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PROPERTIES OF EIGENVALUES AND ESTIMATION OF EIGENFUNCTIONS TO THE SOME TYPE OF THE DELAY SPECTRAL PROBLEM

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Abstract. This paper was devoted to the study of the properties of eigenvalues and estimating the eigenfunctions to the some type of the delay spectral problem, and also the eigenfunction was found where the eigenvalue is real with constant weight function.

Keywords: delay spectral problem; boundary value problems; eigenvalues; eigenfunctions.

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1. Introduction

Many topics in mathematical physics lead to the problem of determining the eigenvalues and eigenfunctions of differential operators and of expanding an arbitray function as a series (or an integral) of eigenfunctions. A problem of this sort is encountered, for example, whenever the Fourier method is used to find a solution of a partial differential equations under prescribed initial and boundary conditions. Consequently a lot of attention has been paid to differential operators and they are the subject of much current research. The spectral theory of differential operators appears as the basic mathematical method for investigating many topics in quantum

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mechanics[10]. Most of the published work deals with the spectral problem with different bounary conditions, bellow we state some of these works.

Karwan and Aryan [5] identified some properties of eigenvalues and they studied the estimation of the eigenfunctions corresponding to the eigenvalues for the spectral problem

$$-y''(x)+y'(x)=\lambda^2\rho(x)y(x), \quad x\in[0,a], \quad a>0,$$

with the boundary conditions:

$$y(0)=y'(0)=y(a)+y'(a)=0, \int_0^a y'(x)\bar{y}(x)dx=\tau^2, \sqrt{\int_0^a \rho(x)|y(x)|^2dx}=1.$$

Karwan and Khelan [6] have studied the estimation of normalized eigenfunctions of the spectral problem

$$-y''+q(x)y=\lambda^2\rho(x)y, \quad x\in(0,a)$$

with the boundary conditions

$$y'(0)=y(a)+y'(a)=0, \sqrt{\int_0^a \rho(x)|y(x)|^2dx}=1,$$

for the different cases of weight function $\rho(x)$ where the eigenvalue λ is real and $q(x)$ is a constant. For the first time an Italian physicist Regge.T [11] has studied the differential equation

$$-y''(x)+q(x)y(x)=\lambda^2\rho(x)y(x), \quad x\in(0,a),$$

with the boundary conditions

$$y(0)=0, y'(a)-i\lambda y(a)=0,$$

and was considered by who showed that the system of eigenfunctions of this problem are completed and studied asymptotic behavior of eigenvalues of this problem when $\rho(x)=1$. Kravitsky [7] specified a class of functions that allowed expansion in uniformly convergent series in eigenfunctions and associated functions in the Regge problem when $\rho(x)=1$. Until the present time there are many Authors studied the estimation of eigenfunctions to the equation

$$-y''(x)+y(x)=\lambda^2\rho(x)y(x),$$

but with different boundary conditions. for more known about their works see [1,2,4,8,9]. Gesztesy and Tkachenko [3] were studied generally, non-self-adjoint Schrodinger operators H^P and H^{AP} in the Hilbert space $L^2([0,\pi];dx)$ associated with the differential expression

$$L = -\frac{d^2}{dx^2} + V(x), x \in [0,\pi]$$

and complex-valued potential V satisfying $V \in (L^2[0,\pi];dx)$, with periodic and antiperiodic boundary conditions defined by

$$(H^P f)(x) = (Lf)(x), x \in [0,\pi],$$

$f \in dom(H^P)$

$$= \{g \in L^2([0,\pi];dx) | g, g' \in AC([0,\pi]); Lg \in L^2([0,\pi];dx); g(\pi) = g(0), g'(\pi) = g'(0)\},$$

and

$$(H^{AP} f)(x) = (Lf)(x), x \in [0,\pi],$$

$f \in dom(H^{AP})$

$$= \{g \in L^2([0,\pi];dx) | g, g' \in AC([0,\pi]); Lg \in L^2([0,\pi];dx); g(\pi) = -g(0), g'(\pi) = -g'(0)\},$$

respectively. In addition to the periodic and antiperiodic Schrodinger operators H^P and $H^A P$ also they invoked the corresponding Dirichlet operator H^D in $L^2([0,\pi];dx)$ which is defined by

$$(H^D f)(x) = (Lf)(x), x \in [0,\pi],$$

$$f \in dom(H^D) = \{g \in L^2([0,\pi];dx) | g, g' \in AC([0,\pi]); Lg \in L^2([0,\pi];dx); g(\pi) = g(0) = 0\},$$

also they mentioned the Neumann operator H^N in $L^2([0,\pi];dx)$ defined by

$$(H^N f)(x) = (Lf)(x), x \in [0,\pi],$$

$$f \in dom(H^N) = \{g \in L^2([0,\pi];dx) | g, g' \in AC([0,\pi]); Lg \in L^2([0,\pi];dx); g'(\pi) = -g'(0) = 0\},$$

where $H^P, H^A P, H^D$, and H^N are closed and densely defined; they are self-adjoint if and only if V is real-valued on$[0,\pi]$. In particular, the boundary conditions in $H^P, H^A P, H^D$, and H^N are all self-adjoint and $AC([0,\pi])$ is the set of absolutely continuous functions on $[0,\pi])$ and The Schrodinger equation associated with L will be written in the form

$$L\psi(\zeta,x) = \zeta^2 \psi(\zeta,x), \zeta \in \mathbb{C}, x \in [0,\pi]$$

with $\psi, \psi' \in AC([0,\pi])$.

The importance of this work appear in the applied fields of engineering and physics because is the first time have been studying the properties of eigenvalues and estimating the eigenfunctions for this type of delay spectral problem which open the doors for studying this type of problem but with different boundary conditions and also for studying this type of problem with higher orders.

this paper is focused on the study of properties of eigenvalues and estimating the eigenfunctions corresponding to the eigenvalues of the delay spectral problem which is defined by:

$$\begin{aligned} -y''(x-\tau)+y'(x-\tau) &= \lambda^2\rho y(x-\tau), \quad x\in(0,a), \quad a,\tau>0, \\ y'(a-\tau) = y'(-\tau)-y(-\tau) &= 0, \\ \int_0^a y'(x-\tau)\bar{y}(x-\tau)dx &= \gamma, \end{aligned} \tag{1.1}$$

where γ is a positive constant, and λ is a spectral parameter.

The plan of the paper is as follows: Section two presents some auxiliary materials. Section three presents properties of eigenvalues of the given problem . Section four present properties of eigenfunctions of the given problem. Finally, Section five presents a short conclusion.

2. Auxiliary materials

The following symbols are used throughout the entire work, and each one is defined in the following.

λ is a spectral parameter and $\lambda = \alpha + i\beta$ where $i = \sqrt{-1}$ and $\alpha, \beta \in \mathbb{R}$. Two positive real constants m and M are chosen so that $0 < m < M$. The symbol ρ is refered to the positive constant weight function such that $0 < m \le \rho \le M$. $\mathbb{R}$ refers to the set of all real numbers. a, τ and γ are psoitive constants.

In this work, we are attempt to determine the features of the eigenvalues and estimating the eigenfunctions corresponding to the eigenvalues of the delay differential equation with the given boundary conditions (1.1).

3. Properties of eigenvalues

This section devoted to specify the features of the eigenvalues of the given delay spectral problem with the defined boundary conditions.

Theorem 3.1. *Let λ be an eigenvalue corresponding to the eigenfunction $y(x-\tau)$ of the given problem, then:*

1) If $\beta \neq 0$, then λ is a complex number.

2) If $\alpha \neq 0$, then λ is a real number.

Proof. Multiplying equation (1.1) by $\bar{y}(x-\tau)$ and integrating with respect to x from 0 to a, yields

$$-\int_0^a y''(x-\tau)\bar{y}(x-\tau)dx+\int_0^a y'(x-\tau)\bar{y}(x-\tau)dx=\lambda^2\int_0^a \rho y(x-\tau)\bar{y}(x-\tau)dx$$

$$-\int_0^a y''(x-\tau)\bar{y}(x-\tau)dx+\gamma=\lambda^2\int_0^a \rho|y(x-\tau)|^2dx.$$

Now, integrating $\int_0^a y''(x-\tau)\bar{y}(x-\tau)dx$ by parts and using the boundary conditions (1.1), gives

$$\int_0^a |y'(x-\tau)|^2dx+|y(-\tau)|^2+\gamma=\lambda^2\int_0^a \rho|y(x-\tau)|^2dx. \tag{3.1}$$

Now, after putting $\bar{y}(x-\tau)$ on place $y(x-\tau)$ in equations (1.1), these equations are transformed to:

$$\begin{gathered}-\bar{y}''(x-\tau)+\bar{y}'(x-\tau)=\bar{\lambda}^2\rho\bar{y}(x-\tau),\\ \bar{y}'(a-\tau)=\bar{y}'(-\tau)-\bar{y}(-\tau)\quad=0,\\ \int_0^a \bar{y}'(x-\tau)y(x-\tau)dx=\gamma.\end{gathered} \tag{3.2}$$

Multiplying the above delay differential equation (3.2) by $y(x-\tau)$ and integrate with respect to x from 0 up to a, we get

$$-\int_0^a \bar{y}''(x-\tau)y(x-\tau)dx+\gamma=\bar{\lambda}^2\int_0^a \rho|y(x-\tau)|^2dx.$$

Now, integrating $\int_0^a \bar{y}''(x-\tau)y(x-\tau)dx$ by parts and using the boundary conditions (3.2), we deduce

$$\int_0^a |y'(x-\tau)|^2 dx + |y(-\tau)|^2 + \gamma = \overline{\lambda}^2 \int_0^a \rho |y(x-\tau)|^2 dx. \tag{3.3}$$

By subtracting equation (3.3) from equation (3.1), we obtain

$$(\lambda^2 - \overline{\lambda}^2) = 0 \longrightarrow (\lambda - \overline{\lambda})(\lambda + \overline{\lambda}) \quad = 0 \longrightarrow (\lambda - \overline{\lambda}) = 0 \quad or \quad (\lambda + \overline{\lambda}) \quad = 0,$$

then we have two cases:
Case 1: If $\beta \neq 0$, so $(\lambda - \overline{\lambda}) \neq 0$, thus $(\lambda + \overline{\lambda}) = 0$, this means that λ is a complex number.
Case 2: If $\alpha \neq 0$, this gives that $(\lambda + \overline{\lambda}) \neq 0$, hence $(\lambda - \overline{\lambda}) = 0$, then $\lambda = \overline{\lambda}$, therefore λ is a real number. Here's the proof of the above theorem is completed.

4. Estimation of Eigenfunctions of the given problem

This section concern to study of the assessment of the eigenfunction $y(x-\tau)$ corresponding to the eigenvalue λ of the given delay spectral problem.

Theorem 4.1. *Let $y(x-\tau)$ be an eigenfunction and λ be an eigenvalue corresponding to $y(x-\tau)$ of the given delay spectral problem, and if $\alpha \neq 0$, then $\max_{x\in(0,a)} |y(x-\tau)| \leq \sqrt{k_2|\lambda| + k_1}$, where $k_1, k_2 > 0$ and $k_1, k_2 \in \mathbb{R}$.*

Proof. Let x be any point in $(0,a)$, and let us consider the identity

$$\begin{aligned} |y(x-\tau)|^2 &= \int_0^x (\bar{y}(s-\tau)y'(s-\tau) + y(s-\tau)\bar{y}'(s-\tau))ds + |y(-\tau)|^2 \\ &= \int_0^x \frac{\sqrt{\rho}(\bar{y}(s-\tau)y'(s-\tau) + y(s-\tau)\bar{y}'(s-\tau))}{\sqrt{\rho}} ds + |y(-\tau)|^2. \end{aligned}$$

From the inequality $\rho \geq m$, we obtain

$$
\begin{aligned}
|y(x-\tau)|^2 &\leq \frac{1}{\sqrt{m}} \int_0^x \sqrt{\rho} |\bar{y}(s-\tau)y'(s-\tau) + y(s-\tau)\bar{y}'(s-\tau)| ds + |y(-\tau)|^2 \\
&\leq \frac{1}{\sqrt{m}} \int_0^x \sqrt{\rho} |\bar{y}(s-\tau)y'(s-\tau)| ds + \int_0^x \sqrt{\rho} |y(s-\tau)\bar{y}'(s-\tau)| ds + |y(-\tau)|^2 \\
&\leq \frac{1}{\sqrt{m}} \{ \int_0^x \sqrt{\rho} |y(s-\tau)||y'(s-\tau)| ds \\
&\quad + \int_0^x \sqrt{\rho} |y(s-\tau)||y'(s-\tau)| ds + |y(-\tau)|^2 \} \\
&\leq \frac{2}{\sqrt{m}} \int_0^a \sqrt{\rho} |y(s-\tau)||y'(s-\tau)| ds + |y(-\tau)|^2.
\end{aligned}
$$

Using Cauchy-Schwartz inequality on the last inequality, we will obtain

$$
|y(x-\tau)|^2 \leq \frac{2}{\sqrt{m}} \sqrt{\int_0^a \rho(s-\tau)|y(s-\tau)|^2 ds} \sqrt{\int_0^a |y'(s-\tau)|^2 ds} + |y(-\tau)|^2. \tag{4.1}
$$

From equation (3.1) (by what we have done in the proof of theorem (3.1)), we have

$$
\int_0^a |y'(x-\tau)|^2 dx = \lambda^2 \int_0^a \rho |y(x-\tau)|^2 dx - |y(-\tau)|^2 - \gamma.
$$

Putting this equation in equation (4.1), we get

$$
|y(x-\tau)|^2 \leq \frac{2}{\sqrt{m}} \sqrt{\int_0^a \rho |y(s-\tau)|^2 ds} \sqrt{\lambda^2 \int_0^a \rho |y(s-\tau)|^2 ds - |y(-\tau)|^2 - \gamma} + |y(-\tau)|^2.
$$

And since $\alpha \neq 0$, then by theorem (3.1) λ is real, so $\lambda^2 = |\lambda|^2$, hence

$$
\begin{aligned}
&|y(x-\tau)|^2 \\
&\leq \frac{2}{\sqrt{m}} \sqrt{\int_0^a \rho |y(s-\tau)|^2 ds} \sqrt{|\lambda|^2 \int_0^a \rho |y(s-\tau)|^2 ds \left(1 - \frac{|y(-\tau)|^2 + \gamma}{|\lambda|^2 \int_0^a \rho |y(s-\tau)|^2 ds} \right)} + |y(-\tau)|^2,
\end{aligned}
$$

or

$$|y(x-\tau)|^2 \leq \frac{2|\lambda|}{\sqrt{m}} \left(\int_0^a \rho |y(s-\tau)|^2 ds \right) \sqrt{1 - \frac{|y(-\tau)|^2 + \gamma}{|\lambda|^2 \int_0^a \rho |y(s-\tau)|^2 ds}} + |y(-\tau)|^2.$$

Since $|y(s-\tau)|^2, \rho > 0$, then $\int_0^a \rho |y(s-\tau)|^2 ds > 0$, so let $\int_0^a \rho |y(s-\tau)|^2 ds$ = positive real constant= k^2, therefore the last inequality becomes

$$|y(x-\tau)|^2 \leq \frac{2|\lambda| k^2}{\sqrt{m}} \sqrt{1 - \frac{|y(-\tau)|^2 + \gamma}{|\lambda|^2 k^2}} + |y(-\tau)|^2$$

$$|y(x-\tau)|^2 \leq k_2 |\lambda| + k_1, \text{ where } k_1 \ = |y(-\tau)|^2 > 0 \text{ and } k_2 = \frac{2k^2}{\sqrt{m}} > 0,$$

or

$$|y(x-\tau)| \leq \sqrt{k_2 |\lambda| + k_1}.$$

And since x is any value in the interval $(0,a)$, hence

$$\max_{x \in (0,a)} |y(x-\tau)| \leq \sqrt{k_2 |\lambda| + k_1}.$$

This completes the proof.

Theorem 4.2. *Let λ be a real eigenvalue corresponding to the eigenfunction $y(x-\tau)$ of the given problem, if $y(a-\tau) = 1$, then $y(x-\tau) = Ae^{\left(\frac{1}{2} + \frac{\sqrt{1-4\lambda^2\rho}}{2}\right)(x-\tau)} + Be^{\left(\frac{1}{2} - \frac{\sqrt{1-4\lambda^2\rho}}{2}\right)(x-\tau)}$, where*

$$A = \frac{e^{\frac{1}{2}(a+\tau)\sqrt{1-4\lambda^2\rho} - \frac{1}{2}(a-\tau)}}{e^{a\sqrt{1-4\lambda^2\rho}} + \frac{\left(\frac{\sqrt{1-4\lambda^2\rho}}{2} - \frac{1}{2}\right)}{\left(\frac{\sqrt{1-4\lambda^2\rho}}{2} + \frac{1}{2}\right)}}, \quad B = \frac{e^{\left(\frac{\sqrt{1-4\lambda^2\rho}}{2} - \frac{1}{2}\right)(a-\tau)}}{\frac{\left(\frac{\sqrt{1-4\lambda^2\rho}}{2} + \frac{1}{2}\right)}{\left(\frac{\sqrt{1-4\lambda^2\rho}}{2} - \frac{1}{2}\right)} e^{a\sqrt{1-4\lambda^2\rho}} + 1}.$$

Proof. Let $z = x - \tau$, then equation (1.1) with the boundary conditions can be rewritten as:

$$\begin{aligned} -y''(z) + y'(z) = \lambda^2 \rho y(z), \quad z \in (-\tau, a-\tau), \quad a, \tau > 0, \\ y'(a-\tau) = y'(-\tau) - y(-\tau) \qquad = 0, \\ \int_{-\tau}^{a-\tau} y'(z) \overline{y(z)} dz = \gamma. \end{aligned} \tag{4.2}$$

The roots of equation (4.2) are of the form

$$\frac{1 \pm \sqrt{1-4\lambda^2\rho}}{2},$$

as we know here, we have three cases.

Case 1: If $4\lambda^2\rho > 1$, this means that $1-4\lambda^2\rho < 0$, hence the eigenfunction can be written as

$$y(z) = c_1 e^{\left(\frac{1}{2}+it\right)z} + c_2 e^{\left(\frac{1}{2}-it\right)z}, \tag{4.3}$$

where c_1, c_2 are constants and $t = \sqrt{\lambda^2\rho - \frac{1}{4}}$, and $i = \sqrt{-1}$.

$$y'(z) = c_1\left(\frac{1}{2}+it\right)e^{\left(\frac{1}{2}+it\right)z} + c_2\left(\frac{1}{2}-it\right)e^{\left(\frac{1}{2}-it\right)z}.$$

In view of boundary conditions (4.2): $y'(-\tau) - y(-\tau) = 0$, we have

$$c_1\left(\frac{1}{2}+it\right)e^{-\left(\frac{1}{2}+it\right)\tau} + c_2\left(\frac{1}{2}-it\right)e^{-\left(\frac{1}{2}-it\right)\tau} - c_1 e^{-\left(\frac{1}{2}+it\right)\tau} - c_2 e^{-\left(\frac{1}{2}-it\right)\tau} = 0,$$

from this equation, we deduce that

$$c_1 = \frac{\left(it+\frac{1}{2}\right)}{\left(it-\frac{1}{2}\right)} c_2 e^{i2t\tau}. \tag{4.4}$$

Again, from equation (4.2): $y'(a-\tau) = 0$, we have

$$c_1\left(\frac{1}{2}+it\right)e^{\left(\frac{1}{2}+it\right)(a-\tau)} + c_2\left(\frac{1}{2}-it\right)e^{\left(\frac{1}{2}-it\right)(a-\tau)} = 0.$$

After putting equation (4.4) in the last equation and doing some algebraic operation, we obtain

$$\left(\left(\frac{\left(\frac{1}{2}+it\right)^2}{\left(\frac{1}{2}-it\right)^2}\right)e^{i2at} - 1\right)c_2 = 0,$$

but $c_2 \neq 0$, (because if $c_2 = 0$, then $c_1 = 0$ this implies that $y(z) = 0$ and this is a contradiction, since $y(z)$ is an eigenfunction), hence $\left(\frac{(\frac{1}{2}+it)^2}{(\frac{1}{2}-it)^2}\right)e^{i2at} - 1 = 0$, and from this equation we get

$$e^{ia\sqrt{4\lambda^2\rho-1}} = \frac{\left(\frac{1}{2} - i\frac{\sqrt{4\lambda^2\rho-1}}{2}\right)^2}{\left(\frac{1}{2} + i\frac{\sqrt{4\lambda^2\rho-1}}{2}\right)^2}.$$

The last equation gives the complex value for the eigenvalue λ, and this contradicts the assumption of the theorem which states that λ is a real eigenvalue, therefore this case can be neglected.

Case 2: If $4\lambda^2\rho = 1$, so we have repeated roots, then eigenfunction will be written as

$$y(z) = c_3 e^{\frac{1}{2}z} + c_4 z e^{\frac{1}{2}z},$$

where c_3, c_4 are constants. And

$$y'(z) = \frac{1}{2} c_3 e^{\frac{1}{2}z} + \left(\frac{1}{2}z + 1\right) c_4 e^{\frac{1}{2}z}.$$

The boundary condition $y'(-\tau) - y(-\tau) = 0$ gives

$$\frac{1}{2} c_3 e^{-\frac{1}{2}\tau} + \left(-\frac{1}{2}\tau + 1\right) c_4 e^{-\frac{1}{2}\tau} - c_3 e^{-\frac{1}{2}\tau} + c_4 \tau e^{-\frac{1}{2}\tau} = 0,$$

or

$$c_3 = (\tau + 2) c_4.$$

Also, from $y'(a-\tau) = 0$, we have $\frac{1}{2} c_3 e^{\frac{1}{2}(a-\tau)} + \left(\frac{1}{2}(a-\tau) + 1\right) c_4 e^{\frac{1}{2}(a-\tau)} = 0$.

The last equation reduced to: $(a+4)c_4 = 0 \longrightarrow c_4 = 0$ because $a + 4 \neq 0$, this implies that $c_3 = 0$, consequently $y(z) = 0$ $(c!)$, thus we ignore this case also.

Case 3: If $4\lambda^2\rho < 1$, this means that $1 - 4\lambda^2\rho > 0$, thus the eigenfunction can be written as

$$y(z) = c_5 e^{\left(\frac{1}{2}+t\right)z} + c_6 e^{\left(\frac{1}{2}-t\right)z}, \tag{4.5}$$

where c_5, c_6 are constants and $t = \frac{\sqrt{1-4\lambda^2\rho}}{2}$. And

$$y'(z) = \left(\frac{1}{2} + t\right) c_5 e^{\left(\frac{1}{2}+t\right)z} + \left(\frac{1}{2} - t\right) c_6 e^{\left(\frac{1}{2}-t\right)z}.$$

Applying the boundary condition: $y'(-\tau) - y(-\tau) = 0$ gives

$$\left(\frac{1}{2} + t\right) c_5 e^{-\left(\frac{1}{2}+t\right)\tau} + \left(\frac{1}{2} - t\right) c_6 e^{-\left(\frac{1}{2}-t\right)\tau} - c_5 e^{-\left(\frac{1}{2}+t\right)\tau} - c_6 e^{-\left(\frac{1}{2}-t\right)\tau} = 0.$$

From the last equation we conclude

$$c_5 = \frac{\left(t + \frac{1}{2}\right)}{\left(t - \frac{1}{2}\right)} e^{2t\tau} c_6. \tag{4.6}$$

Due to the condition $y'(a-\tau) = 0$ we have

$$\left(\frac{1}{2} + t\right) c_5 e^{\left(\frac{1}{2}+t\right)(a-\tau)} + \left(\frac{1}{2} - t\right) c_6 e^{\left(\frac{1}{2}-t\right)(a-\tau)} = 0.$$

Using equation (4.6) in the obtained equation and simplifying the algebraic steps, we get

$$\left(1-\frac{\left(t+\frac{1}{2}\right)^2}{\left(t-\frac{1}{2}\right)^2}e^{2at}\right)c_6=0. \quad c_6\neq 0,$$

because if so, then $c_5=0$ which implies that $y(z)=0$ (c!) since $y(z)$ is an eigenfunction and must be a non zero function, therefore

$$1-\frac{\left(t+\frac{1}{2}\right)^2}{\left(t-\frac{1}{2}\right)^2}e^{2at}=0.$$

From this equation, and replacing the value of t by $\frac{\sqrt{1-4\lambda^2\rho}}{2}$, we obtain

$$e^{a\sqrt{1-4\lambda^2\rho}}=\frac{\left(\sqrt{1-4\lambda^2\rho}-1\right)^2}{\left(\sqrt{1-4\lambda^2\rho}+1\right)^2}. \tag{4.7}$$

Equation (4.7) can be used for finding the eigenvalues λ to the given problem.

Now, next work is allocated to find c_6 in case (3). By inserting equation (4.6) in equation (4.5), we acquire

$$y(z)=\frac{\left(t+\frac{1}{2}\right)}{\left(t-\frac{1}{2}\right)}e^{2t\tau}c_6e^{\left(\frac{1}{2}+t\right)z}+c_6e^{\left(\frac{1}{2}-t\right)z}.$$

From the condition $y(a-\tau)=1$ we can reach to

$$\frac{\left(t+\frac{1}{2}\right)}{\left(t-\frac{1}{2}\right)}e^{2t\tau}c_6e^{\left(\frac{1}{2}+t\right)(a-\tau)}+c_6e^{\left(\frac{1}{2}-t\right)(a-\tau)}=1,$$

or

$$c_6=\frac{e^{\left(t-\frac{1}{2}\right)(a-\tau)}}{\frac{\left(t+\frac{1}{2}\right)}{\left(t-\frac{1}{2}\right)}e^{2at}+1}, \tag{4.8}$$

using equation (4.8) in equation (4.6) gives us c_5

$$c_5=\frac{e^{(a+\tau)t-\frac{1}{2}(a-\tau)}}{e^{2at}+\frac{\left(t-\frac{1}{2}\right)}{\left(t+\frac{1}{2}\right)}}. \tag{4.9}$$

Hence, placing equation (4.8) and equation (4.9) in equation (4.5) gives the required eigenfunction.

5. Conclusions

Properties of eigenvalues and estimation of the eigenfunctions can be studied when the eigenvalues are complex, and also the boundness of eigenfunctions may be discussed when the weight function be the function of the argument $x-\tau$.

Conflict of Interests

The authors declare that there is no conflict of interests.

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