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SOME PROPERTIES OF ANALYTIC FUNCTIONS DEFINED BY A NEW GENERALIZED MULTIPLIER TRANSFORMATION

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Abstract. The object of the present paper is to derive some properties of analytic functions in the open unit disc which are defined by using new generalized multiplier transformations**,** applying a lemma due to Miller and Mocanu.

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1. INTRODUCTION

Let $A(p,n)$ denote the class of functions f(z) of the form $f(z) = z^p + \sum_{j=p+n}^{\infty} a_j z^j$ $p,n \in N = \{1, 2, 3\ldots\}$,which are analytic in the open unit disc $U = \{z : z \in C, |z| < 1\}$. In particular, we set $A(p,1) = A_p$, $A(1,n) = A(n)$ and $A(1,1) = A = A_1 = A(1)$, which are well known classes of analytic functions in U.

We consider the following new generalized multiplier transformation.

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Definition 1.1[17]. Let $f(z) \in A(p,n)$.The new generalized multiplier transformation $I^{\delta}_{p,\alpha,\beta}$ on $A(p,n)$ is defined by the following infinite series:

(1.1) $$I^{\delta}_{p,\alpha,\beta} f(z) = z^{p} + \sum_{j=p+n}^{\infty} \left(\frac{\alpha + k\beta}{\alpha + p\beta} \right)^{\delta} a_j z^{j},$$

where $p,n \in N$, $\delta \geq 0, \beta \geq 0, \alpha$ a real number such that $\alpha + p\beta > 0$.

It follows from (1.1) that

$$I^{\delta}_{p,\alpha,0} f(z) = f(z) \text{ and } I^{\delta}_{p,0,\beta} f(z) = zf'(z)/p,$$

(1.2) $$(\alpha + p\beta) I^{\delta+1}_{p,\alpha,\beta} f(z) = \alpha I^{\delta}_{p,\alpha,\beta} f(z) + \beta z (I^{\delta}_{p,\alpha,\beta} f(z))'.$$

We note that for $\delta = m \in N_0 = N \cup \{0\}$ (n = 1 in some cases)

- $I^{m}_{1,\alpha,\beta} f(z) = I^{m}_{\alpha,\beta} f(z)$ (See [16]).
- $I^{m}_{p,\alpha,1} f(z) = I^{m}_{p}(\alpha) f(z), \alpha > -p$ (See [1], [13] and [14]).
- $I^{m}_{p,l+p-p\beta,\beta} f(z) = I^{m}_{p}(\beta,l) f(z), l > -p, \beta \geq 0$ (See [6]).
- $I^{m}_{p,0,\beta} f(z) = D^{m}_{p} f(z)$ (See [4], [9] and [11]).
- $I^{m}_{p,1,\beta} f(z) = N^{m}_{p,\beta} f(z)$, where $N^{m}_{p,\beta} f(z)$ is a new operator defined by

$$N^{m}_{p,\beta} f(z) = z^{p} + \sum_{j=p+n}^{\infty} \left(\frac{1 + k\beta}{1 + p\beta} \right)^{m} a_j z^{j}, (f \in A(p,n), \beta \geq 0).$$

Remark 1.2. i) $I^{m}_{p}(\alpha) f(z)$ was considered in [1], [13] and [14] for $\alpha \geq 0$ and $I^{m}_{p}(\beta,l) f(z)$ was defined in [6] for $l \geq 0, \beta \geq 0$, ii) $I^{m}_{p}(l) f(z) = I^{m}_{p}(1,l) f(z), l > -p$, iii) $I^{m}_{p}(\beta,0) f(z) = D^{m}_{p}(\beta) f(z)$, $\beta \geq 0$, was mentioned in Aouf et.al. [3], iv) $D^{m}_{1}(\beta), \beta \geq 0$, was introduced by Al-Oboudi [2], v) $D^{m}_{1}(1) f(z) = D^{m} f(z)$ was defined by Salagean [12] and was considered for m ≥ 0 in [5], vi) $I^{m}_{1}(\alpha) f(z), \alpha \geq 0$, was investigated in [7] and [8] and vii) $I^{m}_{1}(1) f(z)$ was due to Uralegaddi and Somanatha [18].

The main object of this paper is to present some interesting properties of analytic functions defined by using the new generalized multiplier transformations $I^{\delta}_{p,\alpha,\beta}f(z)$ associated with the class $A(p,n)$.

In order to prove our main results, we will make use of the following lemma.

Lemma 1.3 [10]. Let Ω be a set in the complex plane C. Suppose that the function $\Psi : C^2 \times U \to C$ satisfies the condition $\Psi(ix_2, y_1; z) \notin \Omega$ for all $z \in U$ and for all real x_2 and y_1 such that

(1.3) $$y_1 \leq -\frac{1}{2} n (1 + x_2^2).$$

If $p(z) = 1 + c_n z^n + \ldots$ is analytic in U and for $z \in U$, $\psi(p(z), zp'(z); z) \subset \Omega$, then $\mathrm{Re}(p(z)) > 0$ in U.

2. MAIN RESULTS

Theorem 2.1. Let λ be a complex number satisfying $\mathrm{Re}(\lambda) > 0$ and $\rho < 1$. Let $p, n \in N$, $\mu > 0, \delta \geq 0, \beta \geq 0, \alpha$ a real number such that $\alpha + p\beta > 0$, $f(z), g(z) \in A(p,n)$ and

(2.1) $$\mathrm{Re}\left\{\lambda \frac{I^{\delta}_{p,\alpha,\beta}g(z)}{I^{\delta+1}_{p,\alpha,\beta}g(z)}\right\} > \gamma, 0 \leq \gamma < \mathrm{Re}(\lambda), z \in U.$$

Then

$$\mathrm{Re}\left\{\left(\frac{I^{\delta}_{p,\alpha,\beta}f(z)}{I^{\delta}_{p,\alpha,\beta}g(z)}\right)^{\mu}\right\} > \frac{2\mu(\alpha + p\beta)\rho + \beta n\gamma}{2\mu(\alpha + p\beta) + \beta n\gamma}, z \in U,$$

whenever

(2.2) $$\mathrm{Re}\left\{(1-\lambda)\left(\frac{I^{\delta}_{p,\alpha,\beta}f(z)}{I^{\delta}_{p,\alpha,\beta}g(z)}\right)^{\mu} + \lambda\left(\frac{I^{\delta+1}_{p,\alpha,\beta}f(z)}{I^{\delta+1}_{p,\alpha,\beta}g(z)}\right)\left(\frac{I^{\delta}_{p,\alpha,\beta}f(z)}{I^{\delta}_{p,\alpha,\beta}g(z)}\right)^{\mu-1}\right\} > \rho, z \in U.$$

Proof. Let $\tau = (2\mu(\alpha + p\beta)\rho + \beta n\gamma)/(2\mu(\alpha + p\beta) + \beta n\gamma)$ and define the function $p(z)$ by

(2.3) $$p(z) = (1-\tau)^{-1}\left\{\left(\frac{I_{p,\alpha,\beta}^{\delta} f(z)}{I_{p,\alpha,\beta}^{\delta} g(z)}\right)^{\mu} - \tau\right\}.$$

Then, clearly, $p(z) = 1 + c_n z^n + c_{n+1} z^{n+1} + \ldots$ and is analytic in U . We set $u(z) = \lambda \frac{I_{p,\alpha,\beta}^{\delta} g(z)}{I_{p,\alpha,\beta}^{\delta+1} g(z)}$ and observe from (2.1) that $\mathrm{Re}(u(z)) > \gamma, z \in U.$ Making use of the identity (1.2), we find from (2.3) that

(2.4)
$$\left\{(1-\lambda)\left(\frac{I_{p,\alpha,\beta}^{\delta} f(z)}{I_{p,\alpha,\beta}^{\delta} g(z)}\right)^{\mu} + \lambda\left(\frac{I_{p,\alpha,\beta}^{\delta+1} f(z)}{I_{p,\alpha,\beta}^{\delta+1} g(z)}\right)\left(\frac{I_{p,\alpha,\beta}^{\delta} f(z)}{I_{p,\alpha,\beta}^{\delta} g(z)}\right)^{\mu-1}\right\} = \tau + (1-\tau)[p(z) + \frac{\beta u(z)}{\mu(\alpha + p\beta)} zp^{'}(z)]$$

If we define $\psi(x, y; z)$ by

(2.5) $$\psi(x, y; z) = \tau + (1-\tau)\left(x + \frac{\beta u(z)}{\mu(\alpha + p\beta)} y\right),$$

then, we obtain from (2.2) and (2.4) that

$$\{\, \psi(p(z), zp'(z); z) : |z| < 1 \,\} \subset \Omega = \{\, w \in C : \mathrm{Re}(w) > \rho \,\}.$$

Now for all $z \in U$ and for all real x_2 and y_1 constrained by the inequality (1.3), we find from (2.5) that

$$\mathrm{Re}\{\psi(ix_2, y_1; z)\} = \tau + (1-\tau)\frac{\beta y_1}{\mu(\alpha + p\beta)} \mathrm{Re}(u(z))$$

$$\leq \tau - (1-\tau)\frac{\beta n\gamma}{2\mu(\alpha + p\beta)} \equiv \rho.$$

Hence $\psi(ix_2, y_1; z) \notin \Omega.$ Thus by Lemma 1.1, $\mathrm{Re}(p(z)) > 0$ and hence $\mathrm{Re}\left\{\left(\frac{I_{p,\alpha,\beta}^{\delta} f(z)}{I_{p,\alpha,\beta}^{\delta} g(z)}\right)^{\mu}\right\} > \tau$ in U .This proves our theorem.

If we set

$$v(z)=\left(\frac{I^{\delta+1}_{p,\alpha,\beta}f(z)}{I^{\delta+1}_{p,\alpha,\beta}g(z)}\right)\left(\frac{I^{\delta}_{p,\alpha,\beta}f(z)}{I^{\delta}_{p,\alpha,\beta}g(z)}\right)^{\mu-1}+(\frac{1}{\lambda}-1)\left(\frac{I^{\delta}_{p,\alpha,\beta}f(z)}{I^{\delta}_{p,\alpha,\beta}g(z)}\right)^{\mu},$$

then for $\delta\geq 0,\beta\geq 0,\mu\geq 0,\alpha$ a real number such that $\alpha+p\beta>0,\ \lambda>0$ and $\rho=0$, Theorem 2.1 reduces to

$$\text{(2.6)}\qquad \operatorname{Re}(v(z))>0, z\in U \text{ implies} \operatorname{Re}\left\{\left(\frac{I^{\delta}_{p,\alpha,\beta}f(z)}{I^{\delta}_{p,\alpha,\beta}g(z)}\right)^{\mu}\right\}>\frac{n\lambda\beta\gamma}{2\mu(\alpha+p\beta)+n\lambda\beta\gamma}, z\in U,$$

whenever $\operatorname{Re}\left\{\frac{I^{\delta}_{p,\alpha,\beta}g(z)}{I^{\delta+1}_{p,\alpha,\beta}g(z)}\right\}>\gamma, 0\leq\gamma\leq 1, z\in U$. Let $\lambda\to\infty$. Then (2.6) is equivalent to

$$\left(\frac{I^{\delta+1}_{p,\alpha,\beta}f(z)}{I^{\delta+1}_{p,\alpha,\beta}g(z)}\right)\left(\frac{I^{\delta}_{p,\alpha,\beta}f(z)}{I^{\delta}_{p,\alpha,\beta}g(z)}\right)^{\mu-1}-\left(\frac{I^{\delta}_{p,\alpha,\beta}f(z)}{I^{\delta}_{p,\alpha,\beta}g(z)}\right)^{\mu}>0 \text{ in } U$$

implies

$$\operatorname{Re}\left\{\left(\frac{I^{\delta}_{p,\alpha,\beta}f(z)}{I^{\delta}_{p,\alpha,\beta}g(z)}\right)^{\mu}\right\}>1 \text{ in } U \text{, whenever } \operatorname{Re}\left\{\frac{I^{\delta}_{p,\alpha,\beta}g(z)}{I^{\delta+1}_{p,\alpha,\beta}g(z)}\right\}>\gamma, 0\leq\gamma\leq 1, z\in U.$$

In the following theorem we shall extend the above result, the proof of which is similar to that of Theorem 2.1.

Theorem 2.2. Let $p,n\in N$, $\mu>0,\delta\geq 0,\beta\geq 0,\alpha$ a real number such that $\alpha+p\beta>0$, $f(z),g(z)\in A(p,n)$ and $\operatorname{Re}\left\{\frac{I^{\delta}_{p,\alpha,\beta}g(z)}{I^{\delta+1}_{p,\alpha,\beta}g(z)}\right\}>\gamma, 0\leq\gamma<1,\ z\in U$. If

$$\operatorname{Re}\left\{\left(\frac{I^{\delta+1}_{p,\alpha,\beta}f(z)}{I^{\delta+1}_{p,\alpha,\beta}g(z)}\right)\left(\frac{I^{\delta}_{p,\alpha,\beta}f(z)}{I^{\delta}_{p,\alpha,\beta}g(z)}\right)^{\mu-1}-\left(\frac{I^{\delta}_{p,\alpha,\beta}f(z)}{I^{\delta}_{p,\alpha,\beta}g(z)}\right)^{\mu}\right\}>-\frac{n\beta\lambda(1-\rho)}{2\mu(\alpha+p\beta)}, z\in U,$$

then

$$\operatorname{Re}\left\{\left(\frac{I^{\delta}_{p,\alpha,\beta}f(z)}{I^{\delta}_{p,\alpha,\beta}g(z)}\right)^{\mu}\right\} > \rho, z \in U.$$

Remark 2.3. For $\mu = 1$, and $\alpha = l + p - p\beta, l > -p$, Theorem 2.1 and Theorem 2.2 agree with Theorem 2.1 and Theorem 2.2, respectively, of the author [15](considered for $l \geq 0$).

In a manner similar to Theorem 2.1, we can easily prove the following theorems.

Theorem 2.4. Let $p, n \in N$, $\delta \geq 0, \beta \geq 0, \alpha$ a real number such that $\alpha + p\beta > 0$, $\mu > 0, \rho < 1$ and $f(z) \in A(p,n)$. Then for λ a complex number with $\operatorname{Re}(\lambda) > 0$, we have

$$\operatorname{Re}\left(\left(\frac{I^{\delta}_{p,\alpha,\beta}f(z)}{z^{p}}\right)^{\mu}\right) > \frac{2\mu(\alpha + p\beta)\rho + n\beta\operatorname{Re}(\lambda)}{2\mu(\alpha + p\beta) + n\beta\operatorname{Re}(\lambda)}, z \in U,$$

whenever

$$\operatorname{Re}\left\{(1-\lambda)\left(\frac{I^{\delta}_{p,\alpha,\beta}f(z)}{z^{p}}\right)^{\mu} + \lambda\left(\frac{I^{\delta+1}_{p,\alpha,\beta}f(z)}{I^{\delta}_{p,\alpha,\beta}f(z)}\right)\left(\frac{I^{\delta}_{p,\alpha,\beta}f(z)}{z^{p}}\right)^{\mu}\right\} > \rho, z \in U.$$

Theorem 2.5. Let $p, n \in N$, $\delta \geq 0, \beta \geq 0, \alpha$ a real number such that $\alpha + p\beta > 0$, $\mu > 0$, λ a complex number with $\operatorname{Re}(\lambda) > 0$ and $\dfrac{n\beta\operatorname{Re}(\lambda)}{2\mu(\alpha + p\beta) + n\beta\operatorname{Re}(\lambda)} \leq \rho < 1$ .If $f(z) \in A(p,n)$ satisfies the condition

$$\operatorname{Re}\left((1-\lambda)\left(\frac{I^{\delta}_{p,\alpha,\beta}f(z)}{z^{p}}\right)^{2\mu} + \lambda\left(\frac{I^{\delta+1}_{p,\alpha,\beta}f(z)}{I^{\delta}_{p,\alpha,\beta}f(z)}\right)\left(\frac{I^{\delta}_{p,\alpha,\beta}f(z)}{z^{p}}\right)^{2\mu}\right) > M(p,n,\lambda,\alpha,\beta,\mu,\rho),$$

($z \in U$), then $\operatorname{Re}\left(\left(\dfrac{I^{\delta}_{p,\alpha,\beta}f(z)}{z^{p}}\right)^{\mu}\right) > \rho, z \in U$, where

$$M(p,n,\lambda,\alpha,\beta,\mu,\rho) = \frac{\rho[(2\mu(\alpha + p\beta) + n\beta\operatorname{Re}(\lambda))\rho - n\beta\operatorname{Re}(\lambda)]}{2\mu(\alpha + p\beta)}.$$

$\rho = \dfrac{n\beta \operatorname{Re}(\lambda)}{2\mu(\alpha + p\beta) + n\beta \operatorname{Re}(\lambda)}$ and $\rho = \dfrac{n\beta \operatorname{Re}(\lambda)}{2[2\mu(\alpha + p\beta) + n\beta \operatorname{Re}(\lambda)]}$ in Theorem 2.4 yields the following:

Corollary 2.6. Let $p, n \in N$, $\delta \geq 0, \beta \geq 0, \alpha$ a real number such that $\alpha + p\beta > 0$, $\mu > 0$, λ a complex number with $\operatorname{Re}(\lambda) > 0$ and $f(z) \in A(p,n)$. Then

(i) $$\operatorname{Re}\left((1-\lambda)\left(\frac{I^{\delta}_{p,\alpha,\beta} f(z)}{z^p}\right)^{2\mu} + \lambda \left(\frac{I^{\delta+1}_{p,\alpha,\beta} f(z)}{I^{\delta}_{p,\alpha,\beta} f(z)}\right)\left(\frac{I^{\delta}_{p,\alpha,\beta} f(z)}{z^p}\right)^{2\mu}\right) > 0, z \in U$$

implies

$$\operatorname{Re}\left(\left(\frac{I^{\delta}_{p,\alpha,\beta} f(z)}{z^p}\right)^{\mu}\right) > \frac{n\beta \operatorname{Re}(\lambda)}{2\mu(\alpha + p\beta) + n\beta \operatorname{Re}(\lambda)}, z \in U,$$

and

(ii)

$$\operatorname{Re}\left((1-\lambda)\left(\frac{I^{\delta}_{p,\alpha,\beta} f(z)}{z^p}\right)^{2\mu} + \lambda \left(\frac{I^{\delta+1}_{p,\alpha,\beta} f(z)}{I^{\delta}_{p,\alpha,\beta} f(z)}\right)\left(\frac{I^{\delta}_{p,\alpha,\beta} f(z)}{z^p}\right)^{2\mu}\right) > M(p,n,\lambda,\alpha,\beta,\mu), z \in U$$

implies

$$\operatorname{Re}\left(\left(\frac{I^{\delta}_{p,\alpha,\beta} f(z)}{z^p}\right)^{\mu}\right) > \frac{n\beta \operatorname{Re}(\lambda)}{2[(2\mu(\alpha + p\beta) + n\beta \operatorname{Re}(\lambda))]}, z \in U,$$

where

$$M(p,n,\lambda,\alpha,\beta,\mu) = -\frac{n^2\beta^2(\operatorname{Re}(\lambda))^2}{8\mu(\alpha + p\beta)[2\mu(\alpha + p\beta) + n\beta \operatorname{Re}(\lambda)]}.$$

Remark 2.7. For $\alpha = l + p - p\beta, l > -p,$ Theorem 2.4, Theorem 2.5 and Corollary 2.6 agree with Theorem 2.4, Theorem 2.5 and Corollary 2.6, respectively, of the author [15] (considered for $l \geq 0$).

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