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SOME VECTOR VALUED MULTIPLIER DIFFERENCE DOUBLE SEQUENCE SPACES IN 2-NORMED SPACES DEFINED BY ORLICZ FUNCTIONS

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Abstract. In this paper, we study certain new difference double sequence spaces using an Orlicz function, a bounded sequence of positive real numbers and a sequence in 2-normed space and we give some relations related to these sequence spaces.

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1. Introduction

Let w, l_∞, c and c_0 denote the spaces of all, bounded, convergent and null sequences $x = (x_k)$ with complex terms, respectively normed by

$$\|x\| = \sup_k |x_k|.$$

Kizmaz [20], defined the difference sequences $l_\infty(\Delta), c(\Delta)$ and $c_0(\Delta)$ as follows:

$$Z(\Delta) = \{x = (x_k) : (\Delta x_k) \in Z\},$$

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for $Z = l_\infty, c$ and c_0, where $\Delta x = (\Delta x_k) = (x_k - x_{k+1})$, for all $k \in \mathbb{N}$.

The above spaces are Banach spaces, normed by

$$\|x\|_\Delta = |x_1| + \sup_k \|\Delta x_k\|.$$

The notion of difference sequence spaces was generalized by Et and Colak[2] as follows:

$$Z(\Delta^n) = \{x = (x_k) : (\Delta^n x_k) \in Z\},$$

for $Z = l_\infty, c$ and c_0, where $n \in \mathbb{N}, (\Delta^n x_k) = (\Delta^{n-1} x_k - \Delta^{n-1} x_{k+1})$ and so that

$$\Delta^n x_k = \sum_{v=0}^{n} (-1)^v \begin{pmatrix} n \\ v \end{pmatrix} x_{k+v}.$$

In 2005, Tripathy and Esi [25], introduced the following new type of difference sequence spaces:

$$Z(\Delta_m) = \{x = (x_k) \in w : \Delta_m x \in Z\}, \text{ for } Z = l_\infty, c \text{ and } c_0$$

where $\Delta_m x = (\Delta_m x_k) = (x_k - x_{k+m})$, for all $k \in \mathbb{N}$.

Later on Tripathy, Esi and Tripathy [26], generalized the above notions and unified them as follows:

Let m, n be non negative integers, then for Z a given sequence space we have

$$Z(\Delta_m^n) = \{x = (x_k) \in w : (\Delta_m^n x_k) \in Z\}$$

where

$$\Delta_m^n x_k = \sum_{v=0}^{n} (-1)^v \begin{pmatrix} n \\ v \end{pmatrix} x_{k+mv}.$$

Taking $m = 1$, we get the spaces $l_\infty(\Delta^n), c(\Delta^n)$ and $c_0(\Delta^n)$ studied by Et and Colak [2]. Taking $n = 1$, we get the spaces $l_\infty(\Delta_m), c(\Delta_m)$ and $c_0(\Delta_m)$ studied by Tripathy and Esi [25]. Taking $m = n = 1$, we get the spaces $l_\infty(\Delta), c(\Delta)$ and $c_0(\Delta)$ introduced and studied by Kizmaz [20]. Difference sequence spaces have been studied by Cigdem and Rifat[1] and V.A.Khan [14 , 15, 16, 17, 18,19] and many others.

Let $\Lambda = (\lambda_k)$ be a sequence of non-zero scalars. Then for E a sequence space, the multiplier sequence $E(\Lambda)$, associated with the multiplier sequence Λ is defined as

$$E(\Delta) = \{(x_k) \in w : (\lambda_k x_k) \in E\}.$$

The concept of 2-normed spaces was initially introduced by Gahler[3,4,5] in the mid of 1960's. Since then, many researchers have studied this concept and obtained various results, see for instance [6].

Let X be a real vector space of dimension d, where $2 \leq d \leq \infty$. A 2-norm on X is a function $\|.,.\| : X \times X \to R^+$ which satisfies the following four conditions:

(1) $\|x_1, x_2\| = 0$ if and only if x_1, x_2 are linearly dependent;
(2) $\|x_1, x_2\| = \|x_2, x_1\|$:
(3) $\|\alpha x_1, x_2\| = \alpha\|x_1, x_2\|$, for any $\alpha \in R^+$:
(4) $\|x + x', x_2\| \leq \|x, x_2\| + \|x', x_2\|$

The pair $(X, \|.,.\|)$ is then called a 2-normed space.

Example 1.1. A standard example of a 2-normed space is R^2 equipped with the following 2-norm
$\|x, y\| :=$ the area of the triangle having vertices $0, x, y$.

Example 1.2. Take $X = R^2$ and consider the function $\|.,.\|$ on X defined as:

$$\|x_1, x_2\| = \text{abs}\left(\begin{vmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{vmatrix}\right).$$

The concept of paranorm is closely related to linear metric spaces. Let X be a linear space. A *paranorm* is a function $g : X \to \mathbb{R}$ which satisfies the following axioms: for any $x, y, x_0 \in X$, $\lambda, \lambda_0 \in \mathbb{C}$,

(i) $g(\theta) = 0$ (where $\theta = (0, 0, \cdots, 0, \cdots)$ is zero of the space);

(ii) $g(x) = g(-x)$;

(iii) $g(x + y) \leq g(x) + g(y)$;

(iv) the scalar multiplication is continuous, that is $\lambda \to \lambda_0$, $x \to x_0$ imply $\lambda x \to \lambda_0 x_0$.

Any function g which satisfies all the condition (i)-(iv) together with the condition

(v) $g(x) = 0$ if only if $x = \theta$,

is called a *total paranorm* on X and the pair (X, g) is called *total paranormed space.* It is well known that the metric of any linear metric space is given by some total paranorm (cf.[27],Theorm 10.42,p183])

An *Orlicz Function* is a function $M : [0, \infty) \to [0, \infty)$ which is continuous, nondecreasing and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \to \infty$, as $x \to \infty$.

If convexity of M is replaced by $M(x + y) \leq M(x) + M(y)$, then it is called a *modulus funtion* .

J. Lindenstrauss and L. Tzafriri [21] used the idea of an Orlicz sequence space;

$$l_M := \left\{ x \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \text{ for some } \rho > 0 \right\}$$

which is a Banach space with the norm

$$\|x\|_M = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\}.$$

The space l_M is closely related to the space l_p, which is an Orlicz sequence space with $M(x) = x^p$ for $1 \leq p < \infty$.

An Orlicz function M satisfies the $\Delta_2 -$ *condition* ($M \in \Delta_2$ for short) if there exist constant $K \geq 2$ and $u_0 > 0$ such that

$$M(2u) \leq KM(u)$$

whenever $|u| \leq u_0$.

Note that an Orlicz function satisfies the inequality

$$M(\lambda x) \leq \lambda M(x) \text{ for all } \lambda \text{ with } 0 < \lambda < 1.$$

Orlicz functions have been studied by V.A.Khan[7,8,9,10,11] and many others.

Throughout, a double sequence $x = (x_{kl})$ is a doubly infinite array of elements x_{kl}. for $k, l \in \mathbb{N}$. Double sequences have been studied by V.A.Khan[12,13], Mursaleen and Osama H.H.Edely [24], Moricz and Rhoades[23] and many others.

A double sequence (x_{jk}) in 2-normed space $(X, \|.,.\|)$ is said to converge to some $L \in X$ in the 2-norm, if

$$\lim_{j,k\to\infty} \|x_{jk} - L, u_1\| = 0, \text{ for every } u_1 \in X.$$

A sequence (x_{jk}) in a 2-normed space $(X, \|., \|)$ is said to be Cauchy with respect to the 2-norm if

$$\lim_{j,p\to\infty} \|x_{jk} - x_{pq}, u_1\| = 0 \rightarrow \text{ for every } u_1 \in X \text{ and } k, q \in \mathbb{N}.$$

If every Cauchy sequence in X converges to some $L \in X$, then X is said to be complete with respect to the 2-norm. Any complete 2-normed space is said to be a 2-Banach space.

Example 1.3. Let w be the linear space of all double sequences of real numbers. For $x = (x_{jk})$, $y = (y_{jk})$ in w, let us define

$$\|x, y\| = 0, \text{ if x,y are linearly dependent,}$$

$$\|x, y\| = \sum_{j,k=1}^{\infty} |x_{jk}||y_{jk}|, \text{ if x,y are linearly independent.}$$

Then it is obvious that $\|.,.\|$ is a 2-norm on w.

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The following inequalities will be used throughout the paper. Let $p = (p_{k,l})$ be a double sequence of positive real numbers with $0 < p_{k,l} \leq \sup_{k,l} p_{k,l} = H$ and let $D = \max\{1, 2^{H-1}\}$. Then for the factorable sequences $\{a_k\}$ and $\{b_k\}$ in the complex plane , we have as in Maddox [22]

$$|a_{k,l} + b_{k,l}|^{p_{k,l}} \leq D\{|a_{k,l}|^{p_{k,l}} + |b_{k,l}|^{p_{k,l}}\}. \qquad [1.1]$$

2. Main results

Let $p = (p_{jk})$ be any bounded sequence of positive numbers and $\Lambda = (\lambda_{jk})$ be a sequence of non-zero reals. Let m, n be non-negative integers, then for a real linear 2-normed space $(X, \|.,.\|)$ and an Orlicz function M we define the following sequence spaces:

$$ {}_2c_0(M, \|.,.\|, \Delta_m^n, \Lambda, p) = \Big\{ x = (x_{jk}) \in w(X) : \lim_{j,k\to\infty} \Big(M\Big(\Big\| \frac{\Delta_m^n \lambda_{jk} x_{jk}}{\rho}, z \Big\| \Big) \Big)^{p_{jk}} = 0, \text{ for every } z \text{ in } X \text{ and for some } \rho > 0 \Big\}, $$

$$ {}_2c(M, \|.,.\|, \Delta_m^n, \Lambda, p) = \Big\{ x = (x_{jk}) \in w(X) : \lim_{j,k\to\infty} \Big(M\Big(\Big\| \frac{\Delta_m^n \lambda_{jk} x_{jk} - L}{\rho}, z \Big\| \Big) \Big)^{p_{jk}} = 0, \text{ for every } z \text{ in } X \text{ and for some } \rho > 0 \text{ and } L \in X \Big\}, $$

$$ {}_2l_\infty(M, \|.,.\|, \Delta_m^n, \Lambda, p) = \Big\{ x = (x_{jk}) \in w(X) : \sup_{j,k\geq 1} \Big(M\Big(\Big\| \frac{\Delta_m^n \lambda_{jk} x_{jk}}{\rho}, z \Big\| \Big) \Big)^{p_{jk}} < \infty, \text{ for every } z \text{ in } X \text{ and for some } \rho > 0 \Big\}, $$

where $(\Delta_m^n \lambda_{jk} x_{jk}) = (\Delta_m^{n-1} \lambda_{jk} x_{jk} - \Delta_m^{n-1} \lambda_{j+1,k} x_{j+1,k} - \Delta_m^{n-1} \lambda_{j,k+1} x_{j,k+1} + \Delta_m^{n-1} \lambda_{j+1,k+1} x_{j+1,k+1})$ and $(\Delta_m^0 \lambda_{jk} x_{jk}) = \lambda_{jk} x_{jk}$ for all $j, k \in N$, which is equivalent to the following binomial representation:

$$\Delta_m^n \lambda_{jk} x_{jk} = \sum_{s=0}^{r} \sum_{v=0}^{n} (-1)^{s+v} \binom{r}{s} \binom{n}{v} \lambda_{j+mv,k+mv} x_{j+mv,k+mv}$$

.

Theorem 2.1. The sets of sequences ${}_2c_0(M, \|.,.\|, \Delta_m^n, \Lambda, p), {}_2c(M, \|.,.\|, \Delta_m^n, \Lambda, p)$ and ${}_2l_\infty(M, \|.,.\|, \Delta_m^n, \Lambda, p)$ are linear spaces over the field $\mathbb{C}$, complex numbers.

Proof. Let $x = (x_{jk})$ and $y = (y_{jk}) \in {}_2l_\infty(M, \|.,.\|, \Delta_m^n, \Lambda, p)$ and $\alpha, \beta \in \mathbb{C}$. Then there exist some positive numbers ρ_1 and ρ_2 such that

$$\sup_{j,k\geq 1}\left(M\left(\left\|\frac{\Delta_m^n \lambda_{jk} x_{jk}}{\rho_1}, z\right\|\right)\right)^{p_{jk}} < \infty,$$

and

$$\sup_{j,k\geq 1}\left(M\left(\left\|\frac{\Delta_m^n \lambda_{jk} y_{jk}}{\rho_2}, z\right\|\right)\right)^{p_{jk}} < \infty.$$

Define $\rho_3 = \max(2|\alpha|\rho_1, 2|\beta|\rho_2)$

$$\sup_{j,k\geq 1}\left(M\left(\left\|\frac{\Delta_m^n \lambda_{jk} x_{jk} + \Delta_m^n \lambda_{jk} y_{jk}}{\rho_3}, z\right\|\right)\right)^{p_{jk}}$$

$$\leq D \sup_{j,k\geq 1}\left(M\left(\left\|\frac{\Delta_m^n \lambda_{jk} x_{jk}}{\rho_1}, z\right\|\right)\right)^{p_{jk}} + D \sup_{J,k\geq 1}\left(M\left(\left\|\frac{\Delta_m^n \lambda_{jk} y_{jk}}{\rho_2}, z\right\|\right)\right)^{p_{jk}} < \infty.$$

Since M is non decreasing convex function using (4) property of $(X.\|.,.\|)$.

This proves that ${}_2l_\infty(M, \|.,.\|, \Delta_m^n, \Lambda, p)$ is a linear space. A similar proof works for ${}_2c$ and ${}_2c_0$.

Theorem 2.2. For $Z = {}_2l_\infty, {}_2c$ and ${}_2c_0$, the spaces $Z(M, \|.,.\|, \Delta_m^n, \Lambda, p)$ are paranormed by

$$g(x) = \inf\left\{\rho^{\frac{p_{jk}}{H}} : \sup_{j,k\geq 1} M\left(\left\|\frac{\Delta_m^n \lambda_{jk} x_{jk}}{\rho}, z\right\|\right) \leq 1\right\}$$

where $H = \max(1, \sup_{j,k\geq 1} p_{jk})$.

Proof. Clearly $g(x) = g(-x)$, $x = \theta$ imply that $g(\theta) = 0$.

Let $x = (x_{jk}), y = (y_{jk}) \in {}_2c_0(M, \|.,.\|, \Delta_m^n, \Lambda, p)$. Then there exist $\rho_1, \rho_2 > 0$ such that

$$\sup_{j,k\geq 1} M\left(\left\|\frac{\Delta_m^n \lambda_{jk} x_{jk}}{\rho_1}, z\right\|\right) \leq 1$$

and

$$\sup_{j,k\geq 1} M\left(\left\|\frac{\Delta_m^n \lambda_{jk} y_{jk}}{\rho_2}, z\right\|\right) \leq 1.$$

Let $\rho = \rho_1 + \rho_2$. Then by convexity of Orlicz functions, we have

$$\sup_{j,k\geq 1}\left(M\left(\left\|\frac{\Delta_m^n \lambda_{jk} x_{jk} + \Delta_m^n \lambda_{jk} y_{jk}}{\rho}, z\right\|\right)\right)$$

$$\leq \left(\frac{\rho_1}{\rho_1+\rho_2}\right)\sup_{j,k\geq 1} M\left(\left\|\frac{\Delta_m^n \lambda_{jk} x_{jk}}{\rho_1}, z\right\|\right)$$

$$+\left(\frac{\rho_2}{\rho_1+\rho_2}\right)\sup_{j,k\geq 1} M\left(\left\|\frac{\Delta_m^n \lambda_{jk} y_{jk}}{\rho_2}, z\right\|\right).$$

Thus we have

$$g(x+y) \leq \inf\left\{\rho^{\frac{p_{jk}}{H}} : \sup_{j,k\geq 1} M\left(\left\|\frac{\Delta_m^n \lambda_{jk} x_{jk}}{\rho_1}, z\right\|\right) \leq 1\right\}$$

$$+\inf\left\{\rho^{\frac{p_{jk}}{H}} : \sup_{j,k\geq 1} M\left(\left\|\frac{\Delta_m^n \lambda_{jk} x_{jk}}{\rho_2}, z\right\|\right) \leq 1\right\}.$$

This implies that $g(x+y) \leq g(x) + g(y)$.

The continuity of the scalar multiplication follows from the following :

$$g(\alpha x) = \inf\left\{\rho^{\frac{p_{jk}}{H}} : \sup_{j,k\geq 1} M\left(\left\|\frac{\Delta_m^n \alpha \lambda_{jk} x_{jk}}{\rho}, z\right\|\right) \leq 1\right\}.$$

Theorem 2.3. If X is a 2- Banach space, then the spaces $Z(M, \|.,.\|, \Delta_m^n, \Lambda, p), Z = {}_2l_\infty, {}_2c$ and ${}_2c_0$ are complete paranormed spaces.

Proof. We prove the theorem for ${}_2l_\infty(M, \|.,.\|, \Delta_m^n, \Lambda, p)$ and the proof for the other cases can be established following similar techniques.

Let $x = (x_{jk}^i)$ be a Cauchy sequence in ${}_2l_\infty(M, \|.,.\|, \Delta_m^n, \Lambda, p)$ and let $\epsilon > 0$ be given. For a fixed $x_0 > 0$, choose $r > 0$ such that $M(\frac{rx_0}{3}) \geq 1$ and $m_0 \in \mathbb{N}$ be such that

$$g((x_{jk}^i - x_{jk}^{i'})) < \frac{\epsilon}{x_0}, \quad \text{for all } i, i' \geq m_0.$$

By the definition of g we have,

$$\inf \left\{ \rho^{\frac{p_{jk}}{H}} : \sup_{j,k \geq 1} M\left(\left\| \frac{\Delta_m^n \lambda_{jk}(x_{jk}^i - x_{jk}^{i'})}{\rho}, z \right\| \right) \leq 1 \right\} \text{ for all } i, i' \geq m_0.$$

Then we get,

$$\sup_{j,k \geq 1} M\left(\left\| \frac{\Delta_m^n \lambda_{jk}(x_{jk}^i - x_{jk}^{i'})}{g((x_{jk}^i - x_{jk}^{i'}))}, z \right\| \right) \leq 1 \leq M(\frac{rx_0}{3}) \text{ for all } i, i' \geq m_0.$$

This implies that

$$\left\| \Delta_m^n \lambda_{jk} x_{jk}^i - \Delta_m^n \lambda_{jk} x_{jk}^{i'}, z \right\| \leq \left(\frac{rx_0}{3} \right) g((x_{jk}^i - x_{jk}^{i'})) \leq \frac{rx_0}{3} \frac{\epsilon}{rx_0} = \frac{\epsilon}{3} \text{ for all } i, i' \geq m_0$$

for every Z in X.

Hence (x_{jk}^i) is a Cauchy sequence in the 2-Banach space X for all $(j, k) \in N \times N$.

Since X is complete this implies that $(\Delta_m^n \lambda_{jk} x_{jk})$, is convergent in X for all $j, k \in N$.

For simplicity, let $\lim_{i \to \infty} \Delta_m^n \lambda_{jk} x_{jk^i} = y_{jk}$ for $(j, k) \in N \times N$.

Let $j = 1$

$$\lim_{i \to \infty} \Delta_m^n \lambda_{jk} x_{jk}^i = \lim_{i \to \infty} \sum_{v=0}^{n} (-1)^v \begin{pmatrix} n \\ v \end{pmatrix} \lambda_{1+mv,k+mv} x_{1+mv,k+mv}^i$$

.

$$= \lim_{i \to \infty} \Delta_m^n \lambda_{1k} x_{1k}^i = y_{1k} \qquad [2.3.1]$$

Let $k = 1$

$$\lim_{i \to \infty} \Delta_m^n \lambda_{j1} x_{j1}^i = \lim_{i \to \infty} \sum_{v=0}^{n} (-1)^v \begin{pmatrix} n \\ v \end{pmatrix} \lambda_{j+mv,1+mv} x_{j+mv,1+mv}^i$$

.

$$= \lim_{i\to\infty} \Delta_m^n \lambda_{j1} x_{j1}^i = y_{j1}. \qquad [2.3.2]$$

Similarly we have

$$\lim_{i\to\infty} \Delta_m^n \lambda_{jk} x_{jk}^i = \lim_{i\to\infty} \lambda_{jk} x_{jk}^i = y_{jk} \text{ for } j,k \in \mathbb{N} \qquad [2.3.3]$$

Thus from [2.3.1], [2.3.2] and [2.3.3] we have

$$\lim_{i\to\infty} x_{jk}^i = x_{jk} \;\text{ exists for all } j,k \in \mathbb{N}.$$

Now we have for all $i \geq m_0$

$$\inf\left\{\rho^{\frac{p_{jk}}{H}} : \sup_{j,k\geq 1} M\left(\left\|\frac{\Delta_m^n \lambda_{jk}(x_{jk}^i - x_{jk}^{i'})}{\rho}, z\right\|\right) \leq 1\right\} < \epsilon.$$

This implies that

$$\lim_{j,k\to\infty} \inf\left\{\rho^{\frac{p_{jk}}{H}} : \sup_{j,k\geq 1} M\left(\left\|\frac{\Delta_m^n \lambda_{jk}(x_{jk}^i - \Delta_m^n \lambda_{jk} x_{jk})}{\rho}, z\right\|\right) \leq 1\right\} < \epsilon \text{ for all } i \geq m_0$$

Hence $(x^i - x) \in {}_2l_\infty(M, \|.,.\|, \Delta_m^n, \Lambda, p)$.

Since $(x^i) \in {}_2l_\infty(M, \|.,.\|, \Delta_m^n, \Lambda, p)$ and ${}_2l_\infty(M, \|.,.\|, \Delta_m^n, \Lambda, p)$ is a linear space, so we have $x = x^i - (x^i - x) \in {}_2l_\infty(M, \|.,.\|, \Delta_m^n, \Lambda, p)$.

Theorem 2.4. If $0 < p_{jk} \leq q_{jk} < \infty$ for each j,k, then $Z(M, \|.,.\|, \Delta_m^n, \Lambda, p) \subseteq Z(M, \|.,.\|, \Delta_m^n, \Lambda, q)$ for $Z = {}_2c_0$ and ${}_2c$.

Proof. We prove the theorem for $Z = {}_2c_0$ and the proof for other cases can be established following similar techniques.

Let $x = (x_{jk}) \in {}_2c_0(M, \|.,.\|, \Delta_m^n, \Lambda, p)$. Then there exists some $\rho > 0$ such that

$$\lim_{j,k\to\infty} \left(M\left(\left\|\frac{\Delta_m^n \lambda_{jk} x_{jk}}{\rho}, z\right\|\right)\right)^{p_{jk}} = 0.$$

This implies that

$$\lim_{j,k\to\infty}\left(M\left(\left\|\frac{\Delta_m^n\lambda_{jk}x_{jk}}{\rho},z\right\|\right)\right)^{p_{jk}} < \epsilon \ (0<\epsilon\leq 1)$$

for sufficiently large j,k.

Hence we get

$$\lim_{j,k\to\infty}\left(M\left(\left\|\frac{\Delta_m^n\lambda_{jk}x_{jk}}{\rho},z\right\|\right)\right)^{q_{jk}} \leq \lim_{j,k\to\infty}\left(M\left(\left\|\frac{\Delta_m^n\lambda_{jk}x_{jk}}{\rho},z\right\|\right)\right)^{p_{jk}} = 0.$$

This implies that

$x=(x_{jk})\in {}_2c_0(M,\|.,.\|,\Delta_m^n,\Lambda,p)$.

This completes the proof.

Corollary 2.5(a). If $0<\inf p_{jk}$ and for each j and k , $p_{jk}\leq 1$, then $Z(M,\|.,.\|,\Delta_m^n,\Lambda,p)\subseteq Z(M,\|.,.\|,\Delta_m^n,\Lambda)$ for $Z={}_2c_0$ and ${}_2c$.

(b). If $0<\inf p_{jk}$ and for each j and k , $p_{jk}\leq 1$, then $Z(M,\|.,.\|,\Delta_m^n,\Lambda)\subseteq Z(M,\|.,.\|,\Delta_m^n,\Lambda,p)$ for $Z={}_2c_0$ and ${}_2c$.

Theorem 2.6. $Z(M,\|.,.\|,\Delta_m^{n-1},\Lambda,p)\subset Z(M,\|.,.\|,\Delta_m^n,\Lambda,p)$, for $i=1,2,3,.....,n-1$ for $Z={}_2l_\infty, {}_2c_0$ and ${}_2c$.

Proof. That the inclusion is strict follows from the following example:

Example 2.7. Let $m=3,n=2,M(x)=x^{10}$ and $x\in[0,\infty)$ and

$$p_{jk}=\begin{cases}3 & \text{for } j \text{ odd and all } k\in N,\\ 2 & \text{otherwise .}\end{cases}$$

Consider the 2-normed space as defined in Example[1.3] and let $\Lambda=\left(\frac{1}{j+k}\right)$ and $x=(x_{jk})=((j+k)^2,(j+k)^2)$.

Then

$$\Delta_3^2 \lambda_{jk} x_{jk} = \sum_{v=0}^{2} (-1)^v \begin{pmatrix} n \\ v \end{pmatrix} \lambda_{j+3v,k+3v} x^i_{j+3v,k+3v}$$

.

$= \lambda_{jk} x_{jk} - 2\lambda_{j+3,k+3} x_{j+3,k+3} + \lambda_{j+6,k+6} x_{j+6,k+6}$

$= \frac{1}{j+k}((j+k)^2, (j+k)^2) - 2\frac{1}{j+k+6}((j+k+6)^2, (j+k+6)^2) + \frac{1}{j+k+12}((j+k+12)^2, (j+k+12)^2)$

$= (j+k, j+k) - 2(j+k+6, j+k+6) + (j+k+12, j+k+12)$

$= \theta$ for all $j, k \in \mathbb{N}$.

Hence $x \in {}_2c_0(M, \|.,.\|, \Delta_3^2, \Lambda, p)$.

Again we have

$$\Delta_3^1 \lambda_{jk} x_{jk} = \sum_{v=0}^{1} (-1)^v \begin{pmatrix} n \\ v \end{pmatrix} \lambda_{j+3v,k+3v} x^i_{j+3v,k+3v}$$

.

$$= \lambda_{jk} x_{jk} - \lambda_{j+3,k+3} x_{j+3,k+3}$$

$$= (j+k, j+k) - (j+k+3, j+k+3)$$

$$= (-3, -3) \text{ for all } j, k \in \mathbb{N}.$$

Hence $x \notin {}_2c_0(M, \|.,.\|, \Delta_3^1, \Lambda, p)$.

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