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A NUMERICAL SOLUTION OF THE BURGERS' EQUATION USING A CRANK-NICOLSON EXPONENTIAL FINITE DIFFERENCE METHOD

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Abstract. This paper presents a new technique of forming improved exponential finite difference solution of Burgers' equation. The technique is called Crank-Nicolson exponential finite difference method for the solution of Burgers' equation. Since the equation is nonlinear the scheme leads to a system of nonlinear equations. At each time-step Newton's method is used to solve this nonlinear system. The results are compared with exact values clearly show that results obtained using the method is precise and reliable.

Keywords: Burgers' equation; finite difference method; exponential finite difference method; Crank-Nicolson exponential finite difference method.

2010 AMS Subject Classification: 65N06, 35Q35.

1. Introduction

In this paper, we consider the one-dimensional non-linear Burgers' equation

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}-\nu\frac{\partial^2 u}{\partial x^2}=0,\quad a<x<b, \tag{1}$$

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with the initial condition

$$u(x,0) = g(x), \;\; a < x < b,$$

and the boundary conditions

$$u(a,t) = h_1(t) \text{ and } u(b,t) = h_2(t), \;\; t > 0,$$

where ν is the positive coefficient of kinematic viscosity and g, h_1 and h_2 are the prescribed functions of the variables.

Burgers' equation is found to describe various kind of phenomena such as mathematical model of turbulence and the approximate theory of flow through a shock wave traveling in a viscous fluid [1].

In literature, many numerical methods have been proposed and implemented for approximating solution of the Burgers' equation. Many authors have used numerical techniques based on finite difference [1-8] finite element [9-13] and boundary element [14] methods in attempting to solve the equation. Kadalbajoo et al. [15] used a parameter-uniform implicit difference scheme for solving time-dependent Burgers' equation. The explicit exponential finite difference method was originally developed by Bhattacharya for solving of heat equation [16]. Bhattacharya [17] and Handschuh and Keith [18] used exponential finite difference method for the solution of Burgers' equation. Bahadır solved the KdV equation by using the exponential finite-difference technique[19]. Implicit exponential finite difference method and fully implicit exponential finite difference method was applied to Burgers?equation by Inan and Bahadir[20]. Inan and Bahadir solved the linearized Burgers' equation by Hopf-Cole transformation using an explicit exponential finite difference method[21].

In this paper, we design a new scheme for solving the Burgers' equation. Some examples are presented to show the ability of this method to solve the equation. It is clearly seen that numerical method is reasonably in good agreement with the exact solution.

2. Crank-Nicolson exponential finite difference scheme

We obtain numerical solution of the Burgers' equation by Crank-Nicolson exponential finite difference method for two standard problems. The accuracy of the proposed method is measured using the L_2 and L_∞ error norms defined by

$$L_2 = \|u-U\|_2 = \left(h\sum_{i=0}^{N}|u_i-U_i|^2\right)^{\frac{1}{2}},$$

$$L_\infty = \|u-U\|_\infty = \max_{0\le i\le N}|u_i-U_i|.$$

The solution domain is discretized into cells described by the nodes set (x_i,t_n) in which $x_i = ih$, $(i = 0,1,2,...,N)$ and $t_n = nk$, $(n = 0,1,2,...)$, $h = \Delta x$ is the spatial mesh size and $k = \Delta t$ is the time step.

Crank-Nicolson exponential finite difference scheme (CN-EFDM) for Eq. (1) takes the following nonlinear form

$$\begin{aligned} U_i^{n+1} &= U_i^n \exp\left\{\frac{\nu\Delta t}{2(\Delta x)^2}\left[-\frac{\Delta x U_i^n}{2\nu}\frac{\left(U_{i+1}^n-U_{i-1}^n+U_{i+1}^{n+1}-U_{i-1}^{n+1}\right)}{U_i^n}\right.\right. \\ &\left.\left.+\frac{\left(U_{i-1}^n-2U_i^n+U_{i+1}^n+U_{i-1}^{n+1}-2U_i^{n+1}+U_{i+1}^{n+1}\right)}{U_i^n}\right]\right\}, \end{aligned} \tag{2}$$

which is valid for values of i lying in the interval $1\le i\le N-1$.

Where U_i^n denotes the exponential finite difference approximation to the exact solution $u(x,t)$. Eq. (2) is system of nonlinear difference equations. Let us consider the nonlinear system of equations in the form

$$\mathbf{F}(\mathbf{V}) = \mathbf{0}, \tag{3}$$

where $\mathbf{F} = [f_1, f_2, ..., f_{N-1}]^T$ and $\mathbf{V} = \left[U_1^{n+1}, U_2^{n+1}, ..., U_{N-1}^{n+1}\right]^T$. Newton's method applied to Eq. (3) results in the following iteration:

1. Set $\mathbf{V}^{(0)}$, an initial guess.
2. For $m = 0,1,2,..$ until convergence do:

Solve $J(\mathbf{V}^{(m)})\delta^{(m)} = -\mathbf{F}(\mathbf{V}^{(m)})$;

Set $\mathbf{V}^{(m+1)} = \mathbf{V}^{(m)} + \delta^{(m)}$, where $J(\mathbf{V}^{(m)})$ is the Jacobian matrix which is evaluated analytically. The solution at the previous time-step is taken as the initial estimate. The Newton's iteration at each time-step is stopped when $\left\|\mathbf{F}(\mathbf{V}^{(m)})\right\|_\infty \leq 10^{-5}$. The convergence is generally obtained in two or three iterations.

3. Numerical Results

Problem 1.

We first solve the Burgers' equation (1) and the initial condition

$$u(x,0) = \sin(\pi x),\ 0 < x < 1$$

with the boundary conditions

$$u(0,t) = u(1,t) = 0,\ t > 0$$

and the exact solution given by

$$u(x,t) = \frac{2\pi\nu \sum_{n=1}^{\infty} a_n \exp\left(-n^2\pi^2\nu t\right) n \sin(n\pi x)}{a_0 + \sum_{n=1}^{\infty} a_n \exp\left(-n^2\pi^2\nu t\right)\cos(n\pi x)} \tag{4}$$

with

$$a_0 = \int_0^1 \exp\left\{-(2\pi\nu)^{-1}\left[1-\cos(\pi x)\right]\right\} dx$$

$$a_n = 2\int_0^1 \exp\left\{-(2\pi\nu)^{-1}\left[1-\cos(\pi x)\right]\right\}\cos(n\pi x)\, dx,\ n = 1,2,3,\ldots$$

The results for Problem 1 are displayed in Table 1-3 and Fig. 1. The numerical solutions obtained by Crank-Nicolson exponential finite difference method and the exact solution for different values of h are presented in Table 1. It is observed from Table 1 that the values of L_2 and L_∞ decrease with decrease of h. Comparison of numerical solutions with exact solutions at different times for $\nu = 1.0, \nu = 0.01$, $h = 0.0125$ and $k = 10^{-5}$ are given in Table 2. The obtained solutions by CN-EFDM are compared with other methods [3, 5, 11, 22] in Table 3. All comparisons show that the present method offer better results than the others. In order to show the behaviour of the numerical solutions of the Problem 1 obtained with Crank-Nicolson exponential finite difference method, we give a graph in Fig. 1.

Table 1. Comparison of the solutions with the exact solution at $t = 0.1$ for $\nu = 1$ and $k = 10^{-5}$ using various mesh sizes.

x	$h = 0.05$	$h = 0.025$	$h = 0.0125$	$h = 0.01$	Exact
0.1	0.109732	0.109590	0.109555	0.109551	0.109538
0.2	0.210174	0.209895	0.209825	0.209817	0.209792
0.3	0.292450	0.292045	0.291944	0.291931	0.291896
0.4	0.348620	0.348110	0.347983	0.347968	0.347924
0.5	0.372366	0.371788	0.371644	0.371626	0.371577
0.6	0.359854	0.359261	0.359113	0.359095	0.359046
0.7	0.310640	0.310101	0.309966	0.309950	0.309905
0.8	0.228381	0.227967	0.227864	0.227851	0.227817
0.9	0.120993	0.120768	0.120712	0.120705	0.120687
L_2	0.000566	0.000151	0.000047	0.000035	
L_∞	0.000809	0.000216	0.000068	0.000050	

Table 2. Comparison of the solutions with the exact solutions at different times for $\nu = 1.0$, $v = 0.01$, $h = 0.0125$ and $k = 10^{-5}$.

		$\nu = 1.0$		$\nu = 0.01$	
x	t	CN-EFDM	Exact	CN-EFDM	Exact
0.25	0.1	0.253678	0.253638	0.566352	0.566328
	0.5	0.005070	0.005065	0.301166	0.301151
	1.0	0.000037	0.000036	0.188200	0.188194
0.50	0.1	0.371644	0.371577	0.947456	0.947414
	0.5	0.007176	0.007169	0.588736	0.588696
	1.0	0.000052	0.000052	0.374435	0.374420
0.75	0.1	0.272636	0.272582	0.860119	0.860129
	0.5	0.005078	0.005073	0.838163	0.838033
	1.0	0.000037	0.000036	0.556081	0.556051

Table 3. Comparison of the results for Problem 1 at different times for $\nu = 0.1$, $h = 0.0125$ and $k = 10^{-4}$.

x	t	RHC [3]	RPA [5]	[11]	[22]	CN-EFDM	Exact
0.25	0.4	0.317062	0.308776	0.31215	0.30415	0.308919	0.308894
	0.6	0.248472	0.240654	0.24360	0.23629	0.240762	0.240739
	0.8	0.202953	0.195579	0.19815	0.19150	0.195700	0.195676
	1.0	0.169527	0.162513	0.16473	0.15861	0.162592	0.162565
0.50	0.4	0.583408	0.569527	0.57293	0.56711	0.569711	0.569632
	0.6	0.461714	0.447117	0.40588	0.44360	0.447291	0.447206
	0.8	0.373800	0.359161	0.36286	0.35486	0.359328	0.359236
	1.0	0.306184	0.291843	0.29532	0.28710	0.292011	0.291916
0.75	0.4	0.638847	0.625341	0.63038	0.61874	0.625677	0.625438
	0.6	0.506429	0.487089	0.49268	0.47855	0.487488	0.487215
	0.8	0.393565	0.373827	0.37912	0.36467	0.374169	0.373922
	1.0	0.305862	0.029726	0.03038	0.27860	0.287679	0.287474

Problem 2.

The initial condition for the current problem is $u(x,0) = 4x(1-x),\ 0 < x < 1$ and the boundary conditions $u(0,t) = u(1,t) = 0,\ t > 0$ with the exact solution also given by Eq. (4) but with following coefficients.

$$a_0 = \int_0^1 \exp\left[-x^2(3\nu)^{-1}(3-2x)\right] dx$$

$$a_n = 2\int_0^1 \exp\left[-x^2(3\nu)^{-1}(3-2x)\right] \cos(n\pi x)\, dx,\ n = 1,2,3,\ldots$$

The numerical solutions obtained by the present method and the exact solution for different values of h are given in Table 4. In Table 5, we compare the numerical results of Problem 2

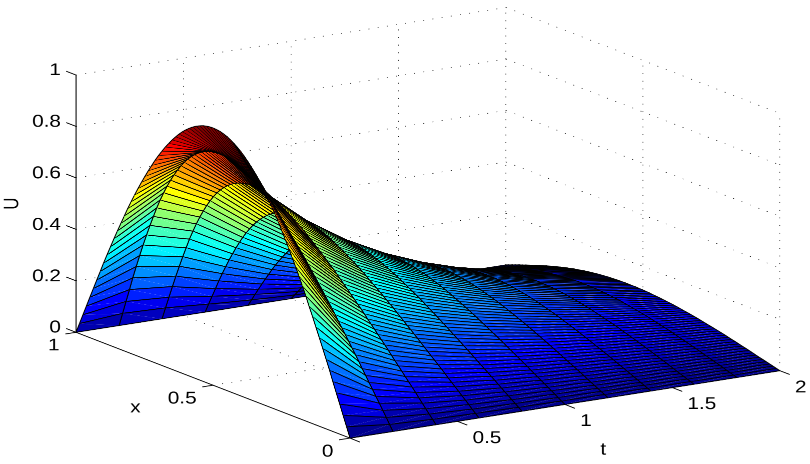

FIGURE 1. Numerical solutions at different times for $\nu = 0.1$, $h = 0.0125$ and $k = 10^{-5}$.

with the exact solutions for both $\nu = 1.0$ and $\nu = 0.01$. The obtained solutions by CN-EFDM are compared with the methods proposed in [3, 5, 11, 22] in Table 6. The comparison showed that the present methods offer better results than the others.

Table 4. Comparison of the solutions with the exact solution at $t = 0.1$ for $\nu = 1$ and $k = 10^{-5}$ using various mesh sizes.

x	$h = 0.05$	$h = 0.025$	$h = 0.0125$	$h = 0.01$	Exact
0.1	0.113091	0.112946	0.112909	0.112905	0.112892
0.2	0.216644	0.216358	0.216286	0.216277	0.216252
0.3	0.301535	0.301119	0.301015	0.301002	0.300966
0.4	0.359579	0.359055	0.358924	0.358908	0.358863
0.5	0.384235	0.383640	0.383491	0.383473	0.383422
0.6	0.371493	0.370881	0.370727	0.370709	0.370658
0.7	0.320828	0.320269	0.320129	0.320112	0.320066
0.8	0.235957	0.235527	0.235419	0.235406	0.235371
0.9	0.125037	0.124803	0.124744	0.124737	0.124718
L_2	0.000584	0.000156	0.000049	0.000036	
L_∞	0.000835	0.000224	0.000070	0.000052	

Table 5. Comparison of the solutions at different times for $\nu = 1.0$, $\nu = 0.01$, $h = 0.0125$ and $k = 10^{-5}$.

		$\nu = 1.0$		$\nu = 0.01$	
x	t	CN-EFDM	Exact	CN-EFDM	Exact
0.25	0.1	0.261521	0.261480	0.607369	0.607363
	0.5	0.005231	0.005227	0.317335	0.317320
	1.0	0.000038	0.000038	0.194699	0.194690
0.50	0.1	0.383491	0.383422	0.956025	0.956007
	0.5	0.007404	0.007398	0.609910	0.609886
	1.0	0.000053	0.000053	0.385688	0.385676
0.75	0.1	0.281629	0.281573	0.886732	0.886707
	0.5	0.005240	0.005235	0.852212	0.852123
	1.0	0.000038	0.000038	0.569339	0.569319

Table 6. Comparison of the results for Problem 2 at different times for $\nu = 0.1$ and $h = 0.0125$.

		$(k = 10^{-5})$		$(k = 10^{-4})$			
x	t	RHC [3]	RPA [5]	[11]	[22]	CN-EFDM	Exact
0.25	0.4	0.306529	0.317399	0.32091	0.31247	0.317549	0.317523
	0.6	0.236051	0.246058	0.24910	0.24148	0.246161	0.246138
	0.8	0.190181	0.199437	0.20211	0.19524	0.199579	0.199555
	1.0	0.156646	0.165529	0.16782	0.16153	0.165626	0.165599
0.50	0.4	0.565994	0.584429	0.58788	0.58176	0.584612	0.584537
	0.6	0.438926	0.457888	0.46174	0.45414	0.458061	0.457976
	0.8	0.348328	0.367320	0.37111	0.36283	0.367491	0.367398
	1.0	0.280038	0.298271	0.30183	0.29336	0.298439	0.298343
0.75	0.4	0.626990	0.645527	0.65054	0.63858	0.645868	0.645616
	0.6	0.477908	0.502564	0.50825	0.49362	0.502961	0.502676
	0.8	0.360630	0.385232	0.39068	0.37570	0.385593	0.385336
	1.0	0.272623	0.295779	0.30057	0.28663	0.296070	0.295857

4. Conclusion

In this paper, an exponential finite difference method was applied to the solution of Burgers' equation. Numerical solutions for two different test problems were given. The results showed that Crank-Nicolson exponential finite difference method offers high accuracy in the numerical solution of the one-dimensional Burgers' equation.

Conflict of Interests

The author declares that there is no conflict of interests.

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