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A FIXED POINT APPROACH TO THE HYPERSTABILITY OF DRYGAS FUNCTIONAL EQUATION IN METRIC SPACES

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Abstract. Piszczek and Szczawińska proved the hyperstability of the Drygas functional equation in Banach spaces. Using the fixed point method, we prove the hyperstability of the Drygas functional equation $f(x+y)+f(x-y)=2f(x)+f(y)+f(-y)$, in the class of functions from a commutative group into a commutative complete metric group.

Keywords: Drygas functional equation; hyperstability; fixed point theory; complete metric space.

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1. Introduction and preliminaries

In 1940, Ulam [32] gave a talk before the Mathematics Club of the University of Wisconsin in which proposed the following stability problem, well-known as Ulam stability problem.

Let G_1 be a group and let G_2 be a metric group with the metric $d(\cdot,\cdot)$. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if a mapping $h : G_1 \to G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G_1$ then there is a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \varepsilon$ for all $x \in G_1$?

In 1941, Hyers [14] gave a first affirmative answer to the question of Ulam for Banach spaces. Hyers' theorem was generalized by Aoki [3] for additive mappings and by Rassias [25] for

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linear mappings by considering an unbounded Cauchy difference. Găvruta [13] provided a further generalization of the Rassias' theorem by using a general control function.

The functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y) \tag{1.1}$$

is called the quadratic functional equation. Quadratic functional equation where used to characterize inner product spaces [1,2,15]. In particular every solution of the quadratic functional equation is said to be quadratic function. It is well known that a function f between real vector spaces is quadratic if and only if there exists a unique symmetric bi-additive function B such that $f(x)=B(x,x)$ for all x (see [1,17]). The bi-additive mapping is given by

$$B(x,y)=\frac{1}{4}\left[f(x+y)-f(x-y)\right].$$

The generalized Hyers-Ulam stability problem for the above quadratic functional equation was proved by Skof [31] for mapping $f: X\to Y$, where X is a normed space and Y is a Banach space. Cholewa [8] noticed that the theorem of Skof is still true if relevant domain X is replaced by an abelian group. In [9], Czerwik proved the generalized Hyers-Ulam of the quadratic functional equation as above. Grabiec [11] has generalized these results mentioned above. Several functional equations have been investigated in [7,20-23,26-30].

Drygas [10] obtained a Jordan and von Neumann type characterization theorem for quasi-inner product spaces. In Drygas's characterization of quasi-inner product spaces the functional equation

$$f(x)+f(y)=f(x-y)+2\left(f(\frac{x+y}{2})-\frac{x-y}{2})\right)$$

played an important role. If we replace y by $-y$ in the above functional equation and add the resulting equation to the above equation, then we obtain the Drygas equation

$$f(x+y)+f(x-y)=2f(x)+f(y)+f(-y). \tag{1.2}$$

The Drygas functional equation (1.2) on an arbitrary group G takes the form

$$f(xy)+f(xy^{-1})-2f(x)-f(y)-f(y^{-1})=0,$$

for all $x,y\in G$. The stability of the equation (1.2) was studied in [16] and [33].

In 2001, Maksa and Páles [19] proved a new type of stability of a class of linear functional equation

$$f(x)+f(y)=\frac{1}{n}\sum_{i=1}^{n}f\left(x\varphi_i(y)\right), \tag{1.3}$$

where f is a real-valued mapping defined on a semigroup $(S,.)$ and where $\varphi_1,...,\varphi_n : S\to S$ are pairwise distinct automorphisme of S. More precisely, they proved that if the error bound for the difference of the two sides of (1.3) satisfies a certain asymptotic property then, in fact, the two sides have to be equal to each other. Such a phenomenon is called the hyperstability of the functional equation on S. Further, Brzdęk and Ciepliński in their paper [5] introduce the following definition, which describes the main ideas of such hyperstability notion for equations in several variables.

Definition 1.2. Let X be a nonempty set, (Y,d) be a metric space, $\varepsilon\in\mathbb{R}_0^{X^n}$ and $\mathscr{F}_1$, $\mathscr{F}_2$ be operators mapping from a nonempty set $\mathscr{D}\subset Y^X$ into Y^{X^n}. We say that the operator equation

$$\mathscr{F}_1\varphi(x_1,\ldots,x_n)=\mathscr{F}_2\varphi(x_1,\ldots,x_n),\quad x_1,\ldots,x_n\in X \tag{1.4}$$

is ε-hyperstable provided every $\varphi_0\in\mathscr{D}$ satisfies the inequality

$$d\left(\mathscr{F}_1\varphi_0(x_1,\ldots,x_n),\mathscr{F}_2\varphi_0(x_1,\ldots,x_n)\right)\le\varepsilon(x_1,\ldots,x_n),\quad x_1,\ldots,x_n\in X \tag{1.5}$$

fulfills equation (1.4).

In [4], Brzdęk proved the hyperstability of the Cauchy functional equation by an idea based on a fixed point theorem for functional equations obtained by Brzdek *et al.* in [6]. Gselmann [12] investigated the hyperstability of parametric fundamental equation of information. Piszczek in [24] proved the hyperstability of the general linear equation. In 2013, Piszczek and Szczawińska in [18] studied the hyperstability of the Drygas equation (1.2) in Banach spaces.

In this paper, using the fixed point method based on a fixed point result ([6]; Theorem 1), we prove the hyperstability of the Drygas functional equation (1.2) in the class of functions from a commutative group to commutative complete metric group.

Throughout this paper, $\mathbb{N}$ denote the set of all non-negative integers, $\mathbb{N}_+$ denote the set of all positive integers, $\mathbb{N}_{n_0}$ denote the set of all integers greater than or equal to $n_0\in\mathbb{N}_+$. By $\mathbb{R}_0$ and

$\mathbb{R}_+$ we will denote the set of all non-negative reals and the set of all positive reals, respectively. A^B denote the family of all functions from a set $B \neq \varnothing$ to a set $A \neq \varnothing$).

Before proceeding to the main results, we will state the following theorem (Theorem (1.3)) which is useful to our purpose.

Theorem 1.3. [6] *Let X be a nonempty set, (Y,d) a complete metric space, $f_1,\dots,f_s : X \to X$ and $L_1,\dots,L_s : X \to \mathbb{R}_0$ be given maps. Let $\Lambda : \mathbb{R}_0^X \to \mathbb{R}_0^X$ be a linear operator defined by*

$$\Lambda\delta(x) := \sum_{i=1}^{s} L_i(x)\delta(f_i(x)), \tag{1.6}$$

for $\delta \in \mathbb{R}_0^X$ and $x \in X$. If $\mathscr{T} : Y^X \to Y^X$ is an operator satisfying the inequality

$$d(\mathscr{T}\xi(x), \mathscr{T}\mu(x)) \leq \sum_{i=1}^{s} L_i(x) d(\xi(f_i(x)), \mu(f_i(x))), \quad \xi,\mu \in Y^X, x \in X, \tag{1.7}$$

and the functions $\varepsilon : X \to \mathbb{R}_0$ and $\varphi : X \to Y$ are such that

$$d(\mathscr{T}\varphi(x), \varphi(x)) \leq \varepsilon(x), \quad x \in X, \tag{1.8}$$

$$\varepsilon^*(x) := \sum_{k=1}^{\infty} \Lambda^k \varepsilon(x) < \infty, \quad x \in X, \tag{1.9}$$

then, for every $x \in X$, the limit

$$\psi(x) := \lim_{n\to\infty} \mathscr{T}^n \varphi(x), \tag{1.10}$$

exists and the function $\psi \in Y^X$ so defined is a unique fixed point of $\mathscr{T}$, with

$$d(\varphi(x), \psi(x)) \leq \varepsilon^*(x), \quad x \in X. \tag{1.11}$$

2. Hyperstability of (1.2)

Given a group $(X,+)$, we denote by $Aut(X)$ the family of all automorphisms of X. Moreover, for each $u \in X^X$ we write $ux := u(x)$ for $x \in X$ and we define u' by $u'x := x - ux$. The following theorem is a result concerning the hyperstability of equation (1.2).

Theorem 2.1. *Let $(X,+)$ and $(Y,+)$ be commutative groups, d be a complete metric in Y that is invariant (i.e., $d(x+z,y+z) = d(x,y)$ for $x,y,z \in X$), $\varepsilon : (X\backslash\{0\})^2 \to \mathbb{R}_+$, and*

$$l(X) := \left\{u \in Aut(X) : u' \in Aut(X), 2\lambda(u') + \lambda(u) + \lambda(-u) + \lambda(-2u) < 1\right\} \neq \emptyset, \tag{2.1}$$

where

$$\lambda(u) := \inf\{t \in \mathbb{R}_+ : \varepsilon(ux,uy) \le t\varepsilon(x,y), \forall x,y \in X\backslash\{0\}\}, \tag{2.2}$$

for $u \in Aut(X)$. Assume that there exists a nonempty subset $\mathscr{U} \subset l(X)$ such that

$$u \circ v = v \circ u, \tag{2.3}$$

for all $u,v \in \mathscr{U}$, and

$$\begin{aligned} &\inf\left\{\varepsilon(u'x,ux) : u \in \mathscr{U}\right\} = 0 \quad \forall x \in X\backslash\{0\}, \\ &\sup\left\{2\lambda(u') + \lambda(u) + \lambda(-u) + \lambda(-2u) : u \in \mathscr{U}\right\} < 1. \end{aligned} \tag{2.4}$$

Then every function $f : X \to Y$ satisfying the inequality

$$d\left(f(x+y), 2f(x) + f(y) + f(-y) - f(x-y)\right) \le \varepsilon(x,y), \tag{2.5}$$

for all $x,y \in X\backslash\{0\}$, satisfies the Drygas functional equation on $X\backslash\{0\}$.

Proof. Let us fix $u \in \mathscr{U} \subset l(X)$. Replacing x with $u'x$ and y with ux in (2.5), we get

$$d\left(f(x), 2f(u'x) + f(ux) + f(-ux) - f(-2ux)\right) \le \varepsilon(u'x,ux) =: \varepsilon_u(x) \tag{2.6}$$

for all $x \in X\backslash\{0\}$. We define the operators $\mathscr{T}_u : Y^X \to Y^X$, $\Lambda_u : \mathbb{R}_+^X \to \mathbb{R}_+^X$ by

$$\mathscr{T}_u\xi(x) := 2\xi(u'x) + \xi(ux) + \xi(-ux) - \xi(-2ux), \tag{2.7}$$

$$\Lambda_u\delta(x) := 2\delta(u'x) + \delta(ux) + \delta(-ux) + \delta(-2ux), \tag{2.8}$$

for all $x \in X$, $\xi \in Y^X$ and $\delta \in \mathbb{R}_+^X$. Then (2.6) becomes

$$d\left(f(x), \mathscr{T}_u f(x)\right) \le \varepsilon_u(x), \tag{2.9}$$

for all $x \in X\backslash\{0\}$. The operator $\Lambda_u : \mathbb{R}_+^X \to \mathbb{R}_+^X$ has the form given by (1.6) with $s = 4$ and $f_1(x) = u'x$, $f_2(x) = ux$, $f_3(x) = -ux$, $f_4(x) = -2ux$, $L_1(x) = 2$, $L_2(x) = L_3(x) = L_4(x) = 1$, for all $x \in X$. Further, we have

$$\begin{aligned}
d\left(\mathscr{T}_u\xi(x), \mathscr{T}_u\mu(x)\right) &= d\big(2\xi(u'x)+\xi(ux)+\xi(-ux)-\xi(-2ux),\\
&\qquad 2\mu(u'x)+\mu(ux)+\mu(-ux)-\mu(-2ux)\big)\,,\\
&\leq 2d\big(\xi(u'x),\mu(u'x)\big)+d\left(\xi(ux),\mu(ux)\right)\\
&\qquad +d\left(\xi(-ux),\mu(-ux)\right)+d\left(\xi(-2ux),\mu(-2ux)\right)\\
&= \sum_{i=1}^{4} L_i(x)d\left(\xi(f_i(x)),\mu(f_i(x))\right)
\end{aligned}$$

for all $x \in X$ and $\xi, \mu \in Y^X$. As $u \in \mathscr{U}$, we have

$$\begin{aligned}
\varepsilon^*(x) &:= \sum_{k=0}^{\infty} \Lambda_u^k \varepsilon_u(x)\\
&\leq \varepsilon(u'x,ux)\sum_{k=0}^{\infty}\big(2\lambda(u')+\lambda(u)+\lambda(-u)+\lambda(-2u)\big)^k\\
&= \frac{\varepsilon(u'x,ux)}{1-2\lambda(u')-\lambda(u)-\lambda(-u)-\lambda(-2u)}\\
&< \infty,
\end{aligned}$$

for all $x \in X\backslash\{0\}$. Now, using the Theorem 1.3, there exists a unique solution $F_u : X\backslash\{0\} \to Y$ of the equation

$$F_u(x) = 2F_u(u'x)+F_u(u)+F_u(-ux)-F_u((-2ux),$$

for all $x \in X\backslash\{0\}$, which is a fixed point of $\mathscr{T}_u$, such that

$$d\left(F_u(x), f(x)\right) \leq \frac{\varepsilon(u'x,ux)}{1-2\lambda(u')-\lambda(u)-\lambda(-u)-\lambda(-2u)},$$

for all $x \in X\backslash\{0\}$. Moreover

$$F_u(x) = \lim_{k\to\infty} \mathscr{T}_u^k f(x),$$

for all $x \in X\backslash\{0\}$. To prove that F_u satisfies the Drygas functional equation (1.2) on $X\backslash\{0\}$, just prove the following inequality

$$\begin{aligned}
&d\left(\mathscr{T}^n f(x+y), 2\mathscr{T}^n f(x)+\mathscr{T}^n f(y)-\mathscr{T}^n f(-y)-\mathscr{T}^n f(x-y)\right)\\
&\leq \varepsilon(x,y)\big(2\lambda(u')+\lambda(u)+\lambda(-u)+\lambda(-2u)\big)^n.
\end{aligned} \tag{2.10}$$

for all $x,y \in X\backslash\{0\}$, and $n \in \mathbb{N}$.

Indeed, if $n=0$ then (2.10) simply becomes (2.5). So, take $n\in\mathbb{N}_+$ and suppose that (2.10) holds for $n\in\mathbb{N}_+$ and $x,y\in X\backslash\{0\}$. Then, by using (2.7) and the triangle inequality, we have

$$\begin{aligned}
&d\left(\mathscr{T}^{n+1}f(x+y),2\mathscr{T}^{n+1}f(x)+\mathscr{T}^{n+1}f(y)-\mathscr{T}^{n+1}f(-y)-\mathscr{T}^{n+1}f(x-y)\right)\\
&=d\big(2\mathscr{T}^{n}f(u'x+u'y)+\mathscr{T}^{n}f(ux+uy)+\mathscr{T}^{n}f(-ux-uy)-\mathscr{T}^{n}f(-2ux-2uy),\\
&\quad 4\mathscr{T}^{n}f(u'x)+2\mathscr{T}^{n}f(ux)+2\mathscr{T}^{n}f(-ux)-2\mathscr{T}^{n}f(-2ux)\\
&\quad +2\mathscr{T}^{n}f(u'y)+\mathscr{T}^{n}f(uy)+\mathscr{T}^{n}f(-uy)-\mathscr{T}^{n}f(-2uy)\\
&\quad +2\mathscr{T}^{n}f(-u'y)+\mathscr{T}^{n}f(-uy)+\mathscr{T}^{n}f(uy)-\mathscr{T}^{n}f(2uy)\\
&\quad -2\mathscr{T}^{n}f(u'x-u'y)-\mathscr{T}^{n}f(ux-uy)-\mathscr{T}^{n}f(-ux+uy)+\mathscr{T}^{n}f(-2ux+2uy)\big)\\
&\le d\left(\mathscr{T}^{n}f(u'x+u'y),2\mathscr{T}^{n}f(u'x)+\mathscr{T}^{n}f(u'y)-\mathscr{T}^{n}f(-u'y)-\mathscr{T}^{n}f(u'x-u'y)\right)\\
&\quad +d\left(\mathscr{T}^{n}f(ux+uy),2\mathscr{T}^{n}f(ux)+\mathscr{T}^{n}f(uy)-\mathscr{T}^{n}f(-uy)-\mathscr{T}^{n}f(ux-uy)\right)\\
&\quad +d\left(\mathscr{T}^{n}f(-ux-uy),2\mathscr{T}^{n}f(-ux)+\mathscr{T}^{n}f(-uy)-\mathscr{T}^{n}f(uy)-\mathscr{T}^{n}f(-ux+uy)\right)\\
&\quad +d\left(\mathscr{T}^{n}f(-2ux-2uy),2\mathscr{T}^{n}f(-2ux)+\mathscr{T}^{n}f(-2uy)-\mathscr{T}^{n}f(2uy)-\mathscr{T}^{n}f(-2ux+2uy)\right)\\
&\le \left(2\varepsilon(u'x,u'y)+\varepsilon(ux,uy)+\varepsilon(-ux,-uy)+\varepsilon(-2ux,-2uy)\right)\\
&\quad \times\left(2\lambda(u')+\lambda(u)+\lambda(-u)+\lambda(-2u)\right)^{n}\\
&\le \varepsilon(x,y)\left(2\lambda(u')+\lambda(u)+\lambda(-u)+\lambda(-2u)\right)\\
&\quad \times\left(2\lambda(u')+\lambda(u)+\lambda(-u)+\lambda(-2u)\right)^{n}\\
&=\varepsilon(x,y)\left(2\lambda(u')+\lambda(u)+\lambda(-u)+\lambda(-2u)\right)^{n+1}.
\end{aligned}\tag{2.11}$$

By induction, we have shown that (2.10) holds for all $x,y\in X\backslash\{0\}$. Letting $n\to\infty$ in (2.10), we get

$$F_u(x+y)+F_u(x-y)=2F_u(x)+F_u(y)+F_u(-y)\tag{2.12}$$

for all $x,y\in X\backslash\{0\}$. Thus, we have proved that for every $u\in\mathscr{U}$ there exists a function $F_u:X\to Y$ solution of the functional equation (1.2) on $X\backslash\{0\}$, and

$$d\left(f(x),F_u(x)\right)\le\frac{\varepsilon(u'x,ux)}{1-2\lambda(u')-\lambda(u)-\lambda(-u)-\lambda(-2u)}$$

for all $x \in X\backslash\{0\}$. Thus

$$\begin{aligned} d\left(f(x), F_u(x)\right) &\leq \inf_{u \in \mathscr{U}} \left\{ \frac{\varepsilon(u'x, ux)}{1 - 2\lambda(u') - \lambda(u) - \lambda(-u) - \lambda(-2u)} \right\} \\ &\leq \frac{\inf_{u \in \mathscr{U}} \varepsilon(u'x, ux)}{1 - 2\lambda(u') + \lambda(u) + \lambda(-u) + \lambda(-2u)} \\ &= 0, \end{aligned}$$

for all $x \in X\backslash\{0\}$, this means that $F_u(x) = f(x)$ for all $x \in X\backslash\{0\}$, hence

$$f(x+y) + f(x-y) = 2f(x) + f(y) + f(-y).$$

for all $x \in X\backslash\{0\}$, which implies that f satisfies the Drygas functional equation on $X\backslash\{0\}$.

The next corollary is an immediate consequence of Theorem 2.1.

Corollary 2.2. *Let E and F be a normed space and a Banach space, respectively. Assume that X is a subgroup of the group $(E, +)$, $p < 0, q < 0$ and $c \geq 0$. If $f : X \to F$ satisfies*

$$\|f(x+y) + f(x-y) - 2f(x) - f(y) - f(-y)\| \leq c\left(\|x\|^p + \|y\|^q\right), \tag{2.13}$$

for all $x, y \in X\backslash\{0\}$, then f satisfies the Drygas equation on $X\backslash\{0\}$.

Proof. The proof follows from Theorem 2.1 by taking

$$\varepsilon(x,y) = c\left(\|x\|^p + \|y\|^q\right), \quad x, y \in X\backslash\{0\},$$

with some real $p < 0, q < 0$, $c \geq 0$, and

$$d(x,y) = \|x - y\|,$$

$$u_m(x) := u_m x := u(x) = -mx, \quad u'_m(x) := u'_m x := u'(x) = (1+m)x \quad m \in \mathbb{N}.$$

So it easily seen that conditions (2.4) are fulfilled with

$$\mathscr{U} := \left\{u_m \in Aut\ X : m \in \mathbb{N}_{n_0}\right\}.$$

Indeed,

$$\begin{aligned}\varepsilon(u_m x, u_k y) &= \varepsilon(-mx, -ky)\\ &= c\left(\|-mx\|^p + \|-ky\|^q\right)\\ &= c\,|m|^p\,\|x\|^p + c\,|k|^q\,\|y\|^q\\ &\leq \left(|m|^p + |k|^q\right) c\left(\|x\|^p + \|y\|^q\right)\\ &= \left(|m|^p + |k|^q\right)\varepsilon(x,y)\end{aligned}$$

for every $x, y \in X\backslash\{0\}$, $k, m \in \mathbb{N}_+$. Hence

$$\begin{aligned}\lim_{m\to\infty} \varepsilon(u_m x, u'_m y) &\leq \lim_{m\to\infty} \left(m^p + (1+m)^p\right)\varepsilon(x,y)\\ &= 0,\end{aligned}$$

for all $x, y \in X\backslash\{0\}$, and there exists $n_0 \in \mathbb{N}_+$ such that

$$2\lambda(u'_m) + \lambda(u_m) + \lambda(-u_m) + \lambda(-2u_m) < 1 \quad m \in \mathbb{N}_{n_0}.$$

Therefore, by Theorem 2.1 every $f: X \to Y$ satisfying (2.13) is solution of Drygas equation on $X\backslash\{0\}$.

Conflict of Interests

The authors declare that there is no conflict of interests.

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