Available online at http://scik.org
J. Math. Comput. Sci. 4 (2014), No. 5, 826-833
ISSN: 1927-5307

A COINCIDENCE POINT THEREM FOR F-CONTRACTIONS ON METRIC SPACES EQUIPPED WITH AN ALTERED DISTANCE

RAKESH BATRA[1], SACHIN VASHISTHA[2], RAJESH KUMAR[2,*]

[1]Department of Mathematics, Hans Raj College, University of Delhi, Delhi-110007

[2]Department of Mathematics, Hindu College, University of Delhi, Delhi-110007

Abstract. A new type of contractive mapping known as an F_w-contraction has been introduced for a metric space equipped with a w-distance recently in 2013. In this paper we extend and generalize the concept of an F_w-contraction to an F_w-g-contraction and prove a coincidence point theorem for an F_w-g-contraction on a metric space equipped with a w-distance. Examples are given in support of usability of our results and to justify that our class of contractions is more generalized.

Keywords: fixed point; complete metric space; w-distance; F-contraction; F_w-contaction.

2010 AMS Subject Classification: 47H10, 54H25, 55M20.

1. Introduction

The theory of fixed points has been an important tool in non-linear analysis since 1930. It is widely used in disciplines such as chemistry, economics, physics, biology, engineering and applied mathematics. This is the basis for the modelling of a system. In dynamical systems it is used to prove several existence and stability results for the strict fixed points of a set-valued dynamic system F, as well as some conditions that guarantee each dynamic process

*Corresponding author

Received June 10, 2014

converges and its limit is a strict fixed point of F. In theoretical economics, such as general equilibrium theory, there comes at point where one needs to know whether the solution to a system of equations necessarily exists; or, more specifically, under which conditions will a solution necessarily exist. The mathematical analysis of this question usually relies on fixed point theorems. In engineering, fixed point technique has been used in areas like image retrieval and signal processing. In game theory it is used to estblish the existence of Nash equilibrium.

The studies of asymmetric structures and their applications in mathematics are important. One of the types of asymmetric structures on a metric space was introduced by Kada *et al.* [9] in 1996 known as a w-distance and he proved some fixed point theorems using it. Since then, many fixed point results have been deveoped by different authors using w-distance on metric spaces or a generalized w-distance such as c-distance on cone metric spaces. For more study in this area one may refer to [1, 2, 6–8]. In 2012, Wardowski [13] introduced the concept of F-contractive mapping on a metric space and proved a fixed point theorem for such a map on a complete metric space. Thereafter Batra *et al.* extended the fixed point result due to Wardowski by introducing an F_w-contraction which is the w-version of an F-contraction. In the present paper we extend the fixed point result due to Batra *et al.* by introducing an F_w-g-contraction which is the more general than an F_w-contraction. For more study on F-contractions one may refer to [3, 4, 11–14].

Throughout the article, denoted by $\mathbb{R}$ is the set of all real numbers, by $\mathbb{R}^+$ is the set of all positive real numbers and by $\mathbb{N}$ is the set of all natural numbers.(X,d), (X for short), is a metric space with a metric d. Let $T : X \to X$ and $g : X \to X$ be any two mappings. T and g are said to have a coincidence point at $x \in X$ if $Tx = gx$ and then gx is called a point of coincidence. Further, a point $x \in X$ is called a fixed point of T if $Tx = x$. For a survey of coincidence point theorey one may refer to [1, 2, 5, 10].

2. Preliminaries

Definition 2.1. [13] Let $F : \mathbb{R}^+ \to \mathbb{R}$ be a mapping satisfying

(F1) F is strictly increasing. That is $\alpha < \beta \Rightarrow F(\alpha) < F(\beta)$ for all $\alpha, \beta \in \mathbb{R}^+$.

(F2) For every sequence $\{\alpha_n\}$ in $\mathbb{R}^+$ we have $\lim_{n\to\infty} \alpha_n = 0$ if and only if $\lim_{n\to\infty} F(\alpha_n) = -\infty$.

(F3) There exists a number $k \in (0,1)$ such that $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.

Let (X,d) be a metric space. A mapping $T : X \to X$ is said to be an F-contraction if there exists a number $\tau > 0$ such that

$$\tau + F(d(Tx,Ty)) \leq F(d(x,y)) \tag{1}$$

for all $x,y \in X$ with $Tx \neq Ty$.

Remark 2.1. Clearly (1) of Definition 2.1. implies that $d(Tx,Ty) < d(x,y)$ for all $x,y \in X$ with $Tx \neq Ty$. Hence every F-contraction mapping is continuous.

Next we give the notation of w-distance of Kada *et al.* [9] with some properties.

Definition 2.2. [9] Let (X,d) be a metric space. A function $p : X \times X \to [0,\infty)$ is called a w-distance on X if the following conditions hold:

(w1) $p(x,z) \preceq p(x,y) + p(y,z)$ for all $x,y,z \in X$,

(w2) $p(x,.)$ is lower semi-continuous for all $x \in X$. That is, if $x \in X$ and $y_n \to y$ in X then $p(x,y) \leq \liminf p(x,y_n)$.

(w3) For all $\varepsilon > 0$, there exists $\delta > 0$ such that $p(z,x) \leq \delta$ and $p(z,y) \leq \delta$ imply $d(x,y) \leq \varepsilon$.

Example 2.1. Let $X = [0,\infty)$ and define a mapping $d : X \times X \to \mathbb{R}$ by $d(x,y) = |x-y|$ for all $x,y \in X$. Then (X,d) is a (complete) metric space. Define a mapping $p : X \times X \to \mathbb{R}$ by $p(x,y) = y$ for all $x,y \in X$. Then p is a w-distance on X.

Example 2.2. Let (X,d) be a metric space. Define a mapping $p : X \times X \to X$ by $p(x,y) = d(x,y)$ for all $x,y \in X$. Then, p is w-distance.

Lemma 2.1. [9] Let (X,d) be a metric space and p be a w-distance on X. Let$\{x_n\}$ and $\{y_n\}$ be sequences in X and $x,y,z \in X$. Suppose that u_n and v_n are sequences in $[0,\infty)$ converging to 0. Then the following hold:

(1) If $p(x_n,y) \preceq u_n$ and $p(x_n,z) \preceq v_n$,then $y = z$. In particular if $p(x,y) = 0$ and $p(x,z) = 0$ then $y = z$.

(2) If $p(x_n,y_n) \preceq u_n$ and $p(x_n,z) \preceq v_n$,then y_n converges to z.

(3) If $p(x_n,x_m) \preceq u_n$ for $m > n$, then $\{x_n\}$ is a Cauchy sequence in X.

(4) If $p(y,x_n) \preceq u_n$, then $\{x_n\}$ is a Cauchy sequence in X.

Remark 2.2.

(i) $p(x,y) = p(y,x)$ may not be true for all $x,y \in X$.

(ii) $p(x,y) = 0$ is not necessarily equivalent to $x = y$ for all $x,y \in X$.

Definition 2.3. [4] Let F be a mapping as defined in Definition 2.1. above. A mapping $T : X \to X$ is said to be an F_w-contraction if

(i) $p(x,y) = 0 \Rightarrow p(Tx,Ty) = 0$

(ii) There exists a number $\tau > 0$ such that $\tau + F(p(Tx,Ty)) \leq F(p(x,y))$ for all $x,y \in X$ with $p(Tx,Ty) > 0$.

Remark 2.3. Clearly, (ii) of Definition 2.3. implies that $p(Tx,Ty) < p(x,y)$ for all $x,y \in X$ with $p(Tx,Ty) > 0$.

Definition 2.4. Let F be a mapping as defined in Definition 2.1. above and $g : X \to X$ be a mapping. A mapping $T : X \to X$ is said to be an F_w-g-contraction if

(i) $p(gx,gy) = 0 \Rightarrow p(Tx,Ty) = 0$

(ii) There exists a number $\tau > 0$ such that $\tau + F(p(Tx,Ty)) \leq F(p(gx,gy))$ for all $x,y \in X$ with $p(Tx,Ty) > 0$.

Remark 2.4. Clearly, (ii) of Definition 2.4. implies that $p(Tx,Ty) < p(gx,gy)$ for all $x,y \in X$ with $p(Tx,Ty) > 0$.

Example 2.3. Define $F : \mathbb{R}^+ \to \mathbb{R}$ by $F(\alpha) = ln\,\alpha$. Then F satisfies $(F1)$, $(F2)$ and $(F3)$ (for all $k \in (0,1)$) of Definition 2.1. and $g : X \to X$ be a mapping. Then a mapping $T : X \to X$ satisfies

$$p(Tx,Ty) \leq \lambda p(gx,gy) \tag{2}$$

for all $x,y \in X$ and some $\lambda \in [0,1)$ if and only if T is an F_w-g-contraction. Let us start with a mapping $T : X \to X$ satisfying (2). If $\lambda = 0$ then (i) and (ii) in Definition 2.4. are vacuously satisfied. For $0 < \lambda < 1$,(i) is obvious and (ii) is satisfied for $\tau = ln\frac{1}{\lambda}$. Thus T is an F_w-g-contraction.

Conversely, if $T : X \to X$ is an F_w-g-contraction then (ii) of Definition 2.4. implies that $p(Tx,Ty) \leq e^{-\tau} p(gx,gy)$ for all $x,y \in X$ with $p(Tx,Ty) > 0$. Clearly it is satisfied even for $p(Tx,Ty) = 0$. Thus $p(Tx,Ty) \leq \lambda p(gx,gy)$ for all $x,y \in X$, where $\lambda = e^{-\tau} \in [0,1)$.

Example 2.4. Consider $G(\alpha) = ln\,\alpha + \alpha$ for all $\alpha > 0$. Then G satisfies $(F1)$, $(F2)$ and $(F3)$ of Definition 2.1. Let $g : X \to X$ be a mapping. A mapping $T : X \to X$ is an G_w-g-contraction if and only if

$$p(Tx,Ty)e^{p(Tx,Ty)-p(gx,gy)} \leq \lambda p(gx,gy) \tag{3}$$

for all $x, y \in X$ and some $\lambda \in [0,1)$. Reason is similar to above example.

Remark 2.5. From $(F1)$ of Definition 2.1. and (ii) of Definition 2.4. , it is clear that every F_w-g-contraction $T : X \to X$ satisfies $p(Tx,Ty) < p(gx,gy)$ for all $x, y \in X$ satisfying $p(Tx,Ty) > 0$.

Remark 2.6. Let F, $G : \mathbb{R}^+ \to \mathbb{R}$ be mappings satisfying $(F1)$, $(F2)$ and $(F3)$ of Definition 2.1. together with $F(\alpha) \leq G(\alpha)$ for all $\alpha > 0$. Let $H = G - F$ be nondecreasing. Then every F_w-g-contraction $T : X \to X$ is an G_w-g-contraction. Indeed for any $x, y \in X$ with $p(Tx,Ty) > 0$, we have, from Remark 2.5.

$$\begin{aligned} \tau + G(p(Tx,Ty)) &= \tau + F(p(Tx,Ty)) + H(p(Tx,Ty)) \\ &\leq F(p(gx,gy)) + H(p(gx,gy)) = G(p(gx,gy)). \end{aligned}$$

3. Main results

Theorem 3.1. *Let (X,d) be a metric space equipped with a w-distance p and $g : X \to X$ be a mapping. Let $T : X \to X$ be an F_w-g-contraction such that $T(X) \subseteq g(X)$. If either (X,d) is complete with T and g as continuous and commuting mappings on X or $g(X)$ is complete then g and T have a coincidence point $x^\star \in X$ with the unique point of coincidence $gx^\star$.*

Proof. For any two coincidence points x^* and y^* of T and g in X with $p(Tx^*,Ty^*) > 0$ we have $\tau \leq F(p(gx^*,gy^*)) - F(p(Tx^*,Ty^*)) = 0$. Thus $p(Tx^*,Ty^*) = p(gx^*,gy^*) = 0$ for any two coincidence points x^* and y^* of T and g in X. In particular $p(Tx^*,Tx^*) = p(gx^*,gx^*) = 0$. So by Lemma 2.1. (1) we obtain $gx^* = gy^*$ for any two coincidence points x^* and y^* of T and g in X. Hence point of coincidence of T and g if exists, is unique and satisfies $p(gx^*,gx^*) = 0$.

Now we show the existence of a coincidence point of T and g. Let $x_0 \in X$ be arbitrary. Define a sequence $\{gx_n\}$ in X by $gx_n = Tx_{n-1}$ for all $n \in \mathbb{N}$. Let $p_n = p(gx_{n-1},gx_n)$ for all $n \in \mathbb{N}$. If there exists $k \in \mathbb{N}$ with $p(gx_{k-1},gx_k) = 0$ then, by (i) of Definition 2.4. , $p(Tx_{k-1},Tx_k) = 0$.

That is $p(gx_k, gx_{k+1}) = 0$. Therefore $p(gx_{k-1}, gx_{k+1}) \leq p(gx_{k-1}, gx_k) + p(gx_k, gx_{k+1}) = 0$. By Lemma 2.1. (1) we have $gx_k = gx_{k+1}$ which implies $Tx_k = gx_k$ and the proof is finished.

Now assume that $p_n = p(gx_{n-1}, gx_n) > 0$ for all $n \in \mathbb{N}$. Then by (ii) of Definition 2.3. we get for all $n \geq 2$

$$F(p_n) \leq F(p_{n-1}) - \tau \leq F(p_{n-2}) - 2\tau \leq \cdots \leq F(p_1) - (n-1)\tau. \tag{4}$$

From (4) we get $\lim_{n\to\infty} F(p_n) = -\infty$ and then by $(F2)$ of Definition 2.1. we have

$$\lim_{n\to\infty} p_n = 0. \tag{5}$$

Now,by $(F3)$ of Definition 2.1. , there exists $k \in (0,1)$ such that

$$\lim_{n\to\infty} p_n^k F(p_n) = 0. \tag{6}$$

By (4), following holds for all $n \geq 2$

$$p_n^k F(p_n) - p_n^k (F(p_1) + \tau) = p_n^k (F(p_n) - F(p_1) - \tau) \leq -n p_n^k \tau. \tag{7}$$

Letting $n \to \infty$ in (7) and using (5) and (6) we have

$$\lim_{n\to\infty} n p_n^k = 0. \tag{8}$$

By (8) there exists a positive integer n_0 such that $np_n^k < 1$ for all $n \geq n_0$. Consequently

$$p_n < \frac{1}{n^{\frac{1}{k}}} \tag{9}$$

for all $n \geq n_0$. Since the series $\sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{k}}}$ is convergent, therefore, by (9), the series $\sum_{n=1}^{\infty} p_n$ is also convergent. Now for any $m > n$ we have

$$p(gx_n, gx_m) \leq p_{n+1} + p_{n+2} + \cdots + p_m < \alpha_n, \tag{10}$$

where $\alpha_n = \sum_{i=n+1}^{\infty} p_i \to 0$ as $n \to \infty$. Thus by Lemma 2.1. (3) $\{gx_n\}$ is a Cauchy sequence in X. Consider the first situation where (X,d) is complete and the mappings g and T both are continuous and commuting. Then there exists $x^* \in X$ such that $lim_{n\to\infty} gx_n = x^*$. Finally, continuity and commutativity of T and g yield $Tx^* = T(lim_{n\to\infty} gx_n) = lim_{n\to\infty} Tgx_n = lim_{n\to\infty} gTx_n = g(\lim_{n\to\infty} Tx_n) = g(lim_{n\to\infty} gx_{n+1}) = gx^*$. That is, T and g have a coincidence point at x^*. In the second situation $g(X)$ is complete. So there exists $x^* \in X$ such that $\lim_{n\to\infty} gx_n = gx^*$. From (10) and (ii) of Definition 2.2. we get

$$p(gx_n, gx^*) \leq \alpha_n. \tag{11}$$

Now for $p(Tx_{n-1}, Tx^*) > 0$, by Remark 2.5. and by (11)

$$\begin{aligned} p(gx_n, Tx^*) &= p(Tx_{n-1}, Tx^*) \\ &< p(gx_{n-1}, gx^*) \\ &\leq \alpha_{n-1}. \end{aligned} \tag{12}$$

Clearly (12) is satisfied even for $p(Tx_{n-1}, Tx^*) = 0$. Thus

$$p(gx_n, Tx^*) \leq \alpha_{n-1} \quad for \quad all \quad n \in \mathbb{N}. \tag{13}$$

From (11), (13) and by using Lemma 2.1. (1) we get $Tx^* = gx^*$.

Example 3.2. *Let $X = [0, \infty)$, $d(x,y) = |x-y|$ for all $x, y \in X$ and $p(x,y) = y$ for all $x, y \in X$. Then (X,d) is a complete metric space and p is a w-distance on X. Define $T : X \to X$ by*

$$Tx = \begin{cases} x^2 \text{ if } 0 \leq x \leq 1, \\ 0 \text{ otherwise} \end{cases} \text{ and } gx = \begin{cases} 2x \text{ if } 0 \leq x \leq 1, \\ 0 \text{ otherwise.} \end{cases}$$

Since T is not continuous, therefore it is not an F-contraction for any mapping F as described in Definition 2.1. Now consider the mapping F as described in Example 2.3. Then T is not an F-g-contraction on X as

$$\frac{d(T(1-\frac{1}{n}), T(1-\frac{1}{m}))}{d(g(1-\frac{1}{n}), g(1-\frac{1}{m}))} = 1 - \frac{1}{2n} - \frac{1}{2m} \to 1$$

as $m, n \to \infty$. Further, T is not even an F_w-contraction for we have $\frac{p(Tx,T1)}{p(x,1)} = 1$. We note that $p(Tx,Ty) = Ty > 0$ if and only if $0 < y \leq 1$. For $x, y \in X$ with $0 < y \leq 1$ we have

$$\frac{p(Tx,Ty)}{p(gx,gy)} = \frac{Ty}{gy} = \frac{y^2}{2y} = \frac{y}{2} \leq \frac{1}{2}.$$

So p satisfies (2) for all $x, y \in X$ and for $\lambda = \frac{1}{2}$. Thus T is an F_w-g-contraction. Next we see that $T(X) = [0,1] \sqsubseteq [0,2] = g(X)$ and $g(X)$ is complete. So all the conditions mentioned in Theorem 3.1. are satisfied. We note that T and g have a coincidence point at 0 and at any number greater *than 1 with 0 as the unique point of coincidence. Also* $p(gx, gx) = 0$ *for any coincidence point* x *of* T *and* g.

Conflict of Interests

The authors declare that there is no conflict of interests.

REFERENCES

[1] R. Batra, S. Vashistha, Coupled coincidence point theorems for nonlinear contractions under c-distance in cone metric spaces, Ann. Funct. Anal. 4 (2013), 138-148.

[2] R. Batra, S. Vashistha, Coupled coincidence point theorems for nonlinear contractions under (F, g)-invariant set in cone metric spaces, J. Nonlinear Sci. Appl. 6 (2013), 86-96.

[3] R. Batra, S. Vashistha, Fixed points of an F-contraction on metric spaces with a graph, Int. J. Comput. Math. (2014), doi: 10.1080/00207160.2014.887700.

[4] R. Batra, S. Vashistha, Fixed point theorem for F_w-contractions in complete metric spaces, J. Nonlinear Anal. Appl. 2013 (2013), doi: 10.5899/2013/jnaa-00211.

[5] T. G. Bhaskar, V. Lakshmikantham, Fixed point theorems in partially ordered metric spaces and applications, Nonlinear Anal. 65 (2006), 1379-1393.

[6] Z.M. Fadail, A.G.B. Ahmad, Common coupled fixed point theorems of single valued mapping for c-distance in cone metric spaces, Abstr. Appl. Anal. 2012 (2012), Article ID 901792.

[7] Z.M. Fadail, A.G.B. Ahmad, Coupled fixed point theorems of single valued mapping for c-distance in cone metric spaces, J. Appl. Math. 2012 (2012), Article ID 246516.

[8] Z.M. Fadail, A.G.B. Ahmad, New fixed point results of single valued mapping for c-distance in cone metric spaces, Abstr. Appl. Anal. 2012 (2012), Article ID 639713.

[9] O. Kada, T. Suzuki, W. Takahashi, Nonconvex minimization theorems and fixed point theorems in complete metric spaces, Math. Japon. 44 (1996), 381-391.

[10] V. Lakshmikantham, L. Ćirić, Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces, Nonlinear Anal. 70 (2009), 4341-4349.

[11] M. Sgroi, C. Vetro, Multi-valued F-contractions and the solution of certain mappingal and integral equations, Filomat 27 (2013), 1259-1268.

[12] S. Shukla, S. Radenović, Some common fixed Point theorems for F-contraction type mappings in 0-complete partial metric spaces, J. Math. 2013 (2013), Article ID 878730.

[13] D. Wardowski, Fixed points of a new type of contractive mappings in complete metric spaces, Fixed Point Theory Appl. 2012 (2012), Article ID 94.

[14] D. Wardowski, N. V. Dung, Fixed Points Of F-Weak Contractions On Complete Metric Spaces. Demonstr. Math. 47 (2014), 146-155.