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NONLINEAR DEGENERATED ELLIPTIC EQUATION HAVING LOWER ORDER TERM IN WEIGHTED ORLICZ-SOBOLEV SPACES AND L^1 DATA

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Abstract. The paper deals with the existence of solutions for a nonlinear degenerated equation in divergence form having lower order term, this problem is associated to elliptic operators in the framework of weighted orlicz-Sobolev spaces and L^1 data.

Keywords: degenerated elliptic equation; orlicz-Sobolev spaces; compact embedding; truncations.

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1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain, and M,P two N-functions such that $P \ll M$, $\overline{M},\overline{P}$ be the complementary functions of M,P, respectively. In this article, we prove the existence of solution for quasilinear degenerate elliptic equations of the form :

$$-\operatorname{div}(\rho(x)a(x,u,\nabla u)) + a_0(x,u,\nabla u) = f, \tag{1}$$

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where $A(u) = -div(\rho(x)a(x,u,\nabla u)) + a_0(x,u,\nabla u)$ is a Leray-Lions operator defined on $D(A) \subset W_0^1 L_M(\Omega)$.
$a : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ and $a_0 : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ are Carathéodorys functions satisfying some naturals growth conditions with respect to u and ∇u and the degenerate ellipticity condition

$$a_0(x,s,\xi)\eta + \rho(x)a(x,s,\xi)\xi \geq \lambda_0[M(\lambda_1 s) + \rho(x)M(\lambda_2|\xi|)].$$

The source term f is supposed in $L^1(\Omega)$. In no degenerate case Gossez [12] solved(1.1) with $f \in W^{-1}E_{\overline{M}}(\Omega)$. An existence theorem have been proved by Benkirane and Elmahi [6,8] with $f \in W^{-1}E_{\overline{M}}(\Omega)$ and $f \in L^1(\Omega)$, respectively, but the result is restricted to N-functions M satisfying a Δ_2 condition. An other work in this direction can be found in [9] in non-weighted case. So for our nonlinear operator $A(u) = -div(\rho(x)a(x,u,\nabla u)) + a_0(x,u,\nabla u)$, with coefficients are singular or degenerated the classic ellipticity conditions are violated one has to change the classical approach introducing weighted spaces. The case where $f \in W^{-1}E_{\overline{M}}(\Omega,\rho)$, and $A(u) = -div(a(x,u,\nabla u)) + a_0(x,u,\nabla u)$ is treated in [3] by using the framework of pseudo-monotones operator in complementary systems introduced by Gossez. Note that this type of equations can be applied in sciences physics. Non-standard example of $M(t)$ which occur in the mechanics of solids and Fluids are $M(t) = tlog(1+t)$. Note that the use of the truncation operator in (1.1) is justified by the fact that the the solution does not in general belong to $L^\infty(\Omega)$ for $f \in L^1(\Omega)$. Specific examples to which our results apply include :

$$-\operatorname{div}(\rho(x)|\nabla u|^{p-2}\nabla u) = f \quad \text{in } \Omega$$

$$-\operatorname{div}(\rho(x)|\nabla u|^{p-2}\nabla u \log^{\beta}(1+|\nabla u|)) = f \quad \text{in } \Omega$$

$$-\operatorname{div}\frac{\rho(x)M(|\nabla u|)\nabla u}{|\nabla u|^2} = f \quad \text{in } \Omega,$$

where $p > 1$, f is function in $L^1(\Omega)$ and ρ is a given weight function on Ω.

2. Preliminaries

In this section we present, some definitions and well-known about N-functions, weighted Orlicz-Sobolev spaces (standard references are in [1], [5] and [8]).

2.1. The N- functions

Let $M : I\!R^+ \to I\!R^+$ be an N- function, ie. M is continuous, convex, with $M(t) > 0$ for $t > 0$, $\quad M(t)/t \to 0$ as $t \to 0$ and $\frac{M(t)}{t} \to \infty$ as $t \to \infty$.
Equivalently. M admits the representation:

$$M(t) = \int_0^t m(\tau)d\tau,$$

where $m : I\!R^+ \to I\!R^+$ is non-decreasing right continuous, with $m(0) = 0$, $m(t) > 0$ for $t > 0$ and $m(t) \to \infty$ as $t \to \infty$ The N-function $\overline{M}$ conjugate to M defined by

$$\overline{M}(t) = \int_0^t \overline{m}(\tau)d\tau,$$

where $\overline{m} : I\!R^+ \to I\!R^+$ is given bay $\overline{m}(t) = \sup\{s : m(s) \leq t\}$. Clearly $\overline{\overline{M}} = M$ and has young's inequality $st \leq M(t) + \overline{M}(s)$ for all $s, t \geq 0$.

It is well known that we can assume that m and $\overline{m}$ are continuous and strictly increasing. We will extend the N-functions into even function on all $I\!R$. The N-function M is said to satisfy the Δ_2-condition every where (resp. infinity) if there exist $k > 0$(resp. $t_0 > 0$) such that, $M(2t) \leq kM(t)$ for all $t \geq 0$ (resp.$t \geq t_0$).

2.2. Orlicz-Sobolev spaces

Let Ω be a open subset of $I\!R^N$, and let M be an N-fonction. The Orlicz classe $K_M(\Omega)$ (resp the Orlicz spaces $L_M(\Omega)$ is the set of all (equivalence classes modulo equality a.e.in Ω of) real-valued measurable functions u defined in Ω and satisfying $\int_\Omega M(u(x))dx < \infty$
(resp) $\int_\Omega M(\frac{|u(x)|}{\lambda})dx < \infty$ for some $\lambda > 0$.

$L_M(\Omega)$ is a Banach space under the Luxemburg's norm:

$$\|u\|_{M,\Omega} = \inf\left\{\lambda > 0 : \int_\Omega M(\frac{|u(x)|}{\lambda})dx \leq 1\right\}. \tag{2.1}$$

The closure in $L_M(\Omega)$ of the set of bounded measurable function with compact support in $\overline{\Omega}$ is denoted by$E_M(\Omega)$. We have $E_M(\Omega) \subset K_M(\Omega) \subset L_M(\Omega))$. The equality $L_M(\Omega) = E_M(\Omega)$ hold if and only if M satisfies the Δ_2-condition, for all t or for t large according to whether Ω has a infinite measure or note. The dual of $E_M(\Omega)$ can be identified with $L_{\overline{M}}(\Omega)$ by means of the pairing $\int_\Omega u(x)v(x)dx$ where $u \in L_M(\Omega)$ and $v \in L_{\overline{M}}(\Omega)$ and the dual norm on $L_{\overline{M}}(\Omega)$ is equivalent to $\|\|_{\overline{M},\Omega}$.

The space $L_M(\Omega)$ is reflexive if and only if M an $\overline{M}$ satisfy the Δ_2-condition for all t or for t large, according to wheither Ω be infinite measure or note.

We return now to the Orlicz-Sobolev spaces $W^1L_M(\Omega)$ (resp $W^1E_M(\Omega)$) is the space of all function u such that u and its distributional derivatives up to order 1 lies in $\in L_M(\Omega)$ (resp $\in E_M(\Omega)$) .

It's Banach space under the norm

$$\|u\|_{1,M} = \sum_{|\alpha|\leq 1} \|D^{\alpha}u\|_{M,\Omega}. \tag{2.2}$$

Thus $W^1L_M(\Omega)$ and $W^1E_M(\Omega)$ can be identified with subspaces of $\prod L_M$ we have the weak topology $\sigma(\prod L_M, \prod E_{\overline{M}})$ and $\sigma(\prod L_M, \prod L_{\overline{M}})$.

The space $W_0^1E_M(\Omega)$ (resp $W_0^1L_M(\Omega)$ is defined by the closure of $D(\Omega)$ in $W^1E_M(\Omega)$ (resp $W^1L_M(\Omega)$ for the norm 2.2 (resp for the topology $\sigma(\prod L_M, \prod E_{\overline{M}})$).

Definition 2.1 The sequence u_n converges to u in $L_M(\Omega)$ for the modular convergence (denoted by $u_n \to u$ (mod) $L_M(\Omega)$) if $\int_{\Omega} M(\frac{|u_n - u|}{\lambda})dx \to 0$ as $n \to \infty$ for some $\lambda > 0$.

2.3. Weighted Orlicz-Sobolev spaces

Now we shall work with weighted Orlicz spaces in the following sense. Let Ω be a domain in $I\!R^N$, and let M be an N-fonction and ρ be a weight function on Ω, ie .measurable positive a.e on Ω. The weighted Orlicz classe $K_M(\Omega,\rho)$ (resp the weighted Orlicz space $L_M(\Omega,\rho)$) is the set of all (equivalence classes modulo equality a.e.in Ω) of real-valued mesurable functions u defined in Ω and satisfying

$$m_{\rho}(u,M) = \int_{\Omega} M(|u(x)|)\rho(x)dx < \infty$$

(resp)

$$m_{\rho}(\frac{u}{\lambda},M) = \int_{\Omega} M(\frac{|u(x)|}{\lambda})\rho(x)dx < \infty \text{ for some} \lambda > 0$$

$L_M(\Omega,\rho)$ is a Banach space under the Luxemburg's norm:

$$\|u\|_{M,\rho} = \inf\left\{\lambda > 0; m_{\rho}(\frac{u}{\lambda},M) \leq 1\right\}. \tag{2.3}$$

The closure in $L_M(\Omega,\rho)$ of the set of bounded measurable function with compact support in $\overline{\Omega}$ is denoted by $E_M(\Omega,\rho)$ we have

$$E_M(\Omega,\rho) \subset K_M(\Omega,\rho) \subset L_M(\Omega,\rho).$$

The equality $L_M(\Omega,\rho) = E_M(\Omega,\rho)$ hold if and only if M satisfies the Δ_2-condition, for all t or for t large according to whether Ω has a infinite measure or note. The dual of $E_M(\Omega,\rho)$ can be identified with $L_{\overline{M}}(\Omega,\rho)$ by means of the pairing

$$\int_\Omega u(x)v(x)\rho(x)dx$$

where $u \in L_M(\Omega,\rho)$ and $v \in L_{\overline{M}}(\Omega,\rho)$. The dual norm on $L_{\overline{M}}(\Omega,\rho)$ is equivalent to $\|\|_{\overline{M},\Omega}$. Giving birth to the so called Orlicz norm on$L_M(\Omega,\rho)$ defined by

$$\|u\|_{(M,\rho)} = \sup\left\{\int_\Omega f(x)g(x)\rho(x)dx; m_\rho(g,\overline{M}) \leq 1\right\}$$

The space $L_M(\Omega,\rho)$ is reflexive if and only if M an $\overline{M}$ satisfy the Δ_2-condition for all t or for t large, according to weighted Ω be infinite measure or note.

We return now to the weighted Orlicz-Sobolev spaces $W^1L_M(\Omega,\rho)$ (resp $W^1E_M(\Omega,\rho)$) is the space of all function u such that $u \in L_M(\Omega)$ (resp $u \in E_M(\Omega)$) and its distributional derivatives up to order 1 lie in $L_M(\Omega,\rho)$ (resp in $E_M(\Omega,\rho)$).

It's Banach space under the norm :

$$\|u\|_{1,M,\rho} = \|u\|_M + \|\nabla u\|_{M,\rho}. \tag{2.4}$$

where $\|u\|_M = \|u\|_{M,\Omega}$.

Thus $W^1L_M(\Omega,\rho)$ and $W^1E_M(\Omega,\rho)$) can be identified with subspaces of $\prod L_{M,\rho} = L_M \times \prod L_M(\Omega,\rho)$ we have the weak topology $\sigma(\prod L_{M,\rho}, \prod E_{\overline{M},\rho})$ and $\sigma(\prod L_{M,\rho}, \prod L_{\overline{M},\rho})$.

The space $W_0^1E_M(\Omega,\rho)$ (resp $W_0^1L_M(\Omega,\rho)$) is defined by the closure of $D(\Omega)$ in $W^1E_M(\Omega,\rho)$) (resp $W^1L_M(\Omega,\rho)$) for the norm (2.4) (resp for the topology $\sigma(\prod L_{M,\rho}, \prod E_{\overline{M},\rho})$.

The space $W_0^1E_M(\Omega,\rho)$ is defined as the (norm) closure of $C_0^\infty(\Omega,\rho)$ in $W^1E_M(\Omega,\rho)$ and the space $W_0^1L_M(\Omega,\rho)$ as the $\sigma(\Pi L_M, \Pi E_{\overline{M}})$ closure of $C_0^\infty(\Omega)$ in $W^1L_M(\Omega)$.

Let $W^{-1}L_{\overline{M}}(\Omega,\rho)$ (resp. $W^{-1}E_{\overline{M}}(\Omega,\rho)$) denote the space of distributions on which can be written as sums of derivatives of order ≤ 1 of functions in $L_{\overline{M}}(\Omega,\rho)$ (resp. $E_{\overline{M}}(\Omega,\rho)$). It is a

Banach space under the usual quotient norm (see). If the open set Ω has the segment property, then the space $C_0^\infty(\Omega)$ is dense in $W_0^1L_M(\Omega)$ for the modular convergence and thus for the topology $\sigma(\Pi L_M, \Pi L_{\overline{M}})$.

Definition 2.3.1 The sequence u_n converges to u in $W^1L_M(\Omega,\rho)$ for the modular convergence (denoted by $u_n \to u$ (mod) $W^1L_M(\Omega,\rho)$) if for some $\lambda > 0$ $\int_\Omega M(\frac{|u_n-u|}{\lambda})dx \to 0$ as $n \to \infty$ and $\int_\Omega M(\frac{|D^\alpha(u_n-u)|}{\lambda})\rho(x)dx \to 0$
as $n \to \infty$ for $|\alpha| = 1$.

3. Basic assumptions and fundamental lemmas

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain, M,P be two N-functions such that $P \ll M$, $\overline{M},\overline{P}$ be the complementary functions of M,P, respectively, $A : D(A) \subset W_0^1L_M(\Omega) \to W^{-1}L_{\bar{M}}(\Omega)$ be a mapping given by $A(u) = -\operatorname{div} a(x,u,\nabla u)$ where $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ be a Caratheodory function satisfying for a.e. $x \in \Omega$ and all $s \in \mathbb{R}$, $\xi,\eta \in \mathbb{R}^N$ with $\xi \neq \eta$ The following lemmas will be applied to the truncation operators, and concerns operators of the Nemytskii type in Orlicz spaces.

Lemma 3.1 *Let $f_n, f \in L^1(\Omega)$ such that*

(1) $f_n \geq 0$ a.e. in Ω.
(2) $f_n \to f$ a.e. in Ω.
(3) $\int_\Omega f_n(x)dx \to \int_\Omega f_n(x)dx$

Then $f_n \to f$ stongly in $L^1(\Omega)$.

Lemma 3.2 *Let $F : \mathbb{R} \to \mathbb{R}$ be uniformly lipschitzian, with $F(0) = 0$. Let M be an N-function and let $u \in W^1L_M(\Omega,\rho)$ (resp. $W^1E_M(\Omega,\rho)$). Then $F(u) \in W^1L_M(\Omega,\rho)$ (resp. $W^1E_M(\Omega,\rho)$). Moreover, if the set of discontinuity points of F' is finite, then*

$$\frac{\partial}{\partial x_i}F(u) = \begin{cases} F'(u)\frac{\partial u}{\partial x_i} & \text{a.e. in } \{x \in \Omega : u(x) \notin D\} \\ 0 & \text{a.e. in } \{x \in \Omega : u(x) \in D\}. \end{cases}$$

Proof. First consider de case $F \in C^1$ and let $W^1L_M(\Omega,\rho)$. Then there exist a sequence $u_n \in D(\Omega)$ such that $u_n \to u$ (mod) $W^1L_M(\Omega,\rho)$). Passing to subsequence, we can assume that $D^\alpha u_n \to D^\alpha u \quad \forall |\alpha| \leq 1$ a.e. in Ω. From the relation $|F(s)| \leq k|s|$,where k denote the lipschitz constant for F, and $\frac{\partial}{\partial x_i}F(u_n) = F'(u_n)\frac{\partial u_n}{\partial x_i}$, we deduce that $F(u_n)$ remains bounded in $W_0^1L_M(\Omega,\rho)$.Thus going to to a further subsequence, we obtain $F(u_n) \to w \in W_0^1L_M(\Omega,\rho)$ for $\sigma(\prod L_M, \prod E_{\overline{M}})$, and also by a local application of the compact imbedding theorem,$F(u_n) \to w$ a.e. in Ω. Consequently $w = F(u)$, and $F(u) \in W_0^1L_M(\Omega,\rho)$. Finally, by the usual chain rule for weak derivatives

$$\frac{\partial}{\partial x_i}F(u) = F'(u)\frac{\partial u}{\partial x_i} \tag{2.5}$$

a.e. in Ω. For the general case. Taking convolution with the mollifiers, we get a sequence $F_n \in C^\infty(I\!R)$ such that $F_n \to F$ uniformly on each compact, $F_n(0) = 0$ and $\left|F_n'\right| \leq k$.
For each n, $F_n \in W_0^1L_M(\Omega,\rho)$,and we have (2.5) with F remplaced by F_n. Finally (2.5) follows from the generalized chain rule for weak derivatives.

Lemma 3.3 *Let $F : \mathbb{R} \to \mathbb{R}$ be uniformly lipschitzian, with $F(0) = 0$. We suppose that the set of discontinuity points of F' is finite. Let M be an N-function, then the mapping $F : W^1L_M(\Omega,\rho) \to W^1L_M(\Omega,\rho)$ is sequentially continuous with respect to the weak* topology $\sigma(\Pi L_M, \Pi E_{\overline{M}})$.*

We use also the following technical lemmas:

Lemma 3.4 *If a sequence u_n converge a.e to u and if u_n remains bonded in $L_M(\Omega)$, then $u \in L_M(\Omega)$ and $u_n \to u$ for$\sigma(L_M(\Omega)$,$E_{\overline{M}}(\Omega))$*

Lemma 3.5 *If a sequence u_n converge a.e to u and if u_n remains bonded in $L_M(\Omega,\rho)$, then $u \in L_M(\Omega,\rho)$ and $u_n \to u$ for$\sigma(L_M(\Omega,\rho)$,$E_{\overline{M}}(\Omega,\rho))$*

Lemma 3.6 *Let Ω be an open subset of $\mathbb{R}^N$ with finite measure. Let M,P and Q be N-functions such that $Q << P$,and let F be a Carathéodorys function such that, for a.e. $x \in \Omega$ and all $s \in R$*

$$|F(x,s)| \leq c(x) + k_1P^{-1}M(k2|s|),$$

where k_1, k_2 are real constants and $c(x) \in E_Q(\Omega)$. Then the Nemytskii operator N_F defined by $NF(u)(x) = F(x,u(x))$ is strongly continuous from

$$P(E_M(\Omega),\frac{1}{k_2}) = \{u \in L_M : d(u,E_M(\Omega)) < \frac{1}{k_2}\}$$

into $E_Q(\Omega)$.

Lemma 3.7 *If the sequence* $u_n \in E_M(\Omega,\rho)$ *Converges a.e in* Ω *with* $\rho \in L^1(\Omega)$*, and the norms are uniformly Absolutely Continuous, i.e. for each* $\varepsilon > 0$ *there existe* $\delta > 0$ *such that*

$$\|u_n\chi(E)\|_{M,\rho} < \varepsilon$$

for all n, and $E \subset \Omega$ *With* $|E| < \delta$*, then it Converge in norm in* $E_M(\Omega,\rho)$.

3.1. Compactness results

Let Ω an bounded open subset of $I\!R^N$ with locally-lipscitzien boundary, ρ a weight function, and the N-function M such that the assumptions **(H)** are satisfied, there is a real $s > 0$ such that:

(H_1) : $(M(t))^{\frac{s}{s+1}}$ be N-function and that $\rho^{-s} \in L^1(\Omega)$

(H_2) : $\displaystyle\int_1^\infty \frac{t}{M(t)^{1+\frac{s}{N(s+1)}}} dM(t) = \infty.$

(H_3) : $\displaystyle lim_{t\to\infty}\frac{1}{M^{-1}(t)}\int_0^{t^{\frac{s+1}{s}}} \frac{M^{-1}(u)}{u^{1+\frac{s}{N(s+1)}}} du = 0.$

Remark 3.1.1 In the particular case where $M(t) = \frac{t^p}{p}$ $(1 < p < \infty)$, the first part of (H_1) is satisfied if $s > \frac{1}{p-1}$.

Theorem 3.1.2(see [2, theorem 9-5]). *Let* Ω *an bounded open subset of* $I\!R^N$ *with locally lipscitzian boundary and* M *an* N *-function.*
Suppose that assumptions ***(H)*** *are satisfied. So we have the following compact injection:* $W^1L_M(\Omega,\rho) \hookrightarrow\hookrightarrow E_M$

Theorem 3.1.2 *Let* Ω *an bounded open subset of* $I\!R^N$ *with locally-lipscitzien boundary,* ρ *a weight function, and* M *a N-function, let* $u \in W_0^1L_M(\Omega)$ *then:*

$$\|u\|_M \le c\|\nabla u\|_{M,\rho}$$

where c is a positive constant which imply that $\|\nabla u\|_{M,\rho}$ *is a equivalent norm of* $\|u\|_{1,M}$ *in* $W_0^1L_{M,\rho}$

Proof. Under the assumption **(H)**, the Sobolev conjugate N-function M_s^* of M_s is well defined, by $M_s^{*-1} = \int_0^s \frac{M^{-1}(t)}{t^{1+\frac{1}{N}}}dt$ we have $W_0^1 L_{M_s} \subset L_{M_s^*}$. And since $M \ll M_s^*$ we have $L_{M_s^*} \subset L_M$ hence

$$\|u\|_M \leq c_1 \|u\|_{M_s^*} \leq c_2 \|u\|_{1,M_s}$$

where c_1 and c_2 are two positives constants, by (gossez74) their exist a positive constant c' such

$$\|u\|_{1,M_s} \leq c' \|\nabla u\|_{M_s}$$

We well show that

$$\|\nabla u\|_{M_s} \leq c \|\nabla u\|_{M,\rho}$$

For that we have

$$\|v\|_{M_s} \leq \int_\Omega M_s(v(x))dx + 1 = \int_\Omega M_s(v(x))\frac{1}{\rho(x)}\rho(x)dx + 1$$

$$\leq \int_\Omega S(M_s(v(x)))\rho(x)dx + \int_\Omega \overline{S}(\frac{1}{\rho(x)})\rho(x)dx + 2$$

$$= \int_\Omega M(v(x))\rho(x)dx + \int_\Omega \rho^{-s}(x)dx + 1$$

witch implies that

$$\|v\|_{M_s} \leq c \|v\|_{M,\rho}$$

for some positive constant C.

In fact if is not true, then there exist a sequence v_n such that $\|v_n\|_{M_s} \to \infty$ and for n large, $\|v_n\|_{M,\rho}\| \leq 1$. Hence, for n sufficiently large we get

$$\int_\Omega M(v_n(x))\rho(x)dx \leq \|v_n\|_{M,\rho} \leq 1$$

then,

$$\|v_n\|_{M_s}\| \leq \int_\Omega M(v(x))\rho(x)dx + \int_\Omega \rho^{-s}(x)dx + 1$$

$$\leq \|v_n\|_{M,\rho}\| + \int_\Omega \rho^{-s}(x)dx + 1$$

witch is contradiction, since the left hand-side tends to infinity while the right hand-side is bonded. Finally taking $v = \nabla u$ we conclude the result. The following lemma is a immediate consequence of the above theorem.

Lemma 3.1.3. *Let Ω an bounded open subset of $\mathbb{R}^N$ with locally-lipscitzien boundary, ρ a weight function, and M a N-function M let $u \in W_0^1 L_M(\Omega)$. Then there exist positives constants c_1 and c_2 such that*

$$\int_\Omega M(|u(x)|)dx \leq c_1 \int_\Omega M(|c_2 \nabla u(x)|)\rho(x)dx$$

4. Main results

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain, M, P be two N-functions such that $P \ll M$, $\overline{M}, \overline{P}$ be the complementary functions of M, P, respectively, $A : D(A) \subset W_0^1 L_M(\Omega, \rho) \to W^{-1} L_{\overline{M}}(\Omega \rho)$ be a mapping (not everywhere defined) given by $A(u) = -\operatorname{div}(\rho(x) a(x, u, \nabla u)) + a_0(x, u, \nabla u)$ where $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ and $a_0 : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ are Carathéodorys functions satisfying for a.e. $x \in \Omega$ and all $s \in \mathbb{R}$, $\xi, \eta \in \mathbb{R}^N$ with $\xi \neq \eta$:

$$|a_0(x,s,\xi)| \leq K_0[g_0(x) + \overline{M}^{-1} M(\alpha_2 \eta) + \overline{M}^{-1}(\rho(x) P(\alpha_1 |\xi|))] \tag{4.1}$$

$$|a(x,s,\xi)| \leq C_0(x) + K_1 \overline{P}^{-1}(\rho^{-1} M(\alpha_2 s)) + K_2 \overline{M}^{-1} M(\alpha_1 |\xi|) \tag{4.2}$$

$$[a(x,s,\xi) - a(x,s,\eta)][\xi - \eta] > 0 \tag{4.3}$$

$$a_0(x,s,\xi)\eta + \rho(x) a(x,s,\xi)\xi \geq \lambda_0[M(\lambda_1 s) + \rho(x) M(\lambda_2 |\xi|)] \tag{4.4}$$

where $\alpha_1, \alpha_2, K_0, K_1, K_2, \lambda_0, \lambda_1, \lambda_2 > 0$, $K_0(x)$ in $L_{\overline{M}}(\Omega)$ and $C_0(x)$ in $(E_{\overline{M}}(\Omega, \rho))^N$
Let T_k the truncation operator at height $k > 0$, defined by $T_k(s) = \max(-k, \min(k, s))$, $\forall s \in \mathbb{R}$, for all $k \geq 0$.
And consider the following nonlinear elliptic problem with Dirichlet boundary condition.

$$A(u) = f \quad \text{in } \Omega \tag{4.5}$$

Finally, we assume that

$$f \in L^1(\Omega) \tag{4.6}$$

In the next section, we will prove the following main result.

Theorem 4.1.. Let Ω an bounded open subset of $\mathbb{R}^N$ with locally lipscitzian boundary, assume that (4.1)-(4.4) holds, and $f \in \mathscr{L}^1(\Omega)$. Then there exists at least one weak solution of the problem

$$\begin{cases} u \in W_0^1 L_M(\Omega, \rho), & * \\ \int_\Omega a(x,u,\nabla u)\nabla v \rho(x)dx + \int_\Omega a_0(x,u,\nabla u)v dx = \langle f, v\rangle, & * \\ \forall v \in W_0^1 E_M(\Omega,\rho) \cap L^\infty(\Omega), & * \end{cases} \tag{4.7}$$

Proof.

Step 1. Approximation problem and a priori estimate.

Let consider the sequence of approximate equations:

$$\begin{cases} u_n \in W_0^1 L_M(\Omega), & * \\ -\operatorname{div}(\rho(x)a(x,u_n,\nabla u_n)) + a_0(x,u_n,\nabla u_n) = f_n, & * \end{cases} \tag{4.8}$$

where f_n is a smooth function which converges to $f$in $L^1(\Omega)$ and $\|f_n\|_{L^1(\Omega)} \leq c_0$. For n fixed by [Theorem 3.1] in [3], there exists at least one solution $\{u_n\}$ to (4.8).

For $k > 0$, by taking $T_k(u_n)$ as test function in (4.8), one has

$$\int_\Omega a(x,u_n,\nabla T_k(u_n))\nabla T_k(u_n)\rho(x)dx + \int_\Omega a_0(x,u_n,\nabla u_n)T_k(u_n)dx =< f_n, T_k(u_n> .$$

In view of the degenerate ellipticity condition (4.4), and the fact that $\|f_n\|_{L^1(\Omega)} \leq c_0$ we get

$$\int_\Omega M(\lambda_2|\nabla T_k(u_n)|)\rho(x)dx \leq \frac{c_0 k}{\lambda_0} \tag{4.9}$$

Thanks to Lemma 3.8, there exist two constants c_1 and $c2$such that

$$\int_\Omega M(|v(x)|)dx \leq c_1 \int_\Omega M(|c_2 \nabla v(x)|)\rho(x)dx$$

Taking $v = \frac{\lambda_2 T_k(u_n)}{c_2}$, we have

$$\int_\Omega M(\frac{\lambda_2|T_k(u_n)|}{c_2}))dx \leq \frac{c_1 c_0 k}{\lambda_0} \tag{4.10}$$

which imply that

$$\operatorname{mes}\{|u_n| \geq k\} \leq \frac{c_1 c_0 k}{\lambda_0 M(\frac{k\lambda}{c_2})}$$

And using $\frac{t}{M(t)} \to 0 as t \to \infty$ hens $\operatorname{mes}\{|u_n| \geq k\} \to 0$ as $k \to +\infty$. for$\delta > 0$ we have :

$$\operatorname{mes}\{|u_n - u_m| > \delta\} \leq \operatorname{mes}\{|u_n| \geq k\} + \operatorname{mes}\{|u_m| \geq k\} + \operatorname{mes}\{|u_n - u_m| > \delta\}$$

by (4.9) $T_k(u_n)$ is bounded in $W_0^1L_M(\Omega,\rho)$, then their exist $\ell_k \in W_0^1L_M(\Omega,\rho)$ such $T_k(u_n) \rightharpoonup \ell_k$ weakly in $W_0^1L_M(\Omega,\rho)$ (for a subsequence still denoted $T_k(u_n)$)thus $\forall\varepsilon > 0$ their exist $k(\varepsilon) > 0$ mes$\{|u_n - u_m| > \delta\} \leqslant \varepsilon$ for$n,m \geq n_0(k(\varepsilon),\delta)$

This proves that u_n is a cauchy sequence in measure, then there exists some measurable function u such that

$$u_n \to u \text{ almost everywhere in } \Omega \tag{4.11}$$

Then

$$T_k(u_n) \rightharpoonup T_k(u) \quad \text{weakly in } W_0^1L_M(\Omega,\rho) \text{ for } \sigma\left(\Pi L_{M,\rho},\Pi E_{\overline{M},\rho}\right) \tag{4.12}$$

And by theorem 3.1.1 we deduce that

$$T_k(u_n) \to T_k(u) \quad \text{strongly in } E_M(\Omega) \tag{4.13}$$

Step 2. boundedness of $a(x,T_k(u_n),\nabla T_k(u_n))$ and $a_0(x,T_k(u_n),\nabla T_k(u_n))$.

In this step we will shows that $a(x,T_k(u_n),\nabla T_k(u_n))$ remains bounded in $(L_{\overline{M}}(\Omega,\rho))^N$. We will use the Orlicz norm. For that, let $\psi \in (L_M(\Omega))^N$with $\|\psi\|_M \leq 1$. In fact, by the monotonicity condition (4.2) we have

$$\int_\Omega [a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\psi)][\nabla T_k(u_n) - \psi]\rho(x)dx \geq 0$$

So that

$$\begin{aligned}\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\psi\rho(x)dx \leq &\int_\Omega a(x,T_k(u_n),\nabla(u_n))\nabla T_k(u_n)\rho(x)dx\\ &-\int_\Omega a(x,T_k(u_n),\psi)\nabla T_k(u_n)\rho(x)dx\\ &+\int_\Omega a(x,T_k(u_n),\psi)\psi\rho(x)dx\end{aligned}$$

And by (4.8) we have:

$$\int_\Omega a(x,T_k(u_n,\nabla T_k(u_n))\nabla T_k(u_n)\rho(x)dx \leq c_0k$$

To estimate the seconde the tiered terms we us Yuong inequality hence

$$\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\psi\rho(x)dx \le c_0k+2\int_\Omega \overline{M}(\frac{|a(x,T_k(u_n),\psi|}{r})\rho(x)dx + \int_\Omega M(r|\nabla T_k(u_n)|)\rho(x)dx+\int_\Omega M(r|\psi|)\rho(x)dx$$

where $r>0$ using the growth conditions (4.1)and the fact that $P<<M$ we conclude for r large and ε small, that

$$\int_\Omega \overline{M}(\frac{|a(x,T_k(u_n)),\psi|}{r})\rho(x)dx \le \frac{1}{r}\int_\Omega \overline{M}(C_0(x))\rho(x)dx + \frac{\varepsilon K_1}{r}\int_\Omega M(\alpha_2 T_k(u_n)dx + \frac{K_2}{r}\int_\Omega M(\alpha_1|\psi|)dx+K'_\varepsilon$$

since $T_k(u_n)$ is bounded in $W_0^1L_M(\Omega,\rho)$ and ψ is bounded in $(L_M(\Omega))^N$ we get

$$\int_\Omega \overline{M}(|a(x,T_k(u_n|),\psi)\rho(x)dx \le C_k$$

for all $\psi\in(L_M(\Omega,))^N$ with $\|\psi\|_{M,}\le 1$

Therefore, we deduce that $a(x,T_k(u_n),\nabla T_k(u_n))$ remains bounded in $(L_{\overline{M}}(\Omega,\rho))^N$

Let now prove that $a_0(x,u_n,\nabla u_n)$ is bounded in $(L_{\overline{M}}(\Omega))$.

First using growth conditions (4.1) it follows that for λ large

$$\int_\Omega \overline{M}(\frac{|a_0(x,u_n,\nabla u_n|}{\lambda})dx \le \frac{1}{2}\int_\Omega \overline{M}(\frac{2}{\lambda}k_0|g_0(x)|dx + \frac{k_0}{\lambda}\int_\Omega M(\alpha_2 u_n)dx + \frac{k_0}{\lambda}\int_\Omega P(\alpha_1|\nabla u_n|)\rho(x)dx$$

Since $P\ll M$ then for large $\lambda$and ε small we get

$$\int_\Omega \overline{M}(\frac{a(x,u_n,\nabla u_n}{\lambda})dx \le \frac{1}{2}\int_\Omega \overline{M}(\frac{2}{\lambda}k_0|g_0(x)|dx + \frac{k_0}{\lambda}\int_\Omega M(\alpha_2 u_n)dx+\frac{k_0}{\lambda}\int_\Omega M(\varepsilon|\nabla u_n|)\rho(x)dx+\frac{k_\varepsilon}{\lambda} \le 1$$

for some positive constant k_ε, which implies that $a_0(x,u_n,\nabla u_n)$ is bounded in $(L_{\overline{M}}(\Omega))$

Remark 4.2 we can easily shows as in [3] that $a(x,u_n,\nabla u_n)$ remains bounded in $(L_{\overline{M}}(\Omega,\rho))^N$

Step 3. Almost everywhere convergence of the gradient.

In this step we prove that $\nabla u_n \to \nabla u$ a.e in Ω for a subsequence. For $k>0$, and n fixed we take $v_n = T_k(u_n) - T_k(u)$ as test function in (4.8) one has :

$$
\begin{aligned}
< f_n, v_n >= &\int_\Omega a(x,u_n,\nabla u_n)(\nabla T_k(u_n) - \nabla T_k(u))\rho(x)dx \\
&+ \int_\Omega a_0(x,u_n,\nabla u_n)(T_k(u_n)-T_k(u))dx
\end{aligned}
\tag{4.14}
$$

let $\Omega_r = \{x \in \Omega / |\nabla T_k(u_n)| \leq r\}$, and let χ_r denoted the characteristic functions of the sets Ω_r hens $\Omega_r \subset \Omega_{r+1}$ and $|\Omega_r \backslash \Omega_{r+1}| \to 0$ as$r \to +\infty$ for hat fix $r>0$ and let $s>r$, by the monotonicity condition (4.2) we get :

$$
\int_{\Omega_r} [a(x,u_n,\nabla T_k(u_n)) - a(x,u_n,\nabla T_k(u))][\nabla T_k(u_n) - \nabla T_k(u)]\rho(x)dx \geq 0 \tag{4.15}
$$

$$
\int_{\Omega_s} [a(x,u_n,\nabla T_k(u_n)) - a(x,u_n,\nabla T_k(u))][\nabla T_k(u_n) - \nabla T_k(u)]\rho(x)dx \geq 0 \tag{4.16}
$$

Then

$$
\int_{\Omega} [a(x,u_n,\nabla T_k(u_n)) - a(x,u_n,\chi_s\nabla T_k(u))][\nabla T_k(u_n) - \chi_s\nabla T_k(u)]\rho(x)dx \geq 0 \tag{4.17}
$$

On the other hand let consider :

$$
< Bu_n, v_n >= \int_\Omega a(x,u_n,\nabla u_n)[\nabla T_k(u_n) - \nabla T_k(u)]\rho(x)dx
$$

We can see that

$$
\begin{aligned}
< Bu_n, v_n > = &\int_\Omega a(x,u_n,\nabla T_k(u_n))[\nabla T_k(u_n) - \nabla T_k(u)\chi_s]\rho(x)dx \\
&- \int_\Omega a(x,u_n,\nabla u_n)\nabla T_k(u)\rho(x)dx \\
&+ \int_\Omega a(x,u_n,\nabla T_k(u_n))\nabla T_k(u)\chi_s\rho(x)dx
\end{aligned}
$$

Hence

$$
\begin{aligned}
< Bu_n, v_n > = &\int_\Omega [a(x,u_n,\nabla T_k(u_n)) - a(x,u_n,\nabla T_k(u)\chi_s)][\nabla T_k(u_n) - \nabla T_k(u)\chi_s]\rho(x)dx \\
&- \int_\Omega [a(x,u_n,\nabla u_n) - a(x,u_n,\nabla T_k(u_n)]\nabla T_k(u)\rho(x)dx \\
&- \int_\Omega a(x,u_n,\nabla T_k(u_n))\nabla T_k(u)\chi_{\Omega\backslash\Omega_s}\rho(x)dx \\
&+ \int_\Omega a(x,u_n,\nabla T_k(u_n)\chi_s)[\nabla T_k(u_n) - \nabla T_k(u)\chi_s]\rho(x)dx
\end{aligned}
$$

Which can be writhed as

$$< Bu_n, v_n >= I_n - I_n^1 - I_n^2 + I_n^3 \tag{4.18}$$

where:

$$I_n = \int_\Omega [a(x,u_n,\nabla T_k(u_n)) - a(x,u_n,\nabla T_k(u)\chi_s)][\nabla T_k(u_n) - \nabla T_k(u)\chi_s]\rho(x)dx$$
$$I_n^1 = \int_\Omega [a(x,u_n,\nabla u_n) - a(x,u_n,\nabla T_k(u_n)]\nabla T_k(u)\rho(x)dx$$
$$I_n^2 = \int_\Omega a(x,u_n,\nabla T_k(u_n))\nabla T_k(u)\chi_{\Omega\setminus\Omega_s}\rho(x)dx$$
$$I_n^3 = \int_\Omega a(x,u_n,\nabla T_k(u_n)\chi_s)[\nabla T_k(u_n) - \nabla T_k(u)\chi_s]\rho(x)dx$$

In the following we will show this intermediate lemma.

Lemma 4.1.1 *Let I_n^1, I_n^2 and I_n^3 as above we have:*

$$\textit{(i)} \quad I_n^1 \to 0$$
$$\textit{(ii)} \quad I_n^2 \to \int_{\Omega\setminus\Omega_s} h\nabla T_k(u)\rho(x)dx$$
$$\textit{(iii)} \quad I_n^3 \to \int_{\Omega\setminus\Omega_s} a(x,u,0)\nabla T_k(u)\rho(x)dx$$

(2)

Proof of (i):

First we can write

$$I_n^1 = \int_\Omega [a(x,u_n,\nabla(u_n)) - a(x,u,0)]\chi_{G_n}\chi_s\nabla T_k(u)\rho(x)dx$$

where: $G_n = \{x \in \Omega; |u_n(x))| > k\}$ we have

$$M(|\chi_{G_n}\chi_s\nabla T_k(u)|)\rho(x) \le M(|\nabla T_k(u)\chi_s|)\rho(x) \in L^1(\Omega)$$

and we have $u_n(x) \to u(x)$ a.e in Ω hence if $|u(x)| < k$ then for n large $|u_n(x)| < k$. which implies that

$$|\nabla T_k(u)|\chi_{G_n}\chi_s \to 0 \text{ a.e in } \Omega$$

then By Lebesgue theorem we deduce that

$$M(|\nabla T_k(u)|\chi_{G_n}\chi_s)\rho(x) \to 0$$

Thus $\chi_s\chi_{G_n}\nabla T_k(u) \to 0$ strongly in $(E_M(\Omega,\rho))^N$ and since $a(x,u_n,\nabla(u_n))$ and $a(x,u,0)$ are bonded in $(L_{\overline{M}}(\Omega,\rho))^N$ we obtain $I_n^1 \to 0$.

Proof of (ii):

We have $a(x,u_n,\nabla T_k(u_n))$ is bonded in $(L_{\overline{M}}(\Omega,\rho))^N$, by the lemma 3.4 we deduce that there exist $h \in L_{\overline{M}}(\Omega,\rho)^N$ and a subsequence also denoted $a(x,u_n,\nabla T_k((u_n)))$ such that $a(x,u_n,\nabla T_k((u_n))) \rightharpoonup h$ weakly in $(L_{\overline{M}}(\Omega,\rho))^N$ we well pass to the limit over n and obtain:

$$I_n^2 \to \int_{\Omega\setminus\Omega_s} h\nabla T_k(u)\rho(x)dx$$

Proof of (iii):

By using (4.12), we have $\nabla T_k(u_n) \rightharpoonup \nabla T_k(u)$ weakly in $(L_M(\Omega,\rho))^N$ for $\sigma(\Pi L_M(\Omega,\rho),\Pi E_{\overline{M}}(\Omega,\rho))$. Thanks to continuity of Nemytsky operator, we get

$$a(x,u_n,\nabla T_k(u_n)\chi_s) \to a(x,u_n,\nabla T_k(u)\chi_s)$$

strongly in $(E_{\overline{M}}(\Omega,\rho))^N$ Then $I_n^3 \to \int_\Omega a(x,u,\nabla T_k(u)\chi_s)[\nabla T_k(u) - \nabla T_k(u)\chi_s]\rho(x)dx$ ie

$$I_n^3 \to \int_{\Omega\setminus\Omega_s} a(x,u,0))\nabla T_k(u)\rho(x)dx$$

which achieved the above Lemma.

Finally we shall prove that $\nabla u_n \to \nabla u$ a.e in Ω, back to approximate problems (4.8), we have

$$< Bu_n, v_n >=< f_n, v_n > - \int_\Omega a_0(x,u_n,\nabla(u_n))v_n dx$$

since $v_n \in W_0^1E_M(\Omega,\rho)\cap L^\infty(\Omega)$ and $v_n \rightharpoonup 0$ weakly* in $L^\infty(\Omega)$ and $f_n \to f$ strongly in $L^1(\Omega)$, while $a_0(x,u_n,\nabla(u_n))$ is bonded in $(L_{\overline{M}}(\Omega,\rho))^N$ then $< f_n, v_n >\to 0$ We conclude that

$$\begin{aligned} 0 &\le \int_\Omega [a(x,u_n,\nabla T_k(u_n)) - a(x,u_n,\nabla T_k(u)\chi_s)][\nabla T_k(u_n) - \nabla T_k(u)\chi_s]\rho(x)dx \\ &\le \int_{\Omega\setminus\Omega_s} (h - a(x,u,0))\nabla T_k(u)\rho(x)dx + \varepsilon(n) \end{aligned}$$

where $\varepsilon(n)$ is a sequence of real numbers which converge to zero as n tends to infinity.

For $r \le s$ we deduce that

$$\begin{aligned} &\int_{\Omega_r} [a(x,u_n,\nabla T_k(u_n)) - a(x,u_n,\nabla T_k(u)\chi_s)][\nabla T_k(u_n) - \nabla T_k(u)\chi_s]\rho(x)dx \\ &\le \int_{\Omega\setminus\Omega_s} (h - a(x,u,0))\nabla T_k(u)\rho(x)dx + \varepsilon(n) \end{aligned} \tag{4.19}$$

By passing to Limit over n and letting $s \to \infty$ since $meas(\Omega \backslash \Omega_s) \to 0$ we get

$$\int_{\Omega_r} [a(x,u_n,\nabla T_k(u_n)) - a(x,u_n,\nabla T_k(u))][\nabla T_k(u_n) - \nabla T_k(u)]\rho(x)dx \to 0 \tag{4.20}$$

Passing to a subsequence, we have

$[a(x,u_n,\nabla T_k(u_n)) - a(x,u_n,\nabla T_k(u))][\nabla T_k(u_n) - \nabla T_k(u)]\rho(x) \to 0$ a.e inΩ_r.

For a subsequence still denote u_n, say for each $x \in \Omega \backslash R$ with $|R| = 0$. Fix $x \in \Omega/R$, one has by using (4.1) and (4.3)

$$\begin{aligned} &[a(x,u_n,\nabla T_k(u_n)) - a(x,u_n,\nabla T_k(u))][\nabla T_k(u_n) - \nabla T_k(u)]\rho(x) \geq \\ &\lambda_0 M(\lambda\,|\nabla T_k(u_n)|)\rho(x) - C_3\left[1 + |\nabla T_k(u_n)| + \overline{M}^{-1} M(\alpha_1\,|\nabla T_k(u_n)|)\right] + C_4 \end{aligned} \tag{4.21}$$

for some positives constants C_3 and C_4, which implies that $\nabla T_k(u_n)$ is bounded in $\mathbb{R}^N$.

Indeed suppose that there exists a subsequence denoted again $\nabla T_k(u_n(x)$ such that $\nabla T_k(u_n(x) \to \infty$ as $n \to \infty$. Writing (4.21) as the form

$$\begin{aligned} &[a(x,u_n,\nabla T_k(u_n)) - a(x,u_n,\nabla T_k(u))][\nabla T_k(u_n) - \nabla T_k(u)]\rho(x) \geq \\ &M(\lambda\,|\nabla u_n|)\left[\lambda_0\rho(x) - C_3\left(\frac{1 + |\nabla T_k(u_n)|}{M(\lambda\,|\nabla T_k(u_n)|)} + \frac{\overline{M}^{-1} M(c_1\,|\nabla T_k(u_n)|)}{M(\lambda\,|\nabla T_k(u_n)|)}\right)\right] + C_4 \end{aligned}$$

which gives the contradiction since the right hand-side converge to infinity while the left hand-side tends to zero as $n \to \infty$. Then, we have for a subsequence $u_{n_p}(x)$,$\nabla T_k(u_{n_p})(x) \to \xi \in \mathbb{R}^N$, and $[a(x,u_{n_p},\nabla T_k(u_{n_p})) - a(x,u_{n_p},\nabla T_k(u))][\nabla T_k(u_n) - \nabla T_k(u)]\rho(x)$, tend to $[a(x,u,\xi) - a(x,u,\nabla T_k(u))][\xi - \nabla T_k(u)]\rho(x)$ as $n_p \to \infty$. hence

$$[a(x,u,\xi) - a(x,u,\nabla T_k(u))][\xi - \nabla T_k(u)]\rho(x) = 0$$

consequently by (4.2) $\nabla T_k(u) = \xi$ and thus $\nabla T_k(u_n(x)) \to \nabla T_k(u(x))$ sice n and k are arbitrary we can construct a subsequence such that

$$\nabla u_n \to \nabla u \text{ a.e in } \Omega \tag{4.22}$$

Step 4. Passage to limit.

Going back to approximate problems (4.8) and taking $v \in D(\Omega))$ as test function we have:

$$\int_\Omega a(x,u,\nabla u)\nabla v\rho(x)dx + \int_\Omega a_0(x,u,\nabla u)v dx = \langle f, v\rangle$$

By (4.22) and (4.11) we deduce that $a(x,u_n,\nabla u_n) \to a(x,u,\nabla u)$ a.e in Ω and $a_0(x,u_n,\nabla u_n) \to a_0(x,u,\nabla(u)$ a.e in Ω moreover, Lemma 3.4 and Lemma 3.5 implies that $a(x,u_n,\nabla u_n \rightharpoonup a(x,u,\nabla u)$ weakly in $(L_M(\Omega,\rho))^N$ for $\sigma(\Pi L_{\overline{M}}(\Omega,\rho),\Pi E_M(\Omega,\rho))$, On the other hand, $f_n \to f$ strongly in $L^1(\Omega)$ and $a_0(x,u_n,\nabla u_n) \to a_0(x,u,\nabla u$ weakly in $(L_M(\Omega))$ for $\sigma(L_{\overline{M}}(\Omega),E_M(\Omega))$. On the other hand, $f_n \to f$ strongly in $L^1(\Omega)$ Finally by passing to the limit in the sequence of approximate problems, we obtain the existence result.

Conflict of Interests

The authors declare that there is no conflict of interests.

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