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COINCIDENCE AND COMMON FIXED POINT THEOREMS ON TVS-VALUED CONE METRIC SPACES

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Abstract. In this paper, we obtained some coincidence and common fixed point theorems for a pair of weakly compatible maps on TVS-valued cone metric spaces, satisfying contractive mapping: $d(fx,fy) \preceq \alpha d(fx,gx) + \beta d(fy,gy) + \gamma d(gx,gy)$ and $d(fx,fy) \preceq \alpha d(gx,gy) + \beta[d(fx,gx) + d(fy,gy)] + \gamma[d(fx,gy) + d(fy,gx)]$. Some corollaries on fixed point theorems on TVS-valued cone metric spaces are also provided.

Keywords: TVS-valued cone metric space; coincidence and common fixed point; weakly compatible.

2010 AMS Subject Classification: 47H10.

1. Introduction

Huang and Zhang [1] generalized the concept of metric space, replacing the set of real numbers by an ordered Banach space and established some fixed point theorems for mappings satisfying different contractive conditions. Afterwards many other authors [2-8] studied different kinds of contractive mappings on cone metric spaces. In addition, several authors [9-12] introduced the notion of TVS-valued cone metric space, which is bigger than that introduced by

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Huang and Zhang, and proved the existence of fixed points. In this paper, we continue to study fixed point theorems on TVS-valued cone metric space, and get some important results.

2. Preliminaries

Let (E,τ) always be a topological vector space (TVS) and P a subset of E. P is called a cone if and only if:

(a) P is closed, nonempty and $P \neq \{\theta\}$;

(b) $\forall a,b \in R, a,b \geq 0, \forall x,y \in P$ imply that ax+$by \in P$;

(c) $P \cap (-P) = \{\theta\}$.

Given a cone $P \subset E$, we define a partial ordering $\preceq$ with respect to P by $x \preceq y$ if and only if $y-x \in P$. We shall write $x \prec y$ to indicate that $x \preceq y$ but $x \neq y$. While $x \prec\prec y$ stands for $y-x \in intP$ (interior of P).

In the following we always suppose E be a topological vector space, θ denotes the zero element, and $\preceq$ is a partial ordering with respect to P.

Definition 2.1. [10] Let X be a nonempty set. Suppose that the mapping $d : X \times X \to E$ satisfies:

(d1) $d(x,y) \succeq \theta$ for all $x,y \in X$ and $d(x,y) = \theta$ if and only if $x = y$;

(d2) $d(x,y) = d(y,x)$ for all $x,y \in X$;

(d3) $d(x,y) \preceq d(x,z) + d(z,y)$ for all $x,y,z \in X$.

Then d is called a TVS-valued cone metric on X , and (X,d) is called a TVS-valued cone metric space.

Definition 2.2. [10] Let (X,d) be a TVS-valued cone metric space, $\{x_n\}$ be a sequence in X and $x \in X$. Then

(i) $\{x_n\}$ converges to x whenever for every $c \in E$ with $\theta \prec\prec c$, there is a natural number N such that for all $n > N, d(x_n,x) \prec\prec c$. We denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$.

(ii) $\{x_n\}$ is a Cauthy sequence whenever for every $c \in E$ with $\theta \prec\prec c$, there is a natural number N such that for all $n,m > N, d(x_n,x_m) \prec\prec c$.

(iii) (X,d) is a complete cone metric space if every Cauchy sequence is convergent in X.

Definition 2.3. [2] Let f and g be self maps of a set X. If $\omega = fx = gx$ for some x in X, then x is called a coincidence point of f and g, and ω is called a point of coincidence of f and g. Self maps f and g are called weakly compatible if they commute at their coincidence point, that is, if $fx = gx$ for some $x \in X$, then $fgx = gfx$.

Lemma 2.4. [2] Let f and g be weakly compatible self maps of a set X. If f and g have a unique point of coincidence $\omega = fx = gx$, then ω is the unique common fixed point of f and g .

Lemma 2.5. [10] Let (X,d) be a TVS-valued cone metric space, and P be a cone. Let $\{x_n\}$ be a sequence in X and $\{a_n\}$ be a sequence in P converging to θ. If $d(x_n,x_m) \preceq a_n$ for every $n \in N$ with $m > n$, then $\{x_n\}$ is a Cauthy sequence.

3. Main results

Theorem 3.1. *Let (X,d) be a TVS-valued cone metric space. Suppose that the mappings f and g are two self-maps of X satisfying*

$$d(fx, fy) \preceq \alpha d(fx, gx) + \beta d(fy, gy) + \gamma d(gx, gy) \tag{3.1}$$

for all $x, y \in X$, where $\alpha, \beta, \gamma \geq 0$ and $\alpha + \beta + \gamma < 1$. If the range of g contains the range of f and $g(X)$ is a complete subspace of X, then f and g have a unique coincidence point in X. Moreover if f and g are weakly compatible, f and g have a unique common fixed point.

Proof. Suppose x_0 be an arbitrary point of X. Since the range of g contains the range of f, choose a point x_1 in X such that $f(x_0) = g(x_1)$. Continuing this process, having chosen x_n in X, we obtain x_{n+1} in X such that $f(x_n) = g(x_{n+1})$. Then

$$\begin{aligned}
d(gx_n, gx_{n+1}) &= d(fx_{n-1}, fx_n) \\
&\preceq \alpha d(fx_{n-1}, gx_{n-1}) + \beta d(fx_n, gx_n) + \gamma d(gx_{n-1}, gx_n) \\
&= \alpha d(gx_n, gx_{n-1}) + \beta d(gx_{n+1}, gx_n) + \gamma d(gx_{n-1}, gx_n),
\end{aligned}$$

which implies that $d(gx_n, gx_{n+1}) \preceq hd(gx_{n-1}, gx_n)$, where $h = \frac{\alpha+\gamma}{1-\beta} < 1$. Therefore, for all n, we have

$$\begin{aligned} d(gx_n, gx_{n+1}) &\preceq hd(gx_{n-1}, gx_n) \\ &\preceq \cdots \preceq h^n d(gx_0, gx_1). \end{aligned}$$

Now, for any $n > m$, we have

$$\begin{aligned} d(gx_n, gx_m) &\preceq d(gx_n, gx_{n-1}) + d(gx_{n-1}, gx_{n-2}) + \cdots + d(gx_{m+1}, gx_m) \\ &\preceq (h^{n-1} + h^{n-2} + \cdots + h^m) d(gx_1, gx_0) \\ &\preceq \frac{h^m}{1-h} d(gx_1, gx_0). \end{aligned}$$

Let $\theta \prec\prec c$ be given. Choose a balanced neighborhood U of θ such that $c + U \subseteq intP$. Since $\frac{h^m}{1-h} d(gx_1, gx_0) \to \theta$ as $m \to \infty$, by Lemma 2.5, we deduce that $\{gx_n\}$ is a Cauthy sequence. Since $g(X)$ is a complete subspace of X, there exists q in $g(X)$ such that $gx_n \to q$ as $n \to \infty$. Consequently, we can find p in X such that $gp = q$. Also, choose a natural number N_1, such that $d(gx_n, q) \prec\prec \frac{1-\beta}{3} c$, for all $n \geq N_1$. Thus

$$\begin{aligned} d(gx_n, fp) &= d(fx_{n-1}, fp) \\ &\preceq \alpha d(fx_{n-1}, gx_{n-1}) + \beta d(fp, gp) + \gamma d(gx_{n-1}, gp) \\ &= \alpha d(gx_n, gx_{n-1}) + \beta d(fp, q) + \gamma d(gx_{n-1}, q) \\ &\preceq \alpha [d(gx_n, q) + d(gx_{n-1}, q)] + \beta d(fp, q) + \gamma d(gx_{n-1}, q). \end{aligned}$$

Since $d(gx_n, fp) \succeq d(fp, q) - d(gx_n, q)$, we can get

$$\begin{aligned} d(fp, q) &\preceq \frac{1}{1-\beta} [(\alpha+\gamma) d(gx_{n-1}, q) + (1+\alpha) d(gx_n, q) \\ &\prec \frac{1}{1-\beta} [d(gx_{n-1}, q) + 2d(gx_n, q)] \\ &\prec\prec \frac{1}{1-\beta} [\frac{1-\beta}{3} c + \frac{2(1-\beta)}{3} c] \\ &= c \end{aligned}$$

for all $n \geq N_1$. Hence, $d(fp,q) \prec\prec \frac{c}{m}$, for all $m \geq 1$. So $\frac{c}{m} - d(fp,q) \in intP$, for all $m \geq 1$. Since $\frac{c}{m} \to \theta$ (as $m \to \infty$) and P is closed, we deduce $-d(fp,q) \in P$. But $d(fp,q) \in P$. Therefore, $d(fp,q) = \theta$, and $fp = q = gp$.

Now we show that f and g have a unique point of coincidence. Assume that there exists another point $u \in X$, such that $fu = gu$. Then

$$d(gu,gp) = d(fu,fp) \preceq \alpha d(fu,gu) + \beta d(fp,gp) + \gamma d(gu,gp),$$

which implies that $(1-\gamma)d(gu,gp) \preceq \theta$. By $1-\gamma > 0$, we have $gu = gp$. From Lemma 1.4, f and g have a unique common fixed point.

Corollary 3.2. *Let (X,d) be a complete TVS-valued cone metric space. Suppose that the mapping f satisfies*

$$d(fx,fy) \preceq \alpha d(x,fx) + \beta d(y,fy) + \gamma d(x,y) \tag{3.2}$$

for all $x,y \in X$, where $\alpha,\beta,\gamma \geq 0$ and $\alpha+\beta+\gamma < 1$. Then f has a unique fixed point in X.

Proof. Inequality (3.2) is obtained from (3.1) by setting $g = I$. I is identity operator. The results then follow from Theorem 3.1.

Corollary 3.3. *Let (X,d) be a complete TVS-valued cone metric space. Suppose that the mapping f satisfies*

$$d(f^n x, f^n y) \preceq \alpha d(x,f^n x) + \beta d(y,f^n y) + \gamma d(x,y) \tag{3.3}$$

for all $x,y \in X$, where $\alpha,\beta,\gamma \geq 0$ and $\alpha+\beta+\gamma < 1$. Then f has a unique fixed point in X.

Proof. Set $f = f^n$ in (3.2), we can obtain that f^n has a unique fixed point x^*. But $f^n(fx^*) = f(f^n x^*) = fx^*$, so fx^* is also a fixed point of f^n. Hence $fx^* = x^*$, x^* is a fixed point of f.

Corollary 3.4. *Let (X,d) be a TVS-valued cone metric space. Suppose that the mappings f and g are two self-maps of X satisfying*

$$d(fx,fy) \preceq k(d(fx,gx) + d(fy,gy)) \tag{3.4}$$

for all $x,y \in X$, where $k \in [0,\frac{1}{2})$ is a constant. If the range of g contains the range of f and $g(X)$ is a complete subspace of X, then f and g have a unique coincidence point in X. Moreover if f and g are weakly compatible, f and g have a unique common fixed point.

Corollary 3.5. *Let (X,d) be a TVS-valued cone metric space. Suppose that the mappings f and g are two self-maps of X satisfying*

$$d(fx,fy) \preceq kd(gx,gy) \tag{3.5}$$

for all $x,y \in X$, where $k \in [0,1)$ is a constant. If the range of g contains the range of f and $g(X)$ is a complete subspace of X, then f and g have a unique coincidence point in X. Moreover if f and g are weakly compatible, f and g have a unique common fixed point.

Theorem 3.6. *Let (X,d) be a TVS-valued cone metric space. Suppose that the mappings f and g are two self-maps of X satisfying*

$$d(fx,fy) \preceq \alpha d(gx,gy) + \beta[d(fx,gx) + d(fy,gy)] + \gamma[d(fx,gy) + d(fy,gx)] \tag{3.6}$$

for all $x,y \in X$, where $\alpha,\beta,\gamma \geq 0$ and $\alpha + 2\beta + 2\gamma < 1$. If the range of g contains the range of f and $g(X)$ is a complete subspace of X, then f and g have a unique coincidence point in X. Moreover if f and g are weakly compatible, f and g have a unique common fixed point.

Proof. Suppose x_0 be an arbitrary point of X. Since the range of g contains the range of f, choose a point x_1 in X such that $f(x_0) = g(x_1)$. Continuing this process, having chosen x_n in X, we obtain x_{n+1} in X such that $f(x_n) = g(x_{n+1})$. Then

$$\begin{aligned} d(gx_n, gx_{n+1}) &= d(fx_{n-1}, fx_n) \\ &\preceq \alpha d(gx_{n-1}, gx_n) + \beta[d(fx_{n-1}, gx_{n-1}) + d(fx_n, gx_n)] \\ &\quad +\gamma[d(fx_{n-1}, gx_n) + d(fx_n, gx_{n-1})] \\ &\preceq \alpha d(gx_{n-1}, gx_n) + \beta[d(gx_n, gx_{n-1}) + d(gx_{n+1}, gx_n)] \\ &\quad +\gamma[d(gx_n, gx_n) + d(gx_{n+1}, gx_n) + d(gx_n, gx_{n-1})] \end{aligned}$$

which implies that $d(gx_n, gx_{n+1}) \preceq hd(gx_{n-1}, gx_n)$, where $h = \frac{\alpha+\beta+\gamma}{1-(\beta+\gamma)} < 1$. Therefore, for all n, we have

$$\begin{aligned} d(gx_n, gx_{n+1}) &\preceq hd(gx_{n-1}, gx_n) \\ &\preceq \cdots \preceq h^n d(gx_0, gx_1). \end{aligned}$$

Now, for any $n > m$, we have

$$\begin{aligned} d(gx_n, gx_m) &\preceq d(gx_n, gx_{n-1}) + d(gx_{n-1}, gx_{n-2}) + \cdots + d(gx_{m+1}, gx_m) \\ &\preceq (h^{n-1} + h^{n-2} + \cdots + h^m) d(x_1, x_0) \\ &\preceq \frac{h^m}{1-h} d(gx_1, gx_0). \end{aligned}$$

Let $\theta \prec\prec c$ be given. Choose a balanced neighborhood U of θ such that $c + U \subseteq intP$. Since $\frac{h^m}{1-h} d(gx_1, gx_0) \to \theta$ as $m \to \infty$, by lemma 2.5, we deduce that $\{gx_n\}$ is a Cauthy sequence. Since $g(X)$ is a complete subspace of X, there exists q in $g(X)$ such that $gx_n \to q$ as $n \to \infty$. Consequently, we can find p in X such that $gp = q$. Also, choose a natural number N_1, such that $d(gx_n, q) \prec\prec \frac{\sigma}{3} c$, where $\sigma = 1 - \beta - \gamma > 0$, for all $n \geq N_1$. Thus

$$\begin{aligned} d(gx_n, fp) &= d(fx_{n-1}, fp) \\ &\preceq \alpha d(gx_{n-1}, gp) + \beta[d(fx_{n-1}, gx_{n-1}) + d(fp, gp)] \\ &\quad + \gamma[d(fx_{n-1}, gp) + d(fp, gx_{n-1})] \\ &= \alpha d(gx_{n-1}, q) + \beta[d(gx_n, gx_{n-1}) + d(fp, q)] \\ &\quad + \gamma[d(gx_n, q) + d(fp, gx_{n-1})] \\ &\preceq \alpha d(gx_{n-1}, q) + \beta[d(gx_n, q) + d(q, gx_{n-1}) + d(fp, q)] \\ &\quad + \gamma[d(gx_n, q) + d(fp, q) + d(q, gx_{n-1})]. \end{aligned}$$

Since $d(gx_n, fp) \succeq d(fp, q) - d(gx_n, q)$, we can get

$$\begin{aligned} d(fp, q) &\preceq \frac{1}{1-\beta-\gamma}[(\alpha+\beta+\gamma) d(gx_{n-1}, q) + (1+\beta+\gamma) d(gx_n, q)] \\ &\prec \frac{1}{\sigma}[d(gx_{n-1}, q) + 2d(gx_n, q)] \\ &\prec\prec \frac{1}{\sigma}[\frac{\sigma}{3} c + \frac{2\sigma}{3} c] \\ &= c \end{aligned}$$

for all $n \geq N_1$. Hence, $d(fp, q) \prec\prec \frac{c}{m}$, for all $m \geq 1$. So $\frac{c}{m} - d(fp, q) \in intP$, for all $m \geq 1$. Since $\frac{c}{m} \to \theta$ (as $m \to \infty$) and P is closed, we deduce $-d(fp, q) \in P$. But $d(fp, q) \in P$. Therefore $d(fp, q) = \theta$, and $fp = q = gp$.

Now we show that f and g have a unique point of coincidence. Assume that there exists another point $u \in X$, such that $fu = gu$. Then

$$\begin{aligned} d(gu,gp) &= d(fu,fp) \\ &\preceq \alpha d(gu,gp) + \beta[d(fu,gu)+d(fp,gp)] + \gamma[d(fu,gp)+d(fp,gu)] \\ &= \alpha d(gu,gp) + 2\gamma d(gu,gp), \end{aligned}$$

which implies that $(1-\alpha-2\gamma)d(gu,gp) \preceq \theta$. By $1-\alpha-2\gamma > 0$, we have $gu = gp$. From Lemma 2.4, f and g have a unique common fixed point.

Corollary 3.7 *Let (X,d) be a complete TVS-valued cone metric space. Suppose that the mapping f satisfies*

$$d(fx,fy) \preceq \alpha d(x,y) + \beta[d(x,fx)+d(y,fy)] + \gamma[d(y,fx)+d(x,fy)] \tag{3.7}$$

for all $x,y \in X$, where $\alpha,\beta,\gamma \geq 0$ and $\alpha+2\beta+2\gamma < 1$. Then f has a unique fixed point.

Proof. Inequality (3.7) is obtained from (3.6) by setting $g = I$. I is identity operator. The results then follow from Theorem 3.6.

Example 3.8 *$X = [0,1]$, E be the set of all complex-valued function on X, then E is a vector space over R under the following operations:*

$$(u+v)(t) = u(t)+v(t), (\alpha u)(t) = \alpha u(t)$$

for all $u,v \in E, \alpha \in R$. Let τ be the topology on E defined by the family $p_x : x \in X$ of semi-norms on E, where $p_x(u) =| u(x) |$, then (X,τ) is a topological vector space. Define $d : X \times X \to E$ as follows:

$$(d(x,y))(t) = (| x-y |, 2 | x-y |)2^t, P = \{(x \in E : x(t) \geq 0, \forall t \in X\}. \tag{3.8}$$

Then (X,d) is a TVS-valued cone metric space. Define $f : X \to X$ as $f(x) = \frac{x^2}{4}$, then all conditions of Corollary 3.7 are satisfied.

Corollary 3.9 *Let (X,d) be a TVS-valued cone metric space. Suppose that the mappings f and g are two self-maps of X satisfying*

$$d(fx,fy) \preceq k(d(fx,gy)+d(fy,gx)) \tag{3.9}$$

for all $x,y \in X$, where $k \in [0, \frac{1}{2})$ is a constant. If the range of g contains the range of f and $g(X)$ is a complete subspace of X, then f and g have a unique coincidence point in X. Moreover if f and g are weakly compatible, f and g have a unique common fixed point.

Proof. Setting $\alpha = \beta = 0$ in Theorem 3.6, and $k = 2\gamma \in [0, \frac{1}{2})$. The results then follow from Theorem 3.6.

Conflict of Interests

The authors declare that there is no conflict of interests.

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