_2}{x_1},\tfrac{-y_1+2x_2-2+2y_2}{x_1})\wedge(1,0))\\
&= (\tfrac{x_1}{2x_2},\tfrac{y_1+2-2y_2}{2x_2})\vee(\tfrac{1}{2},1).
\end{aligned}$$

Since $n_1I_2((x_1,y_1),(x_2,y_2)) = n_2I_1(n_1(x_2,y_2),n_1(x_1,y_1))$, then $\mathbf{T}_1=\mathbf{T}_4$ and $\mathbf{T}_2=\mathbf{T}_3$.

(6) The converse of Theorem 3.1(5) cannot be true because $\mathbf{T}_1=\mathbf{T}_4$ and $\mathbf{T}_2=\mathbf{T}_3$ but

$$[(\frac{2}{3},-1),(\frac{3}{4},0)] \leq [(\frac{2}{3},1),(\frac{3}{4},1)]$$

$$\mathbf{I}_1([(\frac{2}{3},-1),(\frac{3}{4},0)],[(\frac{2}{3},1),(\frac{3}{4},1)]) = [(\frac{8}{9},\frac{5}{9}),(1,0)] \neq [(1,0),(1,0)].$$

(7)

$$\begin{aligned}
&\mathbf{S}_1([(x_1,y_1),(x_2,y_2)],[(z_1,w_1),(z_2,w_2)])\\
&=\mathbf{I}_1(\mathbf{N}_2([(x_1,y_1),(x_2,y_2)]),[(z_1,w_1),(z_2,w_2)])\\
&=\mathbf{I}_1([(\tfrac{1}{2x_2},1-\tfrac{y_2}{2x_2}),(\tfrac{1}{2x_1},1-\tfrac{y_1}{2x_1})],[(z_1,w_1),(z_2,w_2)])\\
&=[(2x_1z_1,2z_1y_1-2z_1+w_1)\wedge(1,0),(2x_2z_2,2y_2z_2-2z_2+w_2)\wedge(1,0)]\\
&=[I_1(n_2(x_1,y_1),(z_1,w_1)),I_1(n_2(x_2,y_2),(z_2,w_2))].
\end{aligned}$$

$$\begin{aligned}
&\mathbf{S}_2([(x_1,y_1),(x_2,y_2)],[(z_1,w_1),(z_2,w_2)])\\
&=\mathbf{I}_1([\mathbf{N}_2((z_1,w_1),(z_2,w_2))],[(x_1,y_1),(x_2,y_2)])\\
&=\mathbf{I}_1([(\tfrac{1}{2z_2},1-\tfrac{w_2}{2z_2}),(\tfrac{1}{2z_1},1-\tfrac{w_1}{2z_1})],[(x_1,y_1),(x_2,y_2)])\\
&=[(2x_1z_1,2x_1w_1-2x_1+y_1)\wedge(1,0),(2x_2z_2,2x_2w_2-2x_2+y_2)\wedge(1,0)]\\
&=[I_1(n_2(z_1,w_1),(x_1,y_1)),I_1(n_2(z_2,w_2),(x_2,y_2))].
\end{aligned}$$

$$\begin{aligned}
&\mathbf{S}_3([(x_1,y_1),(x_2,y_2)],[(z_1,w_1),(z_2,w_2)])\\
&=\mathbf{I}_2(\mathbf{N}_1([(x_1,y_1),(x_2,y_2)]),[(z_1,w_1),(z_2,w_2)])\\
&=\mathbf{I}_2([(\tfrac{1}{2x_2},\tfrac{1-y_2}{x_2}),(\tfrac{1}{2x_1},\tfrac{1-y_1}{x_1})],[(z_1,w_1),(z_2,w_2)])\\
&=[(2x_1z_1,2x_1w_1-2x_1+y_1)\wedge(1,0),(2x_2z_2,2x_2w_2-2x_2+y_2)\wedge(1,0)]\\
&=[I_2(n_1(x_1,y_1),(z_1,w_1)),I_2(n_1(x_2,y_2),(z_2,w_2))].
\end{aligned}$$

$$\begin{aligned}
&\mathbf{S}_4([(x_1,y_1),(x_2,y_2)],[(z_1,w_1),(z_2,w_2)])\\
&=\mathbf{I}_2(\mathbf{N}_1([(z_1,w_1),(z_2,w_2)]),[(x_1,y_1),(x_2,y_2)])\\
&=\mathbf{I}_2([(\tfrac{1}{2z_2},\tfrac{1-w_2}{z_2}),(\tfrac{1}{2z_1},\tfrac{1-w_1}{z_1})],[(x_1,y_1),(x_2,y_2)])\\
&=[(2x_1z_1,2z_1y_1-2z_1+w_1)\wedge(1,0),(2x_2z_2,2y_2z_2-2z_2+w_2)\wedge(1,0)]\\
&=[I_2(n_1(z_1,w_1),(x_1,y_1)),I_2(n_1(z_2,w_2),(x_2,y_2))].
\end{aligned}$$

Conflict of Interests

The author declares that there is no conflict of interests.

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