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A SMALLNESS REGULARITY CRITERION FOR THE 3D NAVIER-STOKES EQUATIONS IN THE LARGEST CLASS

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Abstract. In this paper, we consider the three-dimensional Navier-Stokes equations, and show that if the $\dot{B}^{-1}_{\infty,\infty}$-norm of the velocity field is sufficiently small, then the solution is in fact classical.

Keywords: Navier-Stokes equations; regularity criterion; Besov spaces.

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1. Introduction

Consider the following three-dimensional (3D) Navier-Stokes equations:

$$\text{(1)}\qquad \begin{cases} \boldsymbol{u}_t+(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}-\triangle\boldsymbol{u}+\nabla\pi=0,\\ \nabla\cdot\boldsymbol{u}=0,\\ \boldsymbol{u}(x,0)=\boldsymbol{u}_0,\end{cases}$$

where $\boldsymbol{u}=(u_1(x,t),u_2(x,t),u_3(x,t))$ is the fluid velocity, $\pi=\pi(x,t)$ is a scalar pressure; and $\boldsymbol{u}_0$ is the prescribed initial velocity filed satisfying the compatibility condition $\nabla\cdot\boldsymbol{u}_0=0$.

The existence of a global weak solution

$$\boldsymbol{u}\in L^\infty(0,T;L^2(\mathbb{R}^3))\cap L^2(0,T;H^1(\mathbb{R}^3))$$

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to (1) has long been established by Leray [10], see also Hopf [9]. But the issue of regularity and uniqueness of $\boldsymbol{u}$ remains open. Initialed by Serrin [15, 16] and Prodi [14], there have been a lot of literatures devoted to finding sufficient conditions to ensure $\boldsymbol{u}$ to be smooth, see, e.g., [1, 2, 3, 4, 5, 6, 7, 8, 12, 13, 18, 19, 21, 20, 22, 23, 24, 25, 26, 27] and references cited therein. Noticeably, the following Ladyzhenskaya-Prodi-Serrin condition ([6, 14, 15, 16]):

$$\boldsymbol{u} \in L^p(0,T;L^q(\mathbb{R}^3)), \text{ with } \frac{2}{p}+\frac{3}{q}=1, \quad 3 \leq q \leq \infty \tag{2}$$

can ensure the smoothness of the solution.

Note that the limiting case $L^\infty(0,T;L^3(\mathbb{R}^3))$ in (2) does not fall into the framework of standard energy method, which was proved by Escauriaza, Seregin and Šverák [6] using backward uniqueness theorem. Due to the fact that

$$L^3(\mathbb{R}^3) \subset \dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3), \text{ but } L^3(\mathbb{R}^3) \neq \dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3),$$

we shall consider in this paper the regularity of solutions of (1) in $\dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3)$. However, we could not prove a regularity criterion as $L^\infty(0,T;\dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3))$, since the function in $\dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3)$ has no decay at infinity, which ensures that the solution is smooth outside an big ball centered at origin so that the backward uniqueness theorem can be applied.

Before stating the precise result, let us recall the weak formulation of (1).

Definition 1. *Let $\boldsymbol{u}_0 \in L^2(\mathbb{R}^3)$ satisfying $\nabla\cdot\boldsymbol{u}_0=0$, $T>0$. A measurable vector-valued function $\boldsymbol{u}$ defined in $[0,T]\times\mathbb{R}^3$ is said to be a weak solution to* (1) *if*

(1) $\boldsymbol{u} \in L^\infty(0,T;L^2(\mathbb{R}^3)) \cap L^2(0,T;H^1(\mathbb{R}^3))$;

(2) $\boldsymbol{u}$ *satisfies* $(1)_{1,2}$ *in the sense of distributions;*

(3) $\boldsymbol{u}$ *satisfies the energy inequality:*

$$\|\boldsymbol{u}(t)\|_{L^2}^2 + 2\int_0^t \|\nabla\boldsymbol{u}(s)\|_{L^2}^2 \, \mathrm{d}s \leq \|\boldsymbol{u}_0\|_{L^2}, \quad \forall\, t\in[0,T].$$

Now, our main result reads:

Theorem 2. *Let $\boldsymbol{u}_0 \in L^2(\mathbb{R}^3)$ satisfying $\nabla\cdot\boldsymbol{u}_0=0$, $T>0$. Assume that $\boldsymbol{u}$ is a weak solution of* (1) *in* $[0,T]$. *If there exists an absolute constant $\varepsilon_0>0$ such that*

$$\|\boldsymbol{u}\|_{L^\infty(0,T;\dot{B}^{-1}_{\infty,\infty})} \leq \varepsilon_0, \tag{3}$$

then $\boldsymbol{u}$ is smooth in $(0,T)$.

The rest of this paper is organized as follows. In Section 2, we recall the definition of Besov spaces and an interpolation inequality. Section 3 is devoted to proving Theorem 2.

2. PRELIMINARIES

We first introduce the Littlewood-Paley decomposition. Let $\mathscr{S}(\mathbb{R}^3)$ be the Schwartz class of rapidly decreasing functions. For $f \in \mathscr{S}(\mathbb{R}^3)$, its Fourier transform $\mathscr{F}f = \hat{f}$ is defined as

$$\hat{f}(\xi) = \int_{\mathbb{R}^3} f(x) e^{-ix\cdot\xi}\,\mathrm{d}x.$$

Let us choose an non-negative radial function $\varphi \in \mathscr{S}(\mathbb{R}^3)$ such that

$$0 \le \hat{\varphi}(\xi) \le 1, \quad \hat{\varphi}(\xi) = \begin{cases} 1, & \text{if } |\xi| \le 1, \\ 0, & \text{if } |\xi| \ge 2, \end{cases}$$

and let

$$\psi(x) = \varphi(x) - 2^{-3}\varphi(x/2),\ \varphi_j(x) = 2^{3j}\varphi(2^j x),\ \psi_j(x) = 2^{3j}\psi(2^j x), \quad j \in \mathbb{Z}.$$

For $j \in \mathbb{Z}$, the Littlewood-Paley projection operators S_j and $\triangle_j$ are, respectively, defined by

$$S_j f = \varphi_j * f, \quad \triangle_j f = \psi_j * f.$$

Observe that $\triangle_j = S_j - S_{j-1}$. Also, it is easy to check that if $f \in L^2(\mathbb{R}^3)$, then

$$S_j f \to 0, \text{ as } j \to -\infty; \quad S_j f \to f, \text{ as } j \to \infty,$$

in the L^2 sense. By telescoping the series, we have the following Littlewood-Paley decomposition

$$f = \sum_{j=-\infty}^{\infty} \triangle_j f,$$

for all $f \in L^2(\mathbb{R}^3)$, where the summation is in the L^2 sense.

Let $s \in \mathbb{R}$; $p,q \in [1,\infty]$, the homogeneous Besov space $\dot{B}^s_{p,q}(\mathbb{R}^3)$ is defined by the full dyadic decomposition such as

$$\dot{B}^s_{p,q} = \left\{ f \in \mathscr{Z}'(\mathbb{R}^3);\ \|f\|_{\dot{B}^s_{p,q}} = \left\| \left\{ 2^{js} \left\| \triangle_j f \right\|_{L^p} \right\}_{j=-\infty}^{\infty} \right\|_{\ell^q} < \infty \right\},$$

where $\mathscr{Z}'(\mathbb{R}^3)$ is the dual space of

$$\mathscr{Z}(\mathbb{R}^3) = \left\{ f \in \mathscr{S}(\mathbb{R}^3);\ D^\alpha \hat{f}(0) = 0, \quad \forall\, \alpha \in \mathbb{N}^3 \right\}.$$

The following interpolatin inequality will be need in Section 3,

$$\|f\|_{L^q} \le C \|f\|^{\frac{2}{q}}_{\dot{H}^{\alpha\left(\frac{q}{2}-1\right)}} \|f\|^{1-\frac{2}{q}}_{\dot{B}^{-\alpha}_{\infty,\infty}}, \quad \forall\, f \in \dot{H}^{\alpha\left(\frac{q}{2}-1\right)}(\mathbb{R}^3) \cap \dot{B}^{-\alpha}_{\infty,\infty}(\mathbb{R}^3), \tag{4}$$

where $2 < q < \infty$ and $\alpha > 0$. See [11] for the proof.

3. Proof of Theorem 2

In this section, we shall prove Theorem 2.

For any $\varepsilon \in (0,T)$, due to the fact that $\boldsymbol{u} \in L^2(0,T;H^1(\mathbb{R}^3))$, we may find a $\delta \in (0,\varepsilon)$, such that $\nabla \boldsymbol{u}(\delta) \in L^2(\mathbb{R}^3)$. Take this $\boldsymbol{u}(\delta)$ as initial data, there exists an $\tilde{\boldsymbol{u}} \in C([\delta,\Gamma^*), H^1(\mathbb{R}^3)) \cap L^2(0,\Gamma^*;H^2(\mathbb{R}^3))$, where $[\delta,\Gamma^*)$ is the life span of the unique strong solution, see [17]. Moreover, $\tilde{\boldsymbol{u}} \in C^\infty(\mathbb{R}^3 \times (\delta,\Gamma^*))$. According to the uniqueness result, $\tilde{\boldsymbol{u}} = \boldsymbol{u}$ on $[\delta,\Gamma^*)$. If $\Gamma^* \ge T$, we have already that $\boldsymbol{u} \in C^\infty(\mathbb{R}^3 \times (0,T))$, due to the arbitrariness of $\varepsilon \in (0,T)$. In case $\Gamma^* < T$, our strategy is to show that $\|\nabla \boldsymbol{u}(t)\|_2$ remains bounded independently of $t \nearrow \Gamma^*$. The standard continuation argument then yields that $[\delta,\Gamma^*)$ can not be the maximal interval of existence of $\tilde{\boldsymbol{u}}$, and consequently $\Gamma^* \ge T$. This concludes the proof.

Multiplying $(1)_1$ by $-\triangle \boldsymbol{u}$, and integrating over $\mathbb{R}^3$, we obtain

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla \boldsymbol{u}\|^2_{L^2} + \|\triangle \boldsymbol{u}\|^2_{L^2} &= \int_{\mathbb{R}^3} [(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}]\cdot \triangle \boldsymbol{u}\,\mathrm{d}x \\ &\equiv I. \end{aligned} \tag{5}$$

By Hölder inequality,

$$I \le \|\boldsymbol{u}\|_{L^6} \|\nabla \boldsymbol{u}\|_{L^3} \|\triangle \boldsymbol{u}\|_{L^2}.$$

Invoking (4) with $q = 6$, $\alpha = 1$; and $q = 3$, $\alpha = 2$, we may further estimate I as

$$\begin{aligned} I &\le C \left(\|\boldsymbol{u}\|^{\frac{1}{3}}_{\dot{H}^2} \|\boldsymbol{u}\|^{\frac{2}{3}}_{\dot{B}^{-1}_{\infty,\infty}} \right) \left(\|\nabla \boldsymbol{u}\|^{\frac{2}{3}}_{\dot{H}^1} \|\nabla \boldsymbol{u}\|^{\frac{1}{3}}_{\dot{B}^{-2}_{\infty,\infty}} \right) \|\triangle \boldsymbol{u}\|_{L^2} \\ &= C \|\boldsymbol{u}\|_{\dot{B}^{-1}_{\infty,\infty}} \|\triangle \boldsymbol{u}\|^2_{L^2}. \end{aligned} \tag{6}$$

Substituting (6) into (5), we see

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla \boldsymbol{u}\|_{L^2}^2+\left(1-C\|\boldsymbol{u}\|_{\dot{B}^{-1}_{\infty,\infty}}\right)\|\triangle \boldsymbol{u}\|_{L^2}^2\leq 0.$$

Thus, if

$$\|\boldsymbol{u}\|_{\dot{B}^{-1}_{\infty,\infty}}\leq \frac{1}{C}\equiv \varepsilon_0,$$

we deduce that $\|\nabla \boldsymbol{u}\|_{L^2}$ is decreasing, and hence

$$\|\nabla \boldsymbol{u}(t)\|_{L^2}\leq \|\nabla \boldsymbol{u}(\delta)\|_{L^2}, \quad \forall\, \delta\leq t<\Gamma^*.$$

The proof of Theorem 2 is completed.

Conflict of Interests

The author declares that there is no conflict of interests.

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References

[1] H. Beirão da Veiga, A new regularity class for the Navier-Stokes equations in $\mathbb{R}^n$, Chinese Ann. Math. Ser. B 16 (1995), 407-412.

[2] H. Beirão da Veiga, L.C. Berselli, On the regularizing effect of the vorticity direction in incompressible viscous flows, Differential Integral Equations 15 (2002), 345-356.

[3] C.S. Cao, E.S. Titi, Global regularity criterion for the 3D Navier-Stokes equations involving one entry of the velocity gradient tensor, Arch. Rational Mech. Anal. 202 (2011), 919-932.

[4] D. Chae, J. Lee, Regularity criterion in terms of pressure for the Navier-Stokes equations, Nonlinear Anal. 46 (2001), 727-735.

[5] P. Constantin, C. Fefferman, Direction of vorticity and the problem of global regularity for the Navier-Stokes equations, Indiana Univ. Math. J. 42 (1993), 775-789.

[6] L. Escauriaza, G. Seregin, V. Šverák, Backward uniqueness for parabolic equations, Arch. Ration. Mech. Anal. 169 (2003), 147-157.

[7] J.S. Fan, S. Jiang, G.X. Ni, On regularity criteria for the n-dimensional Navier-Stokes equations in terms of the pressure, J. Differential Equations 244 (2008), 2963-2979.

[8] X.W. He, S. Gala, Regularity criterion for the weak solutions to the Navier-Stokes equations in terms of the pressure in the class $L^2(0,T;\dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3))$, Nonlinear Anal. 12 (2011), 3602-3607.

[9] E. Hopf, Über die Anfangswertaufgabe für die hydrodynamischen Grundgleichungen, Math. Nachr. 4 (1951), 213-231.

[10] J. Leray, Sur le mouvement d'un liquide visqueux emplissant l'espace, Acta Math. 63 (1934), 193-248.

[11] Y. Meyer, P. Gerard, F. Oru, Inégalités de Sobolev précisées, Séminaire É quations aux dérivées partielles (Polytechnique) (1996-1997), Exp. No. 4, 8 pp.

[12] J. Neustupa, P. Penel, Anisotropic and geometric criteria for interior regularity of weak solutions to the 3D Navier-Stokes equations, in Mathematical Fluid Mechanics (Recent Results and Open Problems), Advances in Mathematical Fluid Mechanics, edited by J. Neustupa, and P. Penel (Birkhäuser, Basel-Boston-Berlin, (2001), 239-267.

[13] P. Penel, M. Pokorný, Some new regularity criteria for the Navier-Stokes equations containing the gradient of velocity, Appl. Math. 49 (2004), 483-493.

[14] G. Prodi, Un teorema di unicitá per le equazioni di Navier-Stokes, Ann. Mat. Pura Appl. 48 (1959), 173-182.

[15] J. Serrin, On the interior regularity of weak solutions of the Navier-Stokes equations. Arch. Rational Mech. Anal. 9 (1962), 187-191.

[16] J. Serrin, The initial value problems for the Navier-Stokes equations. in Nonlinear Problems, edited by R. E. Langer (University of Wisconsin Press, Madison, WI, 1963.

[17] R. Temam, Navier-Stokes equations, Theory and numerical analysis, North-Holland, 1977.

[18] Z.F. Zhang and Q.L. Chen, Regularity criterion via two components of vorticity on weak solutions to the Navier-Stokes equations in $\mathbb{R}^3$, J. Differential Equations 216 (2005), 470-481.

[19] Z.J. Zhang, A Serrin-type regularity criterion for the Navier-Stokes equations via one velocity component, Commu. Pure Appl. Anal. (2013), 117-124.

[20] Z.J. Zhang, A remark on the regularity criterion for the 3D Navier-Stokes equations involving the gradient of one velocity component, J. Math. Anal. Appl. **414** (2014), 472–479.

[21] Z.J. Zhang, Z.A. Yao, P. Li, C.C. Guo, M. Lu, Two new regularity criteria for the 3D Navier-Stokes equations via two entries of the velocity gradient, Acta Appl. Math. 123 (2013), 43-52.

[22] Z.J. Zhang, P. Li, D.X. Zhong, Navier-Stokes equations with regularity in two entries of the velocity gradient tensor, Appl. Math. Comput. 228 (2014), 546-551.

[23] Z.J. Zhang, D.X. Zhong, L. Hu, A new regularity criterion for the 3D Navier-Stokes equations via two entries of the velocity gradient tensor, Acta Appl. Math. 129 (2014), 175–181.

[24] Z.J. Zhang, Z.A. Yao, M. Lu, L.D. Ni, Some Serrin-type regularity criteria for weak solutions to the Navier-Stokese equations, J. Math. Phys. 52 (2011), 053103.

[25] Y. Zhou, A new regularity criterion for weak solutions to the Navier-Stokes equations, J. Math. Pures Appl. 84 (2005), 1496-1514.

[26] Y. Zhou, M. Pokorný, On a regularity criterion for the Navier-Stokes equations involving gradient of one velocity component, J. Math. Phys. 50 (2009), 123514.

[27] Y. Zhou, M. Pokorný, On the regularity of the solutions of the Navier-Stokes equations via one velocity component, Nonlinearity 23 (2010), 1097-1107.