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A NOTE ON A SUBCLASS OF ANALYTIC FUNCTIONS DEFINED BY RUSCHEWEYH DERIVATIVE AND A NEW GENERALISED MULTIPLIER TRANSFORMATION

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Abstract: In this paper, we consider the operator $RI^{m}_{\alpha,\beta,\lambda} : A(n) \to A(n)$ defined by $RI^{m}_{\alpha,\beta,\lambda} f(z) = (1-\lambda)R^m f(z) + \lambda I^{m}_{\alpha,\beta} f(z),$ where $A(n)$ denote the class of analytic functions in the unit disc $U = \{z : z \in C, |z| < 1\}$, of the form $f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k$, $R^m f(z), m \in N_0 = N \cup \{0\}$ is the Ruscheweyh operator and $I^{m}_{\alpha,\beta} f(z) = z + \sum_{k=n+1}^{\infty}\left(\frac{\alpha + k\beta}{\alpha+\beta}\right)^m a_k z^k$, $n \in N, m \in N_0 = N \cup \{0\},$ $\lambda \geq 0, \beta \geq 0,$ and α a real number with $\alpha + \beta > 0.$ The new subclass $\Re\mathrm{I}^{\lambda}_{n}(m,\mu,\rho,\alpha,\beta)$ of $A(n)$, involving the operator $RI^{m}_{\alpha,\beta,\lambda}$ is introduced. Some interesting properties of the class $\Re\mathrm{I}^{\lambda}_{n}(m,\mu,\rho,\alpha,\beta)$ are established by making use of the concept of differential subordination.

Keywords: Analytic function, starlike function, convex function, Ruscheweyh derivative, multiplier transformation, differential subordination.

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1. INTRODUCTION

Let $A(n)$ denote the class of functions of the form $f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k, n \in N = \{1,2,3...\}$,

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which are analytic in the open unit disc $U=\{z:z\in C,|z|<1\}$. Clearly $A(1)=A$ is a well-known class of normalized analytic functions in U. If f and g are analytic in U, we say that f is subordinate to g ,written $f\prec g$, if there exists a Schwarz function $w(z)$, which (by definition)is analytic in U with $w(0)=0$ and $|w(z)|<1, z\in U,$ such that $f(z)=g(w(z)), z\in U.$ Further, if the function g is univalent in U ,then we have the following equivalence $f\prec g\Leftrightarrow f(0)=g(0)$ and $f(U)\subset g(U).$

For $0\le\rho<1,$ we denote $S_n^*(\rho)$ and $K_n(\rho)$ the subclasses of $A(n)$ consisting of all analytic functions which are respectively, starlike of order ρ and convex of order ρ in U. It is well known that $K_n(\rho)\subset S_n^*(\rho)\subset S$, where S is the class of univalent functions in U. We also denote by $R_n(\rho)$ the subclass of functions in $A(n)$ which satisfy $\operatorname{Re}(f'(z)>\rho, z\in U.$

Definition 1.1([16]). For $f\in A(n), m\in N_0=N\cup\{0\}, \beta\ge 0$ and α a real number with $\alpha+\beta>0,$ a new generalized multiplier transformation, denoted by $I_{\alpha,\beta}^m$, is defined by the following infinite series:

(1.1) $$I_{\alpha,\beta}^m f(z)=z+\sum_{k=n+1}^{\infty}\left(\frac{\alpha+k\beta}{\alpha+\beta}\right)^m a_k z^k ., z\in U.$$

It follows from (1.1) that

(1.2) $$I_{\alpha,0}^m f(z)=f(z),$$

(1.3) $$(\alpha+\beta)I_{\alpha,\beta}^{m+1}f(z)=\alpha I_{\alpha,\beta}^m f(z)+\beta z(I_{\alpha,\beta}^m f(z))',$$

We note that

- $I_{\alpha,1}^m f(z)=I_\alpha^m f(z), \alpha>-1$ (See Cho and Srivastava [10] and Cho and Kim [11]).
- $I_{1-\beta,\beta}^m f(z)=D_\beta^m f(z), \beta\ge 0$ (See Al-Oboudi [6]).
- $I_{l+1-\beta,\beta}^m f(z)=I_{l,\beta}^m f(z), l>-1, \beta\ge 0$ (See Catas [9]).

Remark 1.2. a) $I_{\alpha}^{m} f(z)$ was defined and investigated in [10] and [11] for $\alpha \geq 0$ and $I_{l,\beta}^{m} f(z)$ was defined and studied in [9] for $l \geq 0, \beta \geq 0$. So our results in this paper are improvement of corresponding results proved earlier for $I_{\alpha}^{m} f(z)$ or $I_{l,\beta}^{m} f(z)$ to $\alpha > -1$ or $l > -1$, respectively.

b) i) $D_{\beta}^{m} f(z), m \geq 0$ was due to Acu and Owa [1], ii) $D_{1}^{m} f(z)$ was introduced by Salagean [15] and was considered for $m \geq 0$ in [7], and iii) $I_{1}^{m} f(z)$ was investigated by Uralegaddi and Somanath [20].

Definition 1.3 ([14]). For $m \in N_0$, $f \in A(n)$, the operator R^m is defined by $R^m : A(n) \to A(n)$,

$$R^0 f(z) = f(z),$$

$$R^1 f(z) = zf'(z),$$

$$\dots$$

$$(m+1)R^{m+1} f(z) = z(R^m f(z))' + mR^m f(z), z \in U.$$

Definition 1.4. Let $m \in N_0, \lambda \geq 0, \beta \geq 0$ and α a real number with $\alpha + \beta > 0$. Denote by $RI_{\alpha,\beta,\lambda}^{m}$ the operator given by $RI_{\alpha,\beta,\lambda}^{m} : A(n) \to A(n)$,

$$RI_{\alpha,\beta,\lambda}^{m} f(z) = (1-\lambda)R^m f(z) + \lambda I_{\alpha,\beta}^{m} f(z), z \in U.$$

Remark 1.5. If $f \in A(n)$, then $RI_{\alpha,\beta,\lambda}^{m} f(z) = z + \sum_{k=n+1}^{\infty} \left\{ (1-\lambda) C_{m+k-1}^{m} + \lambda \left(\frac{\alpha + k\beta}{\alpha + \beta} \right)^m \right\} a_k z^k, z \in U.$

Remark 1.6. The operator $I_{\alpha,\beta}^{m}$ is introduced and investigated in [16] and [17]. The operator $RI_{\alpha,\beta,\lambda}^{m}$ is studied in [18] and [19].

For $\lambda = 0$, $RI_{\alpha,\beta,0}^{m} f(z) = R^m f(z), z \in U$, and for $\lambda = 1$, $RI_{\alpha,\beta,1}^{m} = I_{\alpha,\beta}^{m} f(z), z \in U$.

To prove our results we need the following lemma.

Lemma 1.7 [13]. Let $\frac{1}{2} \le \rho < 1, u$ be analytic in U with $u(0) = 1$ and suppose that

(1.4) $$\operatorname{Re}\left(1 + \frac{zu'(z)}{u(z)}\right) > \frac{3\rho - 1}{2\rho}, z \in U.$$

Then $\operatorname{Re}(u(z)) > \rho, z \in U.$

2. MAIN RESULTS

Definition 2.1. We say that a function $f \in A(n)$ is in the class $\mathrm{I}_n(m, \mu, \rho, \alpha, \beta), m \in N_{0,}$, $n \in N, \mu \ge 0, \rho \in [0,1), \alpha$ a real number with $\alpha + \beta > 0,$ if

(2.1) $$\left|\left(\frac{I_{\alpha,\beta}^{m+1} f(z)}{z}\right)\left(\frac{z}{I_{\alpha,\beta}^{m} f(z)}\right)^{\mu} - 1\right| < 1 - \rho, z \in U.$$

Definition 2.2. We say that a function $f \in A(n)$ is in the class $\Re\mathrm{I}_n^{\lambda}(m, \mu, \rho, \alpha, \beta), m \in N_{0,}$, $n \in N, \mu \ge 0, \rho \in [0,1), \alpha$ a real number with $\alpha + \beta > 0,$ if

(2.2) $$\left|\left(\frac{RI_{\alpha,\beta,\lambda}^{m+1} f(z)}{z}\right)\left(\frac{z}{RI_{\alpha,\beta,\lambda}^{m} f(z)}\right)^{\mu} - 1\right| < 1 - \rho, z \in U.$$

For $\lambda = 1$, (2.2) reduces to (2.1).

Remark 2.3. The family $\Re\mathrm{I}_n^{\lambda}(m, \mu, \rho, \alpha, \beta)$ is a new comprehensive class of analytic functions which includes various well known classes of analytic univalent functions as well as some new ones. For example, i) $\Re\mathrm{I}_n^{\lambda}(m, \mu, \rho, l + 1 - \beta, \beta) = \Re D_n^{\lambda}(m, \mu, \rho, l, \beta), l > -1$, was studied in [2] for $l \ge 0$, ii) $\Re\mathrm{I}_1^{\lambda}(m, \mu, \rho, 1 - \beta, \beta) = \Re D_1^{\lambda}(m, \mu, \rho, 0, \beta) = \Re D^{\lambda}(m, \mu, \rho, \beta)$ was due to Lupas [3], iii) $\Re\mathrm{I}_n^{1}(m, \mu, \rho, \alpha, \beta) = \mathrm{I}_n(m, \mu, \rho, \alpha, \beta)$ (Definition 2.1), iv) $\mathrm{I}_n(m, \mu, \rho, 1 - \beta, \beta) = D_n(m, \mu, \rho, \beta)$ was introduced in [4], v) $D_n(0,1, \rho, 1) = S_n^{*}(\rho)$, $D_n(1,1, \rho, 1) = K_n(\rho)$ and $D_n(0,0, \rho, 1) = R_n(\rho)$, vi) $D_1(m, \mu, \rho, 1) = D(m, n, \rho)$ was introduced in [5,8] , vii) $D_1(0, \mu, \rho, 1) = D(\mu, \rho)$ was introduced by Frasin and Jahangiri [13] and viii) $D_1(0,2,\rho,1) = D(\rho)$ which has been investigated by Frasin and Darus [12].

In this note we provide a sufficient condition for functions to be in the class $\Re \mathrm{I}_n^{\lambda}(m,\mu,\rho,\alpha,\beta)$.

Theorem 2.4. Let $m \in N_0, n \in N, \lambda \geq 0, \mu \geq 0, \frac{1}{2} \leq \rho < 1, \gamma = \frac{3\rho - 1}{2\rho}, \beta > 0, \alpha$ a real number with $\alpha + \beta > 0$ and $f \in A(n)$. If

$$(2.3) \quad (m+2)\frac{RI_{\alpha,\beta,\lambda}^{m+2} f(z)}{RI_{\alpha,\beta,\lambda}^{m+1} f(z)} - \mu(m+1)\frac{RI_{\alpha,\beta,\lambda}^{m+1} f(z)}{RI_{\alpha,\beta,\lambda}^{m} f(z)} + \lambda\left(\frac{\alpha+\beta}{\beta} - m - 2\right)\frac{I_{\alpha,\beta}^{m+2} f(z)}{RI_{\alpha,\beta,\lambda}^{m+1} f(z)} -$$

$$- \lambda\mu\left(\frac{\alpha+\beta}{\beta} - m - 1\right)\frac{I_{\alpha,\beta}^{m+1} f(z)}{RI_{\alpha,\beta,\lambda}^{m} f(z)} - \lambda\left(\frac{\alpha}{\beta} - m - 1\right)\frac{I_{\alpha,\beta}^{m+1} f(z)}{RI_{\alpha,\beta,\lambda}^{m+1} f(z)} +$$

$$+ \lambda\mu\left(\frac{\alpha}{\beta} - m\right)\frac{I_{\alpha,\beta}^{m} f(z)}{RI_{\alpha,\beta,\lambda}^{m} f(z)} + (m+1)(\mu - 1) \prec 1 + \gamma z, z \in U,$$

then $f \in \Re \mathrm{I}_n^{\lambda}(m,\mu,\rho,\alpha,\beta)$.

Proof. Define the function u(z) by

$$(2.4) \quad u(z) = \left(\frac{RI_{\alpha,\beta,\lambda}^{m+1} f(z)}{z}\right)\left(\frac{z}{RI_{\alpha,\beta,\lambda}^{m} f(z)}\right)^{\mu}.$$

Then the function $u(z)$ is analytic in U with $u(0) = 1$. Differentiating (2.4) logarithmically with respect to z and using (1.3), we obtain

$$\frac{zu'(z)}{u(z)} = (m+2)\frac{RI^{m+2}_{\alpha,\beta,\lambda} f(z)}{RI^{m+1}_{\alpha,\beta,\lambda} f(z)} - \mu(m+1)\frac{RI^{m+1}_{\alpha,\beta,\lambda} f(z)}{RI^{m}_{\alpha,\beta,\lambda} f(z)} + \lambda\left(\frac{\alpha+\beta}{\beta} - m - 2\right)\frac{I^{m+2}_{\alpha,\beta} f(z)}{RI^{m+1}_{\alpha,\beta,\lambda} f(z)} -$$

$$-\lambda\mu\left(\frac{\alpha+\beta}{\beta} - m - 1\right)\frac{I^{m+1}_{\alpha,\beta} f(z)}{RI^{m}_{\alpha,\beta,\lambda} f(z)} - \lambda\left(\frac{\alpha}{\beta} - m - 1\right)\frac{I^{m+1}_{\alpha,\beta} f(z)}{RI^{m+1}_{\alpha,\beta,\lambda} f(z)} +$$

$$+\ \lambda\mu\left(\frac{\alpha}{\beta} - m\right)\frac{I^{m}_{\alpha,\beta} f(z)}{RI^{m}_{\alpha,\beta,\lambda} f(z)} + (m+1)(\mu-1) - 1$$

From (1.4) and (2.3) we get $\mathrm{Re}\left(1+\frac{zu'(z)}{u(z)}\right) > \frac{3\rho-1}{2\rho}, z \in U.$ Applying Lemma 1.4 we deduce that

$$\mathrm{Re}\left\{\left(\frac{RI^{m+1}_{\alpha,\beta,\lambda} f(z)}{z}\right)\left(\frac{z}{RI^{m}_{\alpha,\beta,\lambda} f(z)}\right)^{\mu}\right\} > \rho, z \in U.$$

Therefore, $f \in \Re\mathrm{I}^{\lambda}_{n}(m,\mu,\rho,\alpha,\beta)$, by Definition 2.3.

Taking $\lambda = 1$ in Theorem 2.4, we obtain

Theorem 2.5. Let $m \in N_0, n \in N, \mu \geq 0, \frac{1}{2} \leq \rho < 1, \gamma = \frac{3\rho-1}{2\rho}$, $\beta > 0, \alpha$ a real number with $\alpha + \beta > 0$ and $f \in A(n)$. If

$$\left(\frac{\alpha+\beta}{\beta}\right)\left[\frac{I^{m+2}_{\alpha,\beta} f(z)}{I^{m+1}_{\alpha,\beta} f(z)} - \mu\frac{I^{m+1}_{\alpha,\beta} f(z)}{I^{m}_{\alpha,\beta} f(z)} + (\mu-1)\right] + 1 \prec 1 + \gamma z, z \in U,$$

then $f \in \mathrm{I}_n(m,\mu,\rho,\alpha,\beta), z \in U.$

As consequences of the above theorem, we have the following interesting corollary:

Corollary 2.6. Let $f \in A(n), \rho = \frac{1}{2}, \lambda = 1, \beta > 0$ and α a real number with $\alpha + \beta > 0$.

(a) Let $m = 1, \mu = 1$ .If $\mathrm{Re}\left\{\left(\frac{\alpha+\beta}{\beta}\right)\left(\frac{I^3_{\alpha,\beta} f(z)}{I^2_{\alpha,\beta} f(z)} - \frac{I^2_{\alpha,\beta} f(z)}{I_{\alpha,\beta} f(z)}\right)\right\} > -\frac{1}{2}, z \in U,$ then $\mathrm{Re}\left(\frac{I^2_{\alpha,\beta} f(z)}{I_{\alpha,\beta} f(z)}\right) > \frac{1}{2}, z \in U.$ That is $f \in \mathrm{I}_n(1,1,\frac{1}{2},\alpha,\beta).$

(b) Let $m = 1, \mu = 0$ If $\mathrm{Re}\left\{\left(\frac{\alpha+\beta}{\beta}\right)\left(\frac{I^3_{\alpha,\beta} f(z)}{I^2_{\alpha,\beta} f(z)} - 1\right)\right\} > -\frac{1}{2}, z \in U,$ the $\mathrm{Re}\left(\frac{I^2_{\alpha,\beta} f(z)}{z}\right) > \frac{1}{2}, z \in U.$ That is $f \in \mathrm{I}_n(1,0,\frac{1}{2},\alpha,\beta).$

(c) Let $m = 0, \mu = 1$ .If $\mathrm{Re}\left\{\left(\frac{\alpha+\beta}{\beta}\right)\left(\frac{I^2_{\alpha,\beta} f(z)}{I_{\alpha,\beta} f(z)} - \frac{I_{\alpha,\beta} f(z)}{f(z)}\right)\right\} > -\frac{1}{2}, z \in U,$ then $\mathrm{Re}\left(\frac{I_{\alpha,\beta} f(z)}{f(z)}\right) > \frac{1}{2}, z \in U.$ That is $f \in \mathrm{I}_n(0,1,\frac{1}{2},\alpha,\beta).$

(d) Let $m = 0, \mu = 0$ If $\mathrm{Re}\left\{\left(\frac{\alpha+\beta}{\beta}\right)\left(\frac{I^2_{\alpha,\beta} f(z)}{I_{\alpha,\beta} f(z)} - 1\right)\right\} > -\frac{1}{2}, z \in U,$ then $\mathrm{Re}\left(\frac{I_{\alpha,\beta} f(z)}{z}\right) > \frac{1}{2}, z \in U.$ That is $f \in \mathrm{I}_n(0,0,\frac{1}{2},\alpha,\beta).$

$\alpha = 0$ in Corollary 2.6, we have

Corollary 2.7. Let $f \in A(n).$

(a) If $\mathrm{Re}\left\{\left(\frac{2zf''(z) + z^2 f'''(z)}{f'(z) + zf''(z)} - \frac{zf''(z)}{f'(z)}\right)\right\} > -\frac{1}{2}, z \in U,$ then f is convex of order $1/2$ (i.e. $f \in K_n(1/2)$).

(b) If $\mathrm{Re}\left\{\left(\frac{2zf''(z) + z^2 f'''(z)}{f'(z) + zf''(z)}\right)\right\} > -\frac{1}{2}, z \in U,$ then $\mathrm{Re}\,(f'(z) + zf''(z)) > \frac{1}{2}, z \in U.$

(c) If $\mathrm{Re}\left\{\left(\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right)\right\}>-\frac{3}{2}, z\in U$, then f is starlike of order $1/2$ (i.e. $f\in S_n^*(1/2)$).

(d) If f is convex of order $1/2$, then $f\in R_n(1/2)$.

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