Available online at http://scik.org
J. Math. Comput. Sci. 4 (2014), No. 4, 740-762
ISSN: 1927-5307

STABILITY ANALYSIS OF THE RIFT VALLEY FEVER DYNAMICAL MODEL

SAUL C. MPESHE[1,*], LIVINGSTONE S. LUBOOBI[1,2], YAW NKANSAH-GYEKYE[1]

[1]School of CoCSE, Nelson Mandela African Institution of Science and Technology,
P.O. Box 447, Arusha, Tanzania
[2]Department of Mathematics, Makerere University, P.O. Box 7062, Kampala, Uganda

Abstract. Stability analysis of a deterministic SEIR model of Rift Valley Fever with climate change parameters has been considered. The computational results show that the disease-free equilibrium point (DFE) is locally asymptotically stable, and using the Metzler stability theory, we find that the DFE is globally asymptotically stable when $\mathcal{R}_0 < 1$. Using the Lyaponuv stability theory and LaSalle's Invariant Principle we find that the endemic equilibrium point (EE) is globally asymptotically stable when $\mathcal{R}_0 > 1$. These results are in conjecture with the results obtained from numerical simulations.

Keywords: Lyapunov function; disease-free equilibrium; endemic equilibrium; local stability; global stability.

2010 AMS Subject Classification: 92B05, 37N205.

1. Introduction

Stability analysis of equilibria is one of the classical problems in mathematical epidemiology and different approaches have been proposed to the stability of the equilibria. Lyapunov Direct Method [1] combined with LaSalle's Invariance Principle [2] has traditionally been a powerful tool for the analysis of stability of autonomous systems of differential equations through

*Corresponding author

Received June 28, 2014

construction of suitable Lyapunov functions. Lyapunov functions are not unique and therefore different forms of Lyapunov functions have been proposed to including: (i) the *logarithmic Lyapunov function*

$$L(x_i) = \sum_{i=1}^{n} c_i (x_i - x_i^* - x_i^* \ln \frac{x_i}{x_i^*}) \tag{1}$$

proposed by Goh [3] for Lotka-Voltera systems, and later applied by Korobeinikov [4] for SIR, SIRS and SIS epidemic models; (ii) the *composite quadratic Lyaponuv function*

$$V(x_i) = \frac{c}{2} \left[\sum_{i=1}^{n} (x_i - x_i^*) \right]^2 \tag{2}$$

proposed by Vargas-De-León [5] also for SIR, SIRS and SIS epidemic models; (iii) the *composite-Volterra function*

$$W(x_i) = c \left[\sum_{i=1}^{n} (x_i - x_i^*) - \sum_{i=1}^{n} x_i^* \ln \frac{\sum_{i=1}^{n} x_i}{\sum_{i=1}^{n} x_i^*} \right] \tag{3}$$

proposed by Vargas-De-León [6] for models with relapse; and (iv) the *explicit Lyapunov function*

$$V(x_i) = \sum a_i (x_i - x_i^* \ln x_i) \tag{4}$$

proposed by Korobeinikov [7-9] for SEIR and SEIS epidemic models.

Construction of Lyapunov functions to establish stability of equilibria is not an easy task and therefore stability analysis through geometric approach proposed by Li and Muldowney [10] has been used to prove the global stability of the endemic equilibrium. On other hand, a method based on the use of stable Metzler matrices has been proposed and proven to be useful to establish the global stability of DFE by Kamgang and Sallet [1]. The importance of the Metzler matrices is well recognised in the stability of dynamical systems and positive systems [12-13] and more generally in biology, engineering and economics [1-2,14-17].

In this paper we consider the model developed by Mpeshe *et al.* [18] to compute the e-quilibrium points and analyse its stability. We establish the global stability of the disease-free equilibrium point using stable Metzler matrix theory. We also establish the global stability of the endemic equilibrium point using Lyapunov Direct Method combined with LaSalle's Invariance Principle. Finally, we perform numerical simulations of the model system in a closed population.

2. Materials and methods

2.1. Model formulation

The model developed by Mpeshe *et al.* [18] considers three populations: mosquitoes, livestock, and humans with disease-dependent death rate for livestock and humans. The mosquito population is subdivided into two: *Aedes* species and *Culex* species. The egg population of *Aedes* spp. consists of uninfected eggs (X_a) and infected eggs (Y_a). The population for adult *Aedes* spp. consists of susceptible adults (S_a), latently infected adults (E_a), and infectious adults (I_a). The egg population of *Culex* spp. consists of uninfected eggs (X_c) only and the population for adult *Culex* spp. consists of susceptible adults (S_c), latently infected adults (E_c), and infectious adults (I_c). The livestock population consists of susceptible livestock (S_l), latently infected livestock (E_l), infectious livestock (I_l), and recovered livestock (R_l). The human population consists of susceptible humans (S_h), latently infected humans (E_h), infectious humans (I_h) and recovered humans (R_h). The model parameters and their description as they have been used in this work are given in Table 1.

The equations of the model are (5), (6), (7), and (8), where T and P represent temperature and precipitation respectively:

Aedes Mosquito

$$\frac{dX_a}{dt} = b_a(T,P)(N_a - f_a I_a) - h_a(T,P)X_a, \tag{5a}$$

$$\frac{dY_a}{dt} = b_a(T,P) f_a I_a - h_a(T,P)Y_a, \tag{5b}$$

$$\frac{dS_a}{dt} = h_a(T,P)X_a - \mu_a(T)S_a - \lambda_{la}(T)\frac{I_l}{N_l}S_a - \lambda_{ha}(T)\frac{I_h}{N_h}S_a, \tag{5c}$$

$$\frac{dE_a}{dt} = \lambda_{la}(T)\frac{I_l}{N_l}S_a + \lambda_{ha}(T)\frac{I_h}{N_h}S_a - (\varepsilon_a(T) + \mu_a(T))E_a, \tag{5d}$$

$$\frac{dI_a}{dt} = h_a(T,P)Y_a + \varepsilon_a(T)E_a - \mu_a(T)I_a, \tag{5e}$$

$$\frac{dN_a}{dt} = h_a(T,P)(X_a + Y_a) - \mu_a(T)N_a. \tag{5f}$$

Culex Mosquito

$$\frac{dX_c}{dt} = b_c(T,P)N_c - h_c(T,P)X_c, \tag{6a}$$

$$\frac{dS_c}{dt} = h_c(T,P)X_c - \mu_c(T)S_c - \lambda_{lc}(T)\frac{I_l}{N_l}S_c - \lambda_{hc}(T)\frac{I_h}{N_h}S_c, \tag{6b}$$

$$\frac{dE_c}{dt} = \lambda_{lc}(T)\frac{I_l}{N_l}S_c + \lambda_{hc}(T)\frac{I_h}{N_h}S_c - (\varepsilon_c(T) + \mu_c(T))E_c, \tag{6c}$$

$$\frac{dI_c}{dt} = \varepsilon_c(T)E_c - \mu_c(T)I_c, \tag{6d}$$

$$\frac{dN_c}{dt} = h_c(T,P)X_c - \mu_c(T)N_c. \tag{6e}$$

Livestock

$$\frac{dS_l}{dt} = b_lN_l - \mu_lS_l - \lambda_{al}(T)\frac{I_a}{N_a}S_l - \lambda_{cl}(T)\frac{I_c}{N_c}S_l, \tag{7a}$$

$$\frac{dE_l}{dt} = \lambda_{al}(T)\frac{I_a}{N_a}S_l + \lambda_{cl}(T)\frac{I_c}{N_c}S_l - (\varepsilon_l + \mu_l)E_l, \tag{7b}$$

$$\frac{dI_l}{dt} = \varepsilon_lE_l - (\mu_l + \phi_l + \gamma_l)I_l, \tag{7c}$$

$$\frac{dR_l}{dt} = \gamma_lI_l - \mu_lR_l, \tag{7d}$$

$$\frac{dN_l}{dt} = (b_l - \mu_l)N_l - \phi_lI_l. \tag{7e}$$

Humans

$$\frac{dS_h}{dt} = b_hN_h - \mu_hS_h - \lambda_{lh}\frac{I_l}{N_l}S_h - \lambda_{ah}(T)\frac{I_a}{N_a}S_h - \lambda_{ch}(T)\frac{I_c}{N_c}S_h, \tag{8a}$$

$$\frac{dE_h}{dt} = \lambda_{lh}\frac{I_l}{N_l}S_h + \lambda_{ah}(T)\frac{I_a}{N_a}S_h + \lambda_{ch}(T)\frac{I_c}{N_c}S_h - (\varepsilon_h + \mu_h)E_h, \tag{8b}$$

$$\frac{dI_h}{dt} = \varepsilon_hE_h - (\mu_h + \phi_h + \gamma_h)I_h, \tag{8c}$$

$$\frac{dR_h}{dt} = \gamma_hI_h - \mu_hR_h, \tag{8d}$$

$$\frac{dN_h}{dt} = (b_h - \mu_h)N_h - \phi_hI_h. \tag{8e}$$

2.2. Equilibrium points

In solving for the equilibria, we omit the expression containing R in livestock and humans because it can be determined when S, E and I are known. This will transform the equilibria from $\mathcal{D} \in \mathbb{R}^{17}_{+}$ to $\mathbb{R}^{13}_{+}$. We therefore consider the following system of ordinary differential equation:

(9a) $$\frac{dX_a}{dt} = b_a(T,P)(N_a - f_a I_a) - h_a(T,P)X_a,$$

(9b) $$\frac{dY_a}{dt} = b_a(T,P)f_a I_a - h_a(T,P)Y_a,$$

(9c) $$\frac{dS_a}{dt} = h_a(T,P)X_a - \mu_a(T)S_a - \lambda_{la}(T)\frac{I_l}{N_l}S_a - \lambda_{ha}(T)\frac{I_h}{N_h}S_a,$$

(9d) $$\frac{dE_a}{dt} = \lambda_{la}(T)\frac{I_l}{N_l}S_a + \lambda_{ha}(T)\frac{I_h}{N_h}S_a - (\varepsilon_a(T) + \mu_a(T))E_a,$$

(9e) $$\frac{dI_a}{dt} = h_a(T,P)Y_a + \varepsilon_a(T)E_a - \mu_a(T)I_a,$$

(9f) $$\frac{dX_c}{dt} = b_c(T,P)N_c - h_c(T,P)X_c,$$

(9g) $$\frac{dS_c}{dt} = h_c(T,P)X_c - \mu_c(T)S_c - \lambda_{lc}(T)\frac{I_l}{N_l}S_c - \lambda_{hc}(T)\frac{I_h}{N_h}S_c,$$

(9h) $$\frac{dE_c}{dt} = \lambda_{lc}(T)\frac{I_l}{N_l}S_c + \lambda_{hc}(T)\frac{I_h}{N_h}S_c - (\varepsilon_c(T) + \mu_c(T))E_c,$$

(9i) $$\frac{dI_c}{dt} = \varepsilon_c(T)E_c - \mu_c(T)I_c,$$

(9j) $$\frac{dS_l}{dt} = b_l N_l - \mu_l S_l - \lambda_{al}(T)\frac{I_a}{N_a}S_l - \lambda_{cl}(T)\frac{I_c}{N_c}S_l,$$

(9k) $$\frac{dE_l}{dt} = \lambda_{al}(T)\frac{I_a}{N_a}S_l + \lambda_{cl}(T)\frac{I_c}{N_c}S_l - (\varepsilon_l + \mu_l)E_l,$$

(9l) $$\frac{dI_l}{dt} = \varepsilon_l E_l - (\mu_l + \phi_l + \gamma_l)I_l,$$

(9m) $$\frac{dS_h}{dt} = b_h N_h - \mu_h S_h - \lambda_{lh}\frac{I_l}{N_l}S_h - \lambda_{ah}(T)\frac{I_a}{N_a}S_h - \lambda_{ch}(T)\frac{I_c}{N_c}S_h,$$

(9n) $$\frac{dE_h}{dt} = \lambda_{lh}\frac{I_l}{N_l}S_h + \lambda_{ah}(T)\frac{I_a}{N_a}S_h + \lambda_{ch}(T)\frac{I_c}{N_c}S_h - (\varepsilon_h + \mu_h)E_h,$$

(9o) $$\frac{dI_h}{dt} = \varepsilon_h E_h - (\mu_h + \phi_h + \gamma_h)I_h.$$

We compute the equilibria $\mathcal{D}$ by setting the left-hand side of the system (9) equal to zero. That is,

$$0 = b_a(T,P)(N_a - f_a I_a) - h_a(T,P)X_a, \tag{10a}$$

$$0 = b_a(T,P) f_a I_a - h_a(T,P)Y_a, \tag{10b}$$

$$0 = h_a(T,P)X_a - \mu_a(T)S_a - \lambda_{la}(T)\frac{I_l}{N_l}S_a - \lambda_{ha}(T)\frac{I_h}{N_h}S_a, \tag{10c}$$

$$0 = \lambda_{la}(T)\frac{I_l}{N_l}S_a + \lambda_{ha}(T)\frac{I_h}{N_h}S_a - (\varepsilon_a(T) + \mu_a(T))E_a, \tag{10d}$$

$$0 = h_a(T,P)Y_a + \varepsilon_a(T)E_a - \mu_a(T)I_a, \tag{10e}$$

$$0 = b_c(T,P)N_c - h_c(T,P)X_c, \tag{10f}$$

$$0 = h_c(T,P)X_c - \mu_c(T)S_c - \lambda_{lc}(T)\frac{I_l}{N_l}S_c - \lambda_{hc}(T)\frac{I_h}{N_h}S_c, \tag{10g}$$

$$0 = \lambda_{lc}(T)\frac{I_l}{N_l}S_c + \lambda_{hc}(T)\frac{I_h}{N_h}S_c - (\varepsilon_c(T) + \mu_c(T))E_c, \tag{10h}$$

$$0 = \varepsilon_c(T)E_c - \mu_c(T)I_c, \tag{10i}$$

$$0 = b_l N_l - \mu_l S_l - \lambda_{al}(T)\frac{I_a}{N_a}S_l - \lambda_{cl}(T)\frac{I_c}{N_c}S_l, \tag{10j}$$

$$0 = \lambda_{al}(T)\frac{I_a}{N_a}S_l + \lambda_{cl}(T)\frac{I_c}{N_c}S_l - (\varepsilon_l + \mu_l)E_l, \tag{10k}$$

$$0 = \varepsilon_l E_l - (\mu_l + \phi_l + \gamma_l)I_l, \tag{10l}$$

$$0 = b_h N_h - \mu_h S_h - \lambda_{lh}\frac{I_l}{N_l}S_h - \lambda_{ah}(T)\frac{I_a}{N_a}S_h - \lambda_{ch}(T)\frac{I_c}{N_c}S_h, \tag{10m}$$

$$0 = \lambda_{lh}\frac{I_l}{N_l}S_h + \lambda_{ah}(T)\frac{I_a}{N_a}S_h + \lambda_{ch}(T)\frac{I_c}{N_c}S_h - (\varepsilon_h + \mu_h)E_h, \tag{10n}$$

$$0 = \varepsilon_h E_h - (\mu_h + \phi_h + \gamma_h)I_h. \tag{10o}$$

Direct computations gives us two equilibrium points on the transformed region $\mathbb{R}^{13}_+$, the disease-free equilibrium (DFE)

$$\begin{aligned} \mathcal{D}^0 &= (X_a^0, Y_a^0, S_a^0, E_a^0, I_a^0, X_c^0, S_c^0, E_c^0, I_c^0, S_l^0, E_l^0, I_l^0, S_h^0, E_h^0, I_h^0) \\ &= \Bigg(\frac{b_a(T,P)}{h_a(T)}N_a^0, 0, \frac{b_a(T,P)}{\mu_a(T)}N_a^0, 0, 0, \frac{b_c(T,P)}{h_c(T)}N_c^0, \frac{b_c(T,P)}{\mu_c(T)}N_c^0, \\ &\quad 0, 0, \frac{b_l}{\mu_l}N_l^0, 0, 0, \frac{b_h}{\mu_h}N_h^0, 0, 0\Bigg), \end{aligned} \tag{11}$$

and the endemic equilibrium

$$\mathcal{D}^* = (X_a^*, Y_a^*, S_a^*, E_a^*, I_a^*, X_c^*, S_c^*, E_c^*, I_c^*, S_l^*, E_l^*, I_l^*, S_h^*, E_h^*, I_h^*)$$

where

$$X_a^* = \frac{b_a(T,P)}{h_a(T,P)}N_a^* - Y_a^*, \tag{12a}$$

$$Y_a^* = \frac{b_a(T,P)f_a\varepsilon_a(T)E_a^*}{h_a(T,P)(b_a(T,P)f_a - \mu_a(T))} \tag{12b}$$

$$S_a^* = \frac{h_a(T,P)X_a^*}{\mu_a(T) + \frac{\lambda_{la}(T)\varepsilon_l E_l^*}{N_l^*(\mu_l+\phi_l+\gamma_l)} + \frac{\lambda_{ha}(T)\varepsilon_h E_h^*}{N_h^*(\mu_h+\phi_h+\gamma_h)}}, \tag{12c}$$

$$E_a^* = \frac{\lambda_{la}\frac{I_l^*}{N_l^*} + \lambda_{ha}\frac{I_h^*}{N_h^*}}{\varepsilon_a(T) + \mu_a(T)}S_a^*, \tag{12d}$$

$$I_a^* = \frac{h_a(T,P)}{b_a(T,P)f_a}Y_a^*, \tag{12e}$$

$$X_c^* = \frac{b_c(T,P)}{h_c(T,P)}N_c^*, \tag{12f}$$

$$S_c^* = \frac{h_c(T,P)X_c^*}{\mu_c(T) + \frac{\lambda_{lc}(T)\varepsilon_l E_l^*}{N_l(\mu_l+\phi_l+\gamma_l)} + \frac{\lambda_{hc}(T)\varepsilon_h E_h^*}{N_h^*(\mu_h+\phi_h+\gamma_h)}}, \tag{12g}$$

$$E_c^* = \frac{\lambda_{lc}\frac{I_l^*}{N_l^*} + \lambda_{hc}\frac{I_h^*}{N_h^*}}{\varepsilon_c(T) + \mu_c(T)}S_c^*, \tag{12h}$$

$$I_c^* = \frac{\varepsilon_c(T)E_c^*}{\mu_c(T)}, \tag{12i}$$

$$S_l^* = \frac{b_l N_l^*}{\mu_l + \frac{\lambda_{al}(T)h_a(T,P)Y_a^*}{N_a^* b_a(T,P)f_a} + \frac{\lambda_{cl}(T)\varepsilon_c(T)E_c^*}{N_c\mu_c(T)}}, \tag{12j}$$

$$E_l^* = \frac{\lambda_{al}\frac{I_a^*}{N_a^*} + \lambda_{cl}\frac{I_c^*}{N_c^*}}{\varepsilon_l + \mu_l}S_l^*, \tag{12k}$$

$$I_l^* = \frac{\varepsilon_l E_l^*}{\mu_l + \phi_l + \gamma_l}, \tag{12l}$$

$$S_h^* = \frac{b_h N_h^*}{\mu_h + \frac{\lambda_{lh}\varepsilon_l E_l^*}{N_l^*(\mu_l+\phi_l+\gamma_l)} + \frac{\lambda_{ah}(T)h_a(T,P)Y_a^*}{N_a^* b_a(T,P)f_a} + \frac{\lambda_{ch}(T)\varepsilon_c(T)E_c^*}{N_c^*\mu_c(T)}}, \tag{12m}$$

$$E_h^* = \frac{\lambda_{lh}\frac{I_l^*}{N_l^*} + \lambda_{ah}\frac{I_a^*}{N_a^*} + \lambda_{ch}\frac{I_c^*}{N_c^*}}{\varepsilon_h + \mu_h}S_h^*, \tag{12n}$$

$$I_h^* = \frac{\varepsilon_h E_h^*}{\mu_h + \phi_h + \gamma_h}. \tag{12o}$$

2.3. Stability analysis of equilibrium points

In this section, we determine the conditions under which the equilibrium points are asymptotically stable or unstable. Asymptotic stability implies that the solution starting sufficiently close to the equilibrium point remains close to it and approaches it as $t \to \infty$, while instability of the equilibrium implies that there are solutions starting arbitrary close to the equilibrium point which do not approach it.

2.3.1. Local stability of the DFE

In general, local asymptotic stability (LAS) implies that trajectories start arbitrary close to the equilibrium point but they do not reach it. We start by evaluating the Jacobian matrix at the DFE.

That is,

$$J(\mathcal{D}^0) = \begin{bmatrix} J_{11} & 0 & J_{13} \\ J_{21} & J_{22} & J_{23} \\ J_{31} & J_{32} & J_{33} \end{bmatrix}, \tag{13}$$

where

$$J_{11} = \begin{bmatrix} -h_a(T,P) & 0 & 0 & 0 & 0 \\ 0 & -h_a(T,P) & 0 & 0 & 0 \\ h_a(T,P) & 0 & -\mu_a(T) & 0 & 0 \\ 0 & 0 & 0 & -(\varepsilon_a(T)+\mu_a(T)) & 0 \\ 0 & h_a(T,P) & 0 & \varepsilon_a(T) & -\mu_a(T) \end{bmatrix}, \tag{14}$$

$$J_{13} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -\lambda_{la}(T)\frac{S_a^0}{N_l^0} & 0 & 0 & -\lambda_{ha}(T)\frac{S_a^0}{N_h^0} \\ 0 & \lambda_{la}(T)\frac{S_a^0}{N_l^0} & 0 & 0 & \lambda_{ha}(T)\frac{S_a^0}{N_h^0} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, J_{21} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\lambda_{al}(T)\frac{S_l^0}{N_a^0} \end{bmatrix}, \tag{15}$$

$$J_{22} = \begin{bmatrix} -h_c(T,P) & 0 & 0 & 0 & 0 \\ h_c(T,P) & -\mu_c(T) & 0 & 0 & 0 \\ 0 & 0 & -(\varepsilon_c(T)+\mu_c(T)) & 0 & 0 \\ 0 & 0 & \varepsilon_c(T) & -\mu_c(T) & 0 \\ 0 & 0 & 0 & -\lambda_{cl}(T)\frac{S_l^0}{N_c^0} & -\mu_l \end{bmatrix}, \tag{16}$$

(17)
$$J_{23} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -\lambda_{lc}(T)\frac{S_c^0}{N_l^0} & 0 & 0 & -\lambda_{hc}(T)\frac{S_c^0}{N_h^0} \\ 0 & \lambda_{lc}(T)\frac{S_c^0}{N_l^0} & 0 & 0 & \lambda_{hc}(T)\frac{S_c^0}{N_h^0} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

(18)
$$J_{31} = \begin{bmatrix} 0 & 0 & 0 & 0 & \lambda_{al}(T)\frac{S_l^0}{N_a^0} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\lambda_{ah}(T)\frac{S_h^0}{N_a^0} \\ 0 & 0 & 0 & 0 & \lambda_{ah}(T)\frac{S_h^0}{N_a^0} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, J_{32} = \begin{bmatrix} 0 & 0 & 0 & \lambda_{cl}(T)\frac{S_l^0}{N_c^0} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\lambda_{ch}(T)\frac{S_h^0}{N_c^0} & 0 \\ 0 & 0 & 0 & \lambda_{ch}(T)\frac{S_h^0}{N_c^0} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

and

(19)
$$J_{33} = \begin{bmatrix} -(\varepsilon_l+\mu_l) & 0 & 0 & 0 & 0 \\ \varepsilon_l & -(\mu_l+\phi_l+\gamma_l) & 0 & 0 & 0 \\ 0 & -\lambda_{lh}\frac{S_h^0}{N_l^0} & -\mu_h & 0 & 0 \\ 0 & \lambda_{lh}\frac{S_h^0}{N_l^0} & 0 & -(\varepsilon_h+\mu_h) & 0 \\ 0 & 0 & 0 & \varepsilon_h & -(\mu_h+\phi_h+\gamma_h) \end{bmatrix}.$$

Combining all together, the matrix $J(\mathcal{D}_0)$ diagonal entries in the third, seventh, tenth, and thirteenth columns. Therefore, the diagonal entries $-\mu_a$, $-\mu_c$, $-\mu_l$, $-\mu_h$ are four eigenvalues of the Jacobian. Excluding these columns and their corresponding rows, we remain with a matrix $\tilde{J}_{11\times 11}$ given by

(20)
$$\tilde{J}(\mathcal{D}_0) = \begin{bmatrix} \tilde{J}_{11} & 0 & \tilde{J}_{13} \\ 0 & \tilde{J}_{22} & \tilde{J}_{23} \\ \tilde{J}_{31} & \tilde{J}_{32} & \tilde{J}_{33} \end{bmatrix},$$

where

(21)
$$\tilde{J}_{11} = \begin{bmatrix} -h_a(T,P) & 0 & 0 & 0 \\ 0 & -h_a(T,P) & 0 & 0 \\ 0 & 0 & -(\varepsilon_a(T)+\mu_a(T)) & 0 \\ 0 & h_a(T,P) & \varepsilon_a(T) & -\mu_a(T) \end{bmatrix},$$

$$\tilde{J}_{13} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \lambda_{la}(T)\frac{S_a^0}{N_l^0} & 0 & \lambda_{ha}(T)\frac{S_a^0}{N_h^0} \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{22}$$

$$\tilde{J}_{22} = \begin{bmatrix} -h_c(T,P) & 0 & 0 \\ 0 & -(\varepsilon_c(T)+\mu_c(T)) & 0 \\ 0 & \varepsilon_c(T) & -\mu_c(T) \end{bmatrix}, \tilde{J}_{23} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \lambda_{lc}(T)\frac{S_c^0}{N_l^0} & 0 & \lambda_{hc}(T)\frac{S_c^0}{N_h^0} \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{23}$$

$$\tilde{J}_{31} = \begin{bmatrix} 0 & 0 & 0 & \lambda_{al}(T)\frac{S_l^0}{N_a^0} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_{ah}(T)\frac{S_h^0}{N_a^0} \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tilde{J}_{32} = \begin{bmatrix} 0 & 0 & \lambda_{cl}(T)\frac{S_l^0}{N_c^0} \\ 0 & 0 & 0 \\ 0 & 0 & \lambda_{ch}(T)\frac{S_h^0}{N_c^0} \\ 0 & 0 & 0 \end{bmatrix}, \tag{24}$$

and

$$\tilde{J}_{33} = \begin{bmatrix} -(\varepsilon_l+\mu_l) & 0 & 0 & 0 \\ \varepsilon_l & -(\mu_l+\phi_l+\gamma_l) & 0 & 0 \\ 0 & \lambda_{lh}\frac{S_h^0}{N_l^0} & -(\varepsilon_h+\mu_h) & 0 \\ 0 & 0 & \varepsilon_h & -(\mu_h+\phi_h+\gamma_h) \end{bmatrix}. \tag{25}$$

Making further computations on $\tilde{J}(\mathcal{D}_0)$, we find that all eigenvalues of the Jacobian matrix $J(\mathcal{D}_0)$ are negative. This implies that the system is asymptotically stable, and hence, the following proposition:

Propositon 2.1. *The disease-free equilibrium point is locally asymptotically stable in $\mathcal{D}$ if $\mathcal{R}_0 < 1$ and unstable if $\mathcal{R}_0 > 1$.*

2.3.2. Global stability of the DFE

The global stability of the DFE is determined by applying Castillo-Chavez *et al.* [19]. We write the system in the form

$$\begin{cases} \frac{dX_n}{dt} &= A_1(x)(X_n - X_{DFE,n}) + A_{12}(x)X_e, \\ \frac{dX_e}{dt} &= A_2(x)X_e, \end{cases} \tag{26}$$

where X_n is the vector representing the non-transmitting compartments, and X_e is the vector representing the transmitting compartments.
Hence,

$$X_n = (X_a, S_a, X_c, S_c, S_l, S_h)^T, \quad Xe = (Y_a, E_a, I_a, E_c, I_c, E_l, I_l, E_h, I_h)^T, \tag{27a}$$

$$X_{DFE,n} = (\frac{b_a(T,P)}{h_a(T,P)}N_a, \frac{b_a(T,P)}{\mu_a(T)}N_a, \frac{b_c(T,P)}{h_c(T,P)}N_c, \frac{b_c(T,P)}{\mu_c(T)}N_c, \frac{b_l}{\mu_l}N_l, \frac{b_h}{\mu_h}N_h,) \tag{27b}$$

with

$$A_1(x) = \begin{bmatrix} -h_a(T,P) & 0 & 0 & 0 & 0 & 0 \\ h_a(T,P) & -\mu_a(T) & 0 & 0 & 0 & 0 \\ 0 & 0 & -h_c(T,P) & 0 & 0 & 0 \\ 0 & 0 & h_c(T,P) & -\mu_c(T) & 0 & 0 \\ 0 & 0 & 0 & 0 & -\mu_l & 0 \\ 0 & 0 & 0 & 0 & 0 & -\mu_h \end{bmatrix}, \tag{28}$$

$$A_{12}(x) = \begin{bmatrix} 0 & 0 & -b_a(T,P)f_a & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\lambda_{la}(T)\frac{S_a}{N_l} & 0 & -\lambda_{ha}(T)\frac{S_a}{N_h} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\lambda_{lc}(T)\frac{S_c}{N_l} & 0 & -\lambda_{hc}(T)\frac{S_c}{N_h} \\ 0 & 0 & -\lambda_{al}(T)\frac{S_l}{N_a} & 0 & -\lambda_{cl}(T)\frac{S_l}{N_c} & 0 & 0 & 0 & 0 \\ 0 & 0 & -\lambda_{ah}(T)\frac{S_h}{N_a} & 0 & -\lambda_{ch}(T)\frac{S_h}{N_c} & 0 & -\lambda_{lh}\frac{S_h}{N_l} & 0 & 0 \end{bmatrix} \tag{29}$$

and

$$A_2(x) = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}, \tag{30}$$

where

$$(31)\qquad M_{11}=\begin{bmatrix} -h_a(T,P) & 0 & b_a(T,P)f_a & 0 & 0\\ 0 & -(\varepsilon_a(T)+\mu_a(T)) & 0 & 0 & 0\\ h_a(T,P) & \varepsilon_a(T) & -\mu_a(T) & 0 & 0\\ 0 & 0 & 0 & -(\varepsilon_c(T)+\mu_c(T)) & 0\\ 0 & 0 & 0 & \varepsilon_c(T) & -\mu_c(T) \end{bmatrix},$$

$$(32)\qquad M_{12}=\begin{bmatrix} 0 & 0 & 0 & 0\\ 0 & \lambda_{la}(T)\frac{S_a}{N_l} & 0 & \lambda_{ha}(T)\frac{S_a}{N_h}\\ 0 & 0 & 0 & 0\\ 0 & \lambda_{lc}(T)\frac{S_c}{N_l} & 0 & \lambda_{hc}(T)\frac{S_c}{N_h}\\ 0 & 0 & 0 & 0 \end{bmatrix}, M_{21}=\begin{bmatrix} 0 & 0 & \lambda_{al}(T)\frac{S_l}{N_a} & 0 & \lambda_{cl}(T)\frac{S_l}{N_c}\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & \lambda_{ah}(T)\frac{S_h}{N_a} & 0 & \lambda_{ch}(T)\frac{S_h}{N_c}\\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

and

$$(33)\qquad M_{22}=\begin{bmatrix} -(\varepsilon_l+\mu_l) & 0 & 0 & 0\\ \varepsilon_l & -(\mu_l+\phi_l+\gamma_l) & 0 & 0\\ 0 & \lambda_{lh}\frac{S_h}{N_l} & -(\varepsilon_h+\mu_h) & 0\\ 0 & 0 & \varepsilon_h & -(\mu_h+\phi_h+\gamma_h) \end{bmatrix}.$$

A direct computation shows that the eigenvalues of $A_1(x)$ are real and negative. Thus, the system

$$\frac{dX_n}{dt}=A_1(x)(X_n-X_{DFE,n})+A_{12}(x)X_e,$$

is globally asymptotically stable at X_{DFE}. Also, combining all the sub-matrices, the matrix $A_2(x)$ is a Metzler stable matrix. Thus, the DFE is GAS and therefore, we have the following proposition:

Propositon 2.2. *The disease-free equilibrium point is globally asymptotically stable in* $\mathcal{D}$ *if* $\mathcal{R}_0<1$ *and unstable if* $\mathcal{R}_0>1$.

2.3.3. Global stability of the endemic equilibrium

The local stability of the DFE suggests local stability of the EE for the reverse condition [20]. Hence we only investigate the global stability of the EE. We explored the global stability of the endemic equilibrium via construction of a suitable Lyapunov function using Korobeinikov approach [7-9]. In this approach, we construct Lyapunov functions of the form

$$V = \sum a_i (x_i - x_i^* \ln x_i), \tag{34}$$

where a_i is properly selected constant, x_i is the population of the i^{th} compartment, and x_i^* is the equilibrium point. The approach has been found to be useful for compartmental epidemic models with any number of compartments [7-9].

Thus, consider the Lyapunov function

$$\begin{aligned} V = {} & w_1(X_a - X_a^* \ln X_a) + w_2(Y_a - Y_a^* \ln Y_a) + w_3(S_a - S_a^* \ln S_a) \\ & + w_4(E_a - E_a^* \ln E_a) + w_5(I_a - I_a^* \ln I_a) + w_6(X_c - X_c^* \ln X_c) \\ & + w_7(S_c - S_c^* \ln S_c) + w_8(E_c - E_c^* \ln E_c) + w_9(I_c - I_c^* \ln I_c) \\ & + w_{10}(S_l - S_l^* \ln S_l) + w_{11}(E_l - E_l^* \ln E_l) + w_{12}(I_l - I_l^* \ln I_l) \\ & + w_{13}(S_h - S_h^* \ln S_h) + w_{14}(E_h - E_h^* \ln E_h) + w_{15}(I_h - I_h^* \ln I_h), \end{aligned} \tag{35}$$

where $w_i > 0$ for $i = 1, 2, \cdots, 15$.

The time derivative of V is then given by

$$\begin{aligned} \frac{dV}{dt} = {} & w_1(1 - \frac{X_a^*}{X_a})\frac{dX_a}{dt} + w_2(1 - \frac{Y_a^*}{Y_a})\frac{dY_a}{dt} + w_3(1 - \frac{S_a^*}{S_a})\frac{dS_a}{dt} \\ & + w_4(1 - \frac{E_a^*}{E_a})\frac{dE_a}{dt} + w_5(1 - \frac{I_a^*}{I_a})\frac{dI_a}{dt} + w_6(1 - \frac{X_c^*}{X_c})\frac{dX_c}{dt} \\ & + w_7(1 - \frac{S_c^*}{S_c})\frac{dS_c}{dt} + w_8(1 - \frac{E_c^*}{E_c})\frac{dE_c}{dt} + w_9(1 - \frac{I_c^*}{I_c})\frac{dI_c}{dt} \\ & + w_{10}(1 - \frac{S_l^*}{S_l})\frac{dS_l}{dt} + w_{11}(1 - \frac{E_l^*}{E_l})\frac{dE_l}{dt} + w_{12}(1 - \frac{I_l^*}{I_l})\frac{dI_l}{dt} \\ & + w_{13}(1 - \frac{S_h^*}{S_h})\frac{dS_h}{dt} + w_{14}(1 - \frac{E_h^*}{E_h})\frac{dE_h}{dt} + w_{15}(1 - \frac{I_h^*}{I_h})\frac{dI_h}{dt} \end{aligned} \tag{36}$$

and from (9) we have

$$
\begin{aligned}
\frac{dV}{dt} &= w_1(1-\frac{X_a^*}{X_a})[b_a(T,P)(N_a-f_aI_a)-h_a(T,P)X_a] \\
&+w_2(1-\frac{Y_a^*}{Y_a})[b_a(T,P)f_aI_a-h_a(T,P)Y_a] \\
&+w_3(1-\frac{S_a^*}{S_a})[h_a(T,P)X_a-\mu_a(T)S_a-\lambda_{la}(T)\frac{I_l}{N_l}S_a-\lambda_{ha}(T)\frac{I_h}{N_h}S_a] \\
&+w_4(1-\frac{E_a^*}{E_a})[\lambda_{la}(T)\frac{I_l}{N_l}S_a+\lambda_{ha}(T)\frac{I_h}{N_h}S_a-(\varepsilon_a(T)+\mu_a(T))E_a] \\
&+w_5(1-\frac{I_a^*}{I_a})[h_a(T,P)Y_a+\varepsilon_a(T)E_a-\mu_a(T)I_a] \\
&+w_6(1-\frac{X_c^*}{X_c})[b_c(T,P)N_c-h_c(T,P)X_c] \\
&+w_7(1-\frac{S_c^*}{S_c})[h_c(T,P)X_c-\mu_c(T)S_c-\lambda_{lc}(T)\frac{I_l}{N_l}S_c-\lambda_{hc}(T)\frac{I_h}{N_h}S_c] \\
&+w_8(1-\frac{E_c^*}{E_c})[\lambda_{lc}(T)\frac{I_l}{N_l}S_c+\lambda_{hc}(T)\frac{I_h}{N_h}S_c-(\varepsilon_c(T)+\mu_c(T))E_c] \\
&+w_9(1-\frac{I_c^*}{I_c})[\varepsilon_c(T)E_c-\mu_c(T)I_c] \\
&+w_{10}(1-\frac{S_l^*}{S_l})[b_lN_l-\mu_lS_l-\lambda_{al}(T)\frac{I_a}{N_a}S_l-\lambda_{cl}(T)\frac{I_c}{N_c}S_l] \\
&+w_{11}(1-\frac{E_l^*}{E_l})[\lambda_{al}(T)\frac{I_a}{N_a}S_l+\lambda_{cl}(T)\frac{I_c}{N_c}S_l-(\varepsilon_l+\mu_l)E_l] \\
&+w_{12}(1-\frac{I_l^*}{I_l})[\varepsilon_lE_l-(\mu_l+\phi_l+\gamma_l)I_l] \\
&+w_{13}(1-\frac{S_h^*}{S_h})[b_hN_h-\mu_hS_h-\lambda_{lh}\frac{I_l}{N_l}S_h-\lambda_{ah}(T)\frac{I_a}{N_a}S_h-\lambda_{ch}(T)\frac{I_c}{N_c}S_h] \\
&+w_{14}(1-\frac{E_h^*}{E_h})[\lambda_{lh}\frac{I_l}{N_l}S_h+\lambda_{ah}(T)\frac{I_a}{N_a}S_h+\lambda_{ch}(T)\frac{I_c}{N_c}S_h-(\varepsilon_h+\mu_h)E_h] \\
&+w_{15}(1-\frac{I_h^*}{I_h})[\varepsilon_hE_h-(\mu_h+\phi_h+\gamma_h)I_h].
\end{aligned} \tag{37}
$$

Assuming constant total mosquito eggs and constant total population for all species, we have the following at $\mathcal{D}^*$: $h_a(T,P)G_a = h_a(T,P)X_a^* + h_a(T,P)Y_a^*$, $h_c(T,P)G_c = h_c(T,P)X_c^*$,

$$b_a(T,P)N_a = h_a(T,P)X_a^* + h_a(T,P)Y_a^*,$$

$b_c(T,P)N_c = h_c(T,P)X_c^*$, $b_lN_l = \mu_l S_l^* + \lambda_{al}(T)\frac{I_a^*}{N_a}S_l^* + \lambda_{cl}(T)\frac{I_c^*}{N_c}S_l^*$, and $b_hN_h = \mu_h S_h^* + \lambda_{lh}\frac{I_l^*}{N_l}S_h^* + \lambda_{ah}(T)\frac{I_a^*}{N_a}S_h^* + \lambda_{ch}(T)\frac{I_c^*}{N_c}S_h^*$. Therefore,

(38)

$$\begin{aligned}\frac{dV}{dt} = {} & w_1(1-\frac{X_a^*}{X_a})[b_a(T,P)I^*a + h_a(T,P)X_a^* - b_a(T,P)f_aI_a) - h_a(T,P)X_a] \\
& + w_2(1-\frac{Y_a^*}{Y_a})[b_a(T,P)f_aI_a - h_a(T,P)Y_a] \\
& + w_3(1-\frac{S_a^*}{S_a})[h_a(T,P)Y_a^* + \mu_a(T)S_a^* + \lambda_{la}(T)\frac{I_l^*}{N_l}S_a^* + \lambda_{ha}(T)\frac{I_h^*}{N_h}S_a^*] \\
& + w_3(1-\frac{S_a^*}{S_a})[-h_a(T,P)Y_a - \mu_a(T)S_a - \lambda_{la}(T)\frac{I_l}{N_l}S_a - \lambda_{ha}(T)\frac{I_h}{N_h}S_a] \\
& + w_4(1-\frac{E_a^*}{E_a})[\lambda_{la}(T)\frac{I_l}{N_l}S_a + \lambda_{ha}(T)\frac{I_h}{N_h}S_a - (\varepsilon_a(T)+\mu_a(T))E_a] \\
& + w_5(1-\frac{I_a^*}{I_a})[h_a(T,P)Y_a + \varepsilon_a(T)E_a - \mu_a(T)I_a] + w_6(1-\frac{X_c^*}{X_c})[h_c(T,P)X_c^* - h_c(T,P)X_c] \\
& + w_7(1-\frac{S_c^*}{S_c})[\mu_c(T)S_c^* + \lambda_{lc}(T)\frac{I_l^*}{N_l}S_c^* + \lambda_{hc}(T)\frac{I_h^*}{N_h}S_c^* - \mu_c(T)S_c - \lambda_{lc}(T)\frac{I_l}{N_l}S_c - \lambda_{hc}(T)\frac{I_h}{N_h}S_c] \\
& + w_8(1-\frac{E_c^*}{E_c})[\lambda_{lc}(T)\frac{I_l}{N_l}S_c + \lambda_{hc}(T)\frac{I_h}{N_h}S_c - (\varepsilon_c(T)+\mu_c(T))E_c] \\
& + w_9(1-\frac{I_c^*}{I_c})[\varepsilon_c(T)E_c - \mu_c(T)I_c] \\
& + w_{10}(1-\frac{S_l^*}{S_l})[\mu_l S_l^* + \lambda_{al}(T)\frac{I_a^*}{N_a}S_l^* + \lambda_{cl}(T)\frac{I_c^*}{N_c}S_l^* - \mu_l S_l - \lambda_{al}(T)\frac{I_a}{N_a}S_l - \lambda_{cl}(T)\frac{I_c}{N_c}S_l] \\
& + w_{11}(1-\frac{E_l^*}{E_l})[\lambda_{al}(T)\frac{I_a}{N_a}S_l + \lambda_{cl}(T)\frac{I_c}{N_c}S_l - (\varepsilon_l+\mu_l)E_l] \\
& + w_{12}(1-\frac{I_l^*}{I_l})[\varepsilon_l E_l - (\mu_l+\phi_l+\gamma_l)I_l] \\
& + w_{13}(1-\frac{S_h^*}{S_h})[\mu_h S_h^* + \lambda_{lh}\frac{I_l^*}{N_l}S_h^* + \lambda_{ah}(T)\frac{I_a^*}{N_a}S_h^* + \lambda_{ch}(T)\frac{I_c^*}{N_c}S_h^*] \\
& + w_{13}(1-\frac{S_h^*}{S_h})[-\mu_h S_h - \lambda_{lh}\frac{I_l}{N_l}S_h - \lambda_{ah}(T)\frac{I_a}{N_a}S_h - \lambda_{ch}(T)\frac{I_c}{N_c}S_h] \\
& + w_{14}(1-\frac{E_h^*}{E_h})[\lambda_{lh}\frac{I_l}{N_l}S_h + \lambda_{ah}(T)\frac{I_a}{N_a}S_h + \lambda_{ch}(T)\frac{I_c}{N_c}S_h - (\varepsilon_h+\mu_h)E_h] \\
& + w_{15}(1-\frac{I_h^*}{I_h})[\varepsilon_h E_h - (\mu_h+\phi_h+\gamma_h)I_h].\end{aligned}$$

Further simplification gives

$$\begin{aligned}\frac{dV}{dt} = {} & -w_1(1-\frac{X_a^*}{X_a})^2 h_a(T,P)X_a - w_3(1-\frac{S_a^*}{S_a})^2\mu_a S_a - w_6(1-\frac{X_c^*}{X_c})^2 h_c(T,P)X_c \\
& - w_7(1-\frac{S_c^*}{S_c})^2\mu_c S_c - w_{10}(1-\frac{S_l^*}{S_l})^2\mu_l S_l - w_{13}(1-\frac{S_h^*}{S_h})^2\mu_h S_h + F(\mathcal{D}),\end{aligned} \tag{39}$$

where

$$\mathcal{D} = \{(X_a, Y_a, S_a, E_a, I_a, X_c, S_c, E_c, I_c, S_l, E_l, I_l, S_h, E_h, I_h) > 0\}$$

and

$$\begin{aligned}
F(\mathcal{D}) = {} & w_1(1-\frac{X_a^*}{X_a})[b_a(T,P)I^*a - b_a(T,P)f_aI_a)] + w_2(1-\frac{Y_a^*}{Y_a})[b_a(T,P)f_aI_a - h_a(T,P)Y_a] \\
& + w_3(1-\frac{S_a^*}{S_a})[h_a(T,P)Y_a^* - h_a(T,P)Y_a + \lambda_{la}(T)\frac{I_l^*}{N_l}S_a^* + \lambda_{ha}(T)\frac{I_h^*}{N_h}S_a^*] \\
& + w_3(1-\frac{S_a^*}{S_a})[-\lambda_{la}(T)\frac{I_l}{N_l}S_a - \lambda_{ha}(T)\frac{I_h}{N_h}S_a] \\
& + w_4(1-\frac{E_a^*}{E_a})[\lambda_{la}(T)\frac{I_l}{N_l}S_a + \lambda_{ha}(T)\frac{I_h}{N_h}S_a - (\varepsilon_a(T)+\mu_a(T))E_a] \\
& + w_5(1-\frac{I_a^*}{I_a})[h_a(T,P)Y_a + \varepsilon_a(T)E_a - \mu_a(T)I_a] \\
& + w_7(1-\frac{S_c^*}{S_c})[\lambda_{lc}(T)\frac{I_l^*}{N_l}S_c^* + \lambda_{hc}(T)\frac{I_h^*}{N_h}S_c^* - \lambda_{lc}(T)\frac{I_l}{N_l}S_c - \lambda_{hc}(T)\frac{I_h}{N_h}S_c] \\
& + w_8(1-\frac{E_c^*}{E_c})[\lambda_{lc}(T)\frac{I_l}{N_l}S_c + \lambda_{hc}(T)\frac{I_h}{N_h}S_c - (\varepsilon_c(T)+\mu_c(T))E_c] \\
& + w_9(1-\frac{I_c^*}{I_c})[\varepsilon_c(T)E_c - \mu_c(T)I_c] \qquad (40) \\
& + w_{10}(1-\frac{S_l^*}{S_l})[\lambda_{al}(T)\frac{I_a^*}{N_a}S_l^* + \lambda_{cl}(T)\frac{I_c^*}{N_c}S_l^* - \lambda_{al}(T)\frac{I_a}{N_a}S_l - \lambda_{cl}(T)\frac{I_c}{N_c}S_l] \\
& + w_{11}(1-\frac{E_l^*}{E_l})[\lambda_{al}(T)\frac{I_a}{N_a}S_l + \lambda_{cl}(T)\frac{I_c}{N_c}S_l - (\varepsilon_l+\mu_l)E_l] \\
& + w_{12}(1-\frac{I_l^*}{I_l})[\varepsilon_l E_l - (\mu_l+\phi_l+\gamma_l)I_l] \\
& + w_{13}(1-\frac{S_h^*}{S_h})[\lambda_{lh}\frac{I_l^*}{N_l}S_h^* + \lambda_{ah}(T)\frac{I_a^*}{N_a}S_h^* + \lambda_{ch}(T)\frac{I_c^*}{N_c}S_h^*] \\
& + w_{13}(1-\frac{S_h^*}{S_h})[-\lambda_{lh}\frac{I_l}{N_l}S_h - \lambda_{ah}(T)\frac{I_a}{N_a}S_h - \lambda_{ch}(T)\frac{I_c}{N_c}S_h] \\
& + w_{14}(1-\frac{E_h^*}{E_h})[\lambda_{lh}\frac{I_l}{N_l}S_h + \lambda_{ah}(T)\frac{I_a}{N_a}S_h + \lambda_{ch}(T)\frac{I_c}{N_c}S_h - (\varepsilon_h+\mu_h)E_h] \\
& + w_{15}(1-\frac{I_h^*}{I_h})[\varepsilon_h E_h - (\mu_h+\phi_h+\gamma_h)I_h].
\end{aligned}$$

$F(\mathcal{D})$ is non-positive by following the approach of McCluskey [21] and Korobeinikov [4,7-9]. Thus, $F(\mathcal{D}) \leq 0$ for all $\mathcal{D}$. Hence, $\frac{dV}{dt} \leq 0$ in $\mathcal{D}$ and is zero when $\mathcal{D} = \mathcal{D}^*$. Therefore, the largest compact invariant set in $\mathcal{D}$ such that $\frac{dV}{dt} = 0$ is the singleton $\{\mathcal{D}^*\}$ which is the endemic equilibrium point. LaSalle's invariant principle then implies that $\mathcal{D}^*$ is globally asymptotically stable (GAS) in the interior of $\mathcal{D}$. Thus, we have established the following result:

Propositon 2.3. *If $\mathcal{R}_0 > 1$, then, system (9) has a unique endemic equilibrium point $\mathcal{D}^*$ which is GAS in $\mathcal{D}$.*

3. Numerical results and discussion

To explore the dynamical behaviour of RVF in the a closed system and illustrate some analytical results, numerical simulations were carried for the case climate change parameter are considered constant and when the climate change parameters are considered to change with temperature and precipitation. The state initial values used for simulations are $X_a(0) = 4999$, $Y_a(0) = 1$, $S_a(0) = 4500$, $E_a(0) = 499$, $I_a(0) = 1$, $X_c(0) = 5000$, $S_c(0) = 4500$, $E_c(0) = 499$, $I_c(0) = 1$, $S_l(0) = 1000$, $E_l(0) = 0$, $I_l(0) = 0$, $R_l(0) = 0$, $S_h(0) = 1000$, $E_h(0) = 0$, $I_h(0) = 0$, and $R_h(0) = 0$. For the case where climate change parameters are considered to be constant, the parameter values in Table 2 were used for simulations.

Figure 1 shows the plot of the RVF model classes with time over a period of one year (365 days) when the climate change parameters were considered to be constant. The simulations results indicates the existence of both DFE and EE of the RVF dynamics, and that these equilibria are stable whenever they exist.

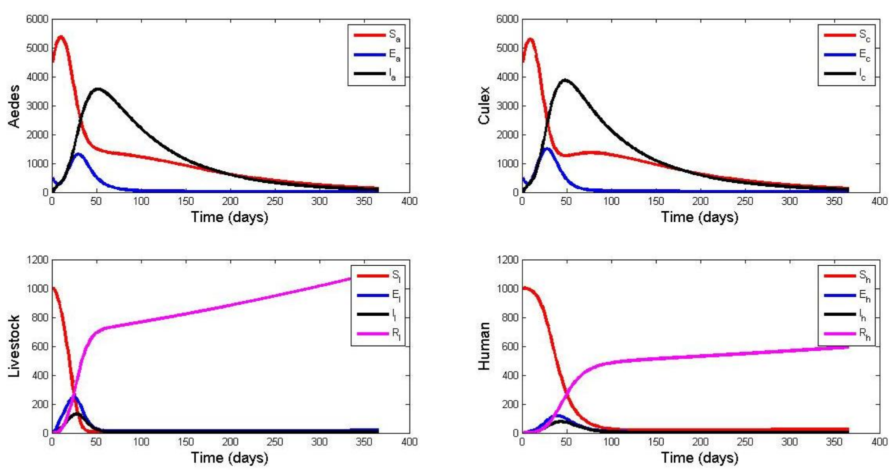

FIGURE 1. Plot of RVF model without impact of climate change

To perform numerical simulations of RVF model for the case of variable climate change parameters, temperature and precipitation data from Tanzania were used. The climate change parameters with their expressions as used in this part are given in Table 3. Detailed explanations on sources and/or derivation of the the climate change parameter expressions can found in Mpeshe *et al.* [18].

Figure 2 shows the plot of the RVF model classes over a period of one year (365 days) when climate change parameters are taken into account. The simulations results also indicate the existence of both DFE and EE of the RVF dynamics, and that these equilibria are stable whenever they exist.

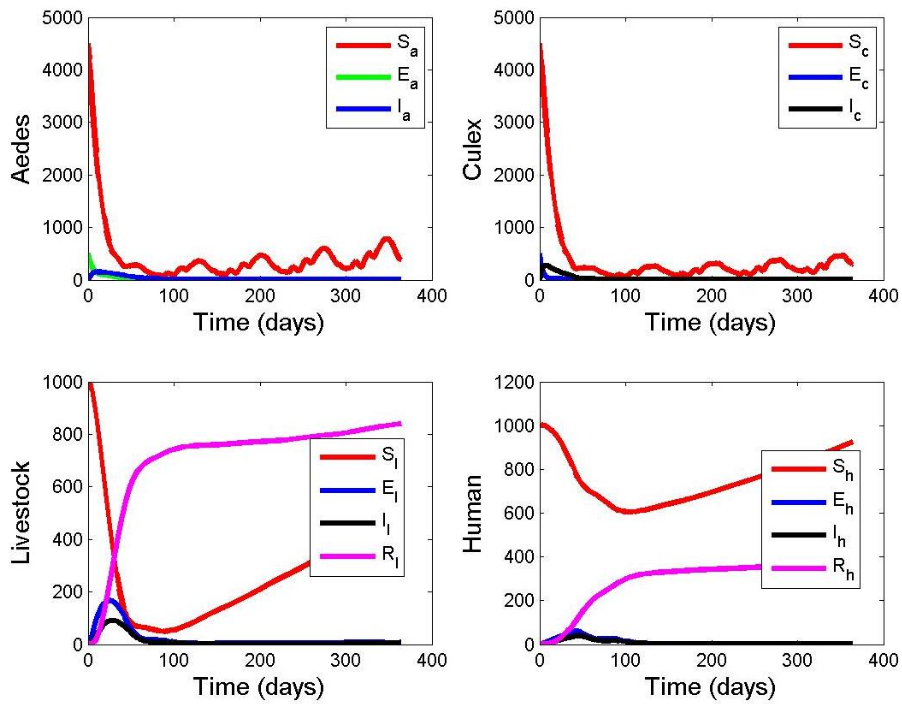

FIGURE 2. Plot of RVF model with impact of climate change

4. Conclusion

Stability analysis of the RVF model has been performed and numerical simulations of the model in a closed population has been performed in the case when there is no impact on the climate change and when there is impact in climate change. Analytical results shows that both the DFE and EE are globally asymptotically stable whenever they exists. These results are in conjecture with the results from numerical simulations when the impact of climate change has been considered and when no climate change is considered.

Conflict of Interests

The authors declare that there is no conflict of interests.

Acknowledgments

The authors would like to thank the University of Iringa and the Nelson Mandela African Institution of Science and Technology (NM-AIST) for their support. However, the conclusions made so far from this study are not influenced by any institution.

REFERENCES

[1] J. P. LaSalle, S. Lefschetz, The stability by Lyapunov's direct method, New York: Academic (1961).

[2] J. P. LaSalle, The stability of dynamical systems, Philadelphia: SIAM (1976).

[3] B.-S. Goh, Management and analysis of biological, Amsterdam: Elsevier Science (1980).

[4] A. Korobeinikov, G. C. Wake, Lyapunov functions and global stability for SIR, SIRS, and SIS epidemiological models, Appl. Math. Lett. 15 (2002), 955-961.

[5] C. Vargas-De-León, Constructions of lyapunov functions for classic SIS, SIR and SIRS epidemic models with variable population size, Revista Electrónica Foro Red. Mat. 26 (2009), 1-12.

[6] C. Vargas-De-León, On the global stability of infectious diseases models with relapse, Abstraction and Application, 9 (2013), 50-61.

[7] A. Korobeinikov, Global properties of basic virus dynamicals models, Bull. Math. Biol. 66 (2004), 879-883.

[8] A. Korobeinikov, Lyapunov functions and global properties for SEIR and SEIS epidemic models, Math. Med. Biol. 21 (2004), 75-83.

[9] A. Korobeinikov, Global properties of infectious disease models with nonlinear incidence, Bull. Math. Biol. 69 (2007), 1871- 1886.

[10] M. Y. Li, J. S. Muldowney, A geometric approach to global stability problems, SIAM J. Math. Anal. 24 (1996), 1070-1083.

[11] J. C. Kamgang, G. Sallet, Computation of threshold conditions for epidemiological models and global stability of the disease-free equilibrium (DFE), Math. Biosci. 213 (2008), 1-12.

[12] M. Twardy, On the alternative stability criteria for positive systems, Bull. Pol. Ac.: Tech. 55 (2007), 379-383.

[13] W. Mitkowski, Dynamical properties of Metzler systems, Bull. Pol. Ac.: Tech. 56 (2008), 309-312.

[14] J. A. Jacquez, C. P. Simon, Qualitative theory of compartmental systems, SIAM Rev. 35 (1993), 43-79.

[15] J. A. Jacquez, C. P. Simon, J. S. Koopman, Core groups and the $\mathcal{R}_0$s for sub groups in heterogeneous SIS models, in: D. Mollison (Ed.), Epidemic models: Their structure and relation to data, UK: Cambridge University Press (1996), pp.279-301.

[16] D. G. Luenberger, Introduction to dynamical systems, New York: John Wileys and Sons (1979).

[17] A. Takayama, Mathematical economics (2nd edition), Cambridge: Cambridge University Press (1985).

[18] S. C. Mpeshe, L. S. Luboobi, Y. Nkansah-Gyekye, Modeling the impact of climate change on the dynamics of Rift Valley fever, Comput. Math. Methods Med. 2014 (2014), 1-12.

[19] C. Castillo-Chavez, Z. Feng, W. Huang, On the computation of $\mathcal{R}_0$ and its role in global stability, in: C. Castillo-Chavez, P. van den Driessche, D. Kirschner, A. A. Yakubu (Eds.), Mathematical approaches for emerging and reemerging infection diseases: an introduction, Vol. 125, New York: Springer (2002), pp.31-65.

[20] P. van den Driessche, J. Watmough, Reproduction numbers and sub-threshold endemic equilibria for compartmental models of disease transmission, Math. Biosci. 180 (2002), 29-48.

[21] C. C. McCluskey, Lyapunov functions for tuberculosis models with fast and slow progression, Math. Biosci. Eng. 3 (2006), 603-614.

TABLE 1. Parameters and their description

Parameter	Description
$1/h_a(T,P)$	development time of *Aedes* mosquitoes
$1/h_c(T,P)$	development time of *Culex* mosquitoes
$b_a(T,P)$	number of *Aedes* eggs laid per day
$b_c(T,P)$	number of *Culex* eggs laid per day
b_h	daily birth rate in humans
b_l	daily birth rate in livestock
$1/\mu_a(T)$	lifespan of *Aedes* mosquitoes
$1/\mu_c(T)$	lifespan of *Culex* mosquitoes
$1/\mu_h$	lifespan of humans
$1/\mu_l$	lifespan of livestock
$1/\varepsilon_a(T)$	Incubation period of *Aedes* mosquitoes
$1/\varepsilon_c(T)$	Incubation period of *Culex* mosquitoes
$1/\varepsilon_h$	Incubation period of humans
$1/\varepsilon_l$	Incubation period of livestock
ϕ_l	Death rate of livestock due to disease
ϕ_h	Death rate of humans due to disease
$1/\gamma_l$	Infectious period in livestock
$1/\gamma_h$	Infectious period in humans
$\lambda_{al}(T)$	Adequate contact rate: *Aedes* to livestock
$\lambda_{cl}(T)$	Adequate contact rate: *Culex* to livestock
$\lambda_{la}(T)$	Adequate contact rate: livestock to *Aedes*
$\lambda_{lc}(T)$	Adequate contact rate: livestock to *Culex*
$\lambda_{ah}(T)$	Adequate contact rate: *Aedes* to humans
$\lambda_{ch}(T)$	Adequate contact rate: *Culex* to humans
$\lambda_{ha}(T)$	Adequate contact rate: humans to *Aedes*
$\lambda_{hc}(T)$	Adequate contact rate: humans to *Culex*
λ_{lh}	Adequate contact rate: livestock to humans
f_a	vertical transmission rate in *Aedes*

TABLE 2. Parameter values used for numerical simulations of RVF model with constant climate change parameters

Parameter	Value	Parameter	Value
$1/b_a$	200	$1/b_c$	200
$1/h_a$	20	$1/h_c$	20
b_h	0.0015	b_l	0.0025
$1/\mu_a$	60 days	$1/\mu_c$	60 days
$1/\mu_h$	60 yrs	$1/\mu_l$	10 yrs
$1/\varepsilon_a$	8 days	$1/\varepsilon_c$	8 days
$1/\varepsilon_h$	6 days	$1/\varepsilon_l$	6 days
ϕ_l	0.10	ϕ_h	0.10
$1/\gamma_l$	5 days	$1/\gamma_h$	7 days
λ_{al}	0.48	λ_{cl}	0.13
λ_{la}	0.395	λ_{lc}	0.56
λ_{ah}	0.025	λ_{ch}	0.065
λ_{ha}	0.0125	λ_{hc}	0.025
λ_{lh}	0.002	f_a	0.1

TABLE 3. Climate change parameters with their expressions

Parameter expression	Values of constants
$h(T,P) = \frac{\rho(T)\rho(P)}{d(T)}$	$1/d_a(T) = \alpha_1 T^3 + \alpha_2 T^2 + \alpha_3 T + \alpha_4$
	$\alpha_1 = -0.0025, \alpha_2 = 0.2069, \alpha_3 = -5.5285, \alpha_4 = 48.2951$
	$1/d_c(T) = \alpha_1 T^2 + \alpha_2 T + \alpha_3$
	$\alpha_1 = 0.0095, \quad \alpha_2 = -0.4684, \quad \alpha_3 = 5.8343$
	$\rho(P) = (1 - e^{-\beta_1(P-P_1)})(1 - e^{-\beta_2(P_2-P)})$
	$\beta_1 a = -0.0015, \quad \beta_2 a = -0.000015, \quad \beta_1 c = -0.0025,$
	$\beta_2 c = -0.000025, \quad P_1 = 10, \quad P_2 = 250,$
	$\rho(P) = 0 \quad \text{for} \quad P < P_1 \quad \text{and} \quad P > P_2$
$b(T,P) = b_0 + \frac{E_{\max}}{1+e^{\{\frac{-(Im-\overline{E})}{E_{var}}\}}}$	$I_m = 100(\frac{r}{E_0} - 1), \quad E_{max} = 20, \quad \overline{E} = 0,$
	$E_{var} = 12, \quad b_0 = 0$
$1/\mu_a(T) = a_0 - a_1 T$	$a_0 = 25.8, a_1 = 0.45$
$1/\mu_c(T) = a_0 - a_1 T$	$a_0 = 69.1, a_1 = 2.14$
$1/\varepsilon_a(T) = \varepsilon_{max} - \varepsilon_{slope} T$	$\varepsilon_{max} = 18.9, \quad \varepsilon_{slope} = 0.30$
$1/\varepsilon_c(T) = \varepsilon_{max} - \varepsilon_{slope} T$	$\varepsilon_{max} = 11.3, \quad \varepsilon_{slope} = 0.30$
$\lambda_i(T) = a_s(T - T_{min})\rho_i$	$i = al, cl, la, lc, ah, ha, ch, hc, a_s = 0.0173, T_{min} = 9.60$
	$\rho_{al} = 0.70, \quad \rho_{cl} = 0.78, \quad \rho_{la} = 0.38, \quad \rho_{lc} = 0.22$
	$\rho_{ah} = 0.01, \quad \rho_{ha} = 0.05, \quad \rho_{ch} = 0.01, \quad \rho_{hc} = 0.015$